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EXCELLENT DOMINATION IN FUZZY GRAPHS

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Abstract. Let G be a fuzzy graph. A subset D of V is said to be Fuzzy dominating set if every vertex $u \in V(G)$ there exists a vertex $v \in V - D$ such that $uv \in E(G)$ and $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$. The minimum Cardinality of fuzzy dominating set is denoted by γ^f. A graph G is said to be fuzzy excellent if every vertex of G belongs to γ^f-sets of G. In this paper, we give a construction to imbedded non-excellent fuzzy graph G in an excellent fuzzy graph H such that $\gamma^f(H) \leqslant \gamma^f(G) + 2$. We also show that for a given non-excellent fuzzy graph G, there is subdivision of G which is fuzzy excellent. Also, we introduce the concept of γ^f-flexible, fuzzy bridge independent dominating number $\gamma^f_{b_i}$ and obtain some interesting results for this new parameter in excellent fuzzy graphs.

1. Introduction

A mathematical frame work to describe the phenomena of uncertainty in real life situation is first suggested by L. A. Zadeh in 1965. Rosenfield [8] introduced the notion of fuzzy graphs and several fuzzy analogs of graph theoretic concepts Such as Path, Cycle and Connectedness. The study of dominating sets in graphs was begun by Orge and Berge. V. R. Kulli [10] wrote on theory of domination in graphs. A. Somasundaram, S. Somasundaram [9] presented the concepts of Domination in fuzzy graphs. Here we introduced the concept of Excellent domination in fuzzy graphs and their related concepts.

2. Preliminaries

Definition 2.1. *A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ where for all $u, v \in V$, we have $\mu(u, v) \leqslant \sigma(u) \wedge \sigma(v)$.*

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DEFINITION 2.2. *The order p and size q of a fuzzy graph $G = (\sigma, \mu)$ are defined to be* $p = \sum_{x \in V} \sigma(x)$ *and* $q = \sum_{xy \in E} \mu(xy)$.

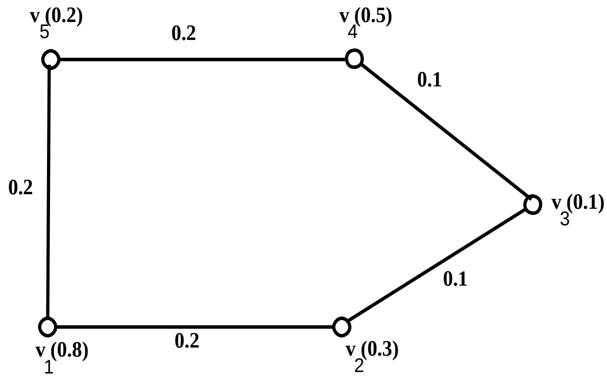
DEFINITION 2.3. *The degree of vertex u is defined as the sum of the weights of the edges incident at u and is denoted by $d(u)$.*

DEFINITION 2.4. *A subset D of V is called an fuzzy dominating set if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ such that $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$. The minimum cardinality of such a dominating set is denoted by γ^f and is called the fuzzy domination number of G.*

3. Main definitions and Results

DEFINITION 3.1. *A fuzzy graph G is said to be fuzzy excellent if for every vertex of G belongs to γ^f-sets of G. A vertex which belongs to γ^f-set is called Fuzzy good. (i.e) A Fuzzy graph G is said to be Fuzzy excellent if for every vertex of G is Fuzzy good.*

EXAMPLE 3.1.



Here γ^f-sets of G are $\{v_1, v_3\}$, $\{v_2, v_5\}$, $\{v_2, v_4\}$. Hence Every vertex is Fuzzy good. Therefore G is Fuzzy excellent.

DEFINITION 3.2. *A Fuzzy graph G is said to be vertex-transitive if given any two vertices u and $v(\neq u)$ of G, there is an automorphism ϕ^f of G such that $\phi^f(u) = v$.*

THEOREM 3.1. *Every vertex transitive fuzzy graph G is fuzzy excellent.*

PROOF. Let G be a vertex transitive fuzzy graph and D be a γ^f-set of G. Let $u \in V(G)$, select any vertex $v \in D$ such that $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$. As G is vertex transitive, there is an automorphism ϕ^f of G which maps v to u. Let $D' = \phi^f(D) = \{\phi^f(w)/w \in D\}$. Now we claim that D' is also γ^f-set of G. As ϕ^f is an automorphism, $|D'| = |D| = \gamma^f(G)$. Let a be a vertex of G not in D' and let $b \in V(G)$ be such that $\phi^f(b) = a$. As $a \notin D'$, $b \notin D$ and D is γ^f-set of G, $b \in N(w_1)$ for some $w_1 \in D$. Then $a = \phi^f(b) \in N[\phi^f(w_1)]$. Hence D' is a dominating set of G and as $|D'| = \gamma^f(G)$, D' is a γ^f-set of G. Thus given any vertex $'a'$ of G, there is a γ^f-set of G containing $'a'$. Therefore G is Excellent fuzzy graph. $\square$

THEOREM 3.2. *Let G be a non-fuzzy excellent graph. Then there exist a fuzzy graph H such that*

(i) *H is γ^f-fuzzy excellent.*
(ii) *$\gamma^f(G) < \gamma^f(H) \leqslant \gamma^f(G) + 2$.*
(iii) *G is an induced subgraph of H.*

PROOF. Let G be a non-fuzzy excellent graph. Let Z be a set of all fuzzy good vertices of G and T be the set of all fuzzy bad vertices of G. Since G is non-fuzzy excellent, $T \neq \phi$. Let $T = \{t_1, t_2, ...t_n\}$ and T^* be non-empty subset of T. Then

$$\gamma^f(G - T^*) \geqslant \gamma^f(G) - |T^*| + 1 \tag{3.1}$$

If $\gamma^f(G - T^*) \geqslant \gamma^f(G) - |T^*| + 1$, then we say that T^* is an optimal fuzzy bad set. If T^* is an optimal fuzzy bad set and $G - T^*$ is γ^f-fuzzy excellent then we say that T is an extreme optimal fuzzy bad set.

Case 1: Let $|T| = 1$ (i.e., $n = 1$) and add a new vertex $'t'$ and join it with t_1 such that $\mu(tt_1) \leqslant \sigma(t) \wedge \sigma(t_1)$. In this case the resulting graph is fuzzy excellent graph H. Clearly G is an induced subgraph of H, $\gamma^f(H) = \gamma^f(G) + 1$. For every γ^f-set D of G, $D \cup \{t\}$ and $D \cup \{t_1\}$ are $\gamma^f(H)$-sets of G. Therefore H is γ^f-fuzzy excellent.

Case 2: Assume that $|T| \geqslant 2$ and $T = \{t_1, t_2, ..., t_n\}$. Now we assume that there is a non-empty subset T^* of T such that T^* is extreme optimal fuzzy bad set. Let $T^* = \{t_1, t_2, ..., t_k\}$. In this case we construct H as follows:

$$V(H) = V(G) \cup \{v_1, v_2, ..., v_k\}$$
$$E(H) = E(G) \cup \{v_i t_i / i = 1, 2, ..., k\} \text{ and } \mu(v_i t_i) \leqslant \sigma(v_i) \wedge \sigma(t_i).$$

Then obviously

(i) G is induced subgraph of H.
(ii) $\gamma^f(G) \leqslant \gamma^f(H)$
(iii) $\gamma^f(H) = \gamma^f(G) + 1$, for each set γ^f-set D of $G - T^*$, $D \cup T^*$ is a dominating set for H.
(iv) H is fuzzy excellent.

Case 3: Let us consider the dominating set D of G such that $T \subset D$ and $|D| = \gamma^f(G) + 1$. We construct a fuzzy graph H_1 as follows. Let $\{t, v_1, v_2, ...v_n, w_1, w_2, ...w_n\}$ be a set disjoint with $V(G)$. Let

$$V(H_0) = V(G) \cup \{t, v_1, v_2, ..., v_n, w_1, w_2, ...w_n\}$$

and

$$E(H_0) = E(G) \cup \{v_i t_i / i = 1, 2, ..., n\}$$
$$\cup \{v_i w_j, w_i w_j / i \neq j,\ i = 1, 2, ..., n\} \cup \{t w_j / j = 1, 2, ..., n\}.$$

Clearly G is an induced subgraph of H_1. Let

$$V_1 = \{v_1, v_2, ..., v_n\},\ V_2 = \{t, w_1, w_2, ..., w_n\} \text{ and } V_0 = V(G).$$

If D is a dominating set of G, then $D \cup \{v_1, w_1\}$ is a dominating set of H_1. Thus

$$\gamma^f(H_1) \leqslant \gamma^f(G) + 2 \tag{3.2}$$

If S is a minimum dominating set of H_1, then $|S \cap V_2| \geqslant 1$. If $|S \cap V_2| = 1$, then either $S \cap V_1 \neq \emptyset$ or $S \cap T \neq \emptyset$. Assume that $S \cap V_1$ is empty, since any $v_i \in S$ can be replaced by t_i. Then $S - V_2$ is a dominating set of G. If $|S \cap V_2| = 1$, then by our assumption $S \cap V_1$ is empty, the set $S - V_2$ contains a vertex from T and $|T - V_2| \geqslant \gamma^f(G) + 1$. If $|T \cap V_2| \geqslant 2$ then $|S \cap V_2| \geqslant \gamma^f(G)$. In either case $|S| \geqslant \gamma^f(G) + 2$ and $\gamma^f(H_1) = \gamma^f(G) + 2$. We now show that H_1 is fuzzy excellent,

(1) If $a \in Z$, there is a $\gamma^f(G)$-set D of G containing $'a'$. Then $D \cup \{v_1, w_1\}$ is a $\gamma^f(H_1)$-set;
(2) For each $i \in \{1, 2, ..., n\}$, $D \cup \{t_i, w_i\}$ and $D \cup \{v_i, w_i\}$ are $\gamma^f(H_1)$ sets of H_1, where D is any $\gamma^f(G)$ set of G;
(3) If T is an optimal fuzzy bad set, take a γ^f-set S for $G - T$, then $|S_1| = \gamma^f(G) - n + 1$ and $S \cup V_1 \cup \{t\}$ is a γ^f-set for H_1.

Case 4: Let G be a fuzzy graph which cannot be considered under any of the above cases. Construct the graph H_1 as in Case 3. Let $H = H_1 - \{t\}$, whenever D is a γ^f-set of G, $D \cup \{v_1, w_1\}$ is a dominating set for H. So $\gamma^f(H) \leqslant \gamma^f(G) + 2$. Let S be a γ^f-set for H. The set S should contain atleast one element from $V_1 \cup V_2$, where $V_1 = \{v_i/i = 1, 2, ..., n\}$, $V_2 = \{w_i/i = 1, 2, ..., n\}$. Let $V_0 = V(G)$.

Subcase 1: Let $S \cap V_1 = \emptyset$. Then either $|S \cap V_2| = 2$ or $|S \cap V_2| = 1$ and $S \cap T \neq \emptyset$. As $S \cap V_1 = \emptyset$, $S \cap V_0$ is a dominating set for G and hence

$$|S \cap V_0| \geqslant \begin{cases} \gamma^f(G) & \text{if } S \cap T = \emptyset \\ \gamma^f(G) + 1 & \text{if } S \cap T \neq \emptyset. \end{cases}$$

Thus in this case $|S| \geqslant \gamma^f(G) + 2$.

Subcase 2: Let $S \cap V_1 \neq \emptyset$. Then $|S \cap (V_1 \cap V_2)| \geqslant 2$. Let $T' = \{t_i/v_i \in S\}$. Then $S \cap V_0$ dominates $G - T'$. Hence $(S \cap V_0) \cup T'$ dominates of G and contains atleast one bad vertex in G. Then $|(S \cap V_0) \cup T'| \geqslant \gamma^f(G) + 1$ and $|(S \cap V_0)| \geqslant \gamma^f(G) + 1 - |T'|$. As $|T'| = |(S \cap V_1)|$ it follows that if $T \cap V_2 \neq \emptyset$, then $|S| \geqslant \gamma^f(G) + 2$. As $S \cap V_1$ does not dominate any vertex in $V_1 - S$, if $S \cap V_2 = \emptyset$, then $S \cap V_0$ must contain $T - T'$. In this case $(S \cap V_0) \cup T'$ is a dominating set for G, containing T. Hence $|(T \cap V_0) \cup T'| \geqslant \gamma^f(G) + 2$. So, $|S| = |S \cap V_0| + |S \cap V_1| = |S \cap V_0| + |T'| \geqslant \gamma^f(G) + 2$. H is γ^f-excellent:

(1) Given any vertex $a \in Z$, let D be any $\gamma^f(G)$ set for G containing $'a'$. then $D \cap \{v_i, w_i\}$ is a γ^f-set for H containing v_i, w_i and a.
(2) Let D be any $\gamma^f(G)$-set for G and $t_i \in T$. Then $D \cap \{t_i, w_i\}$ is a γ^f-set for H containing t_i and w_i.

□

Definition 3.3. (Subdivision of fuzzy graph). *Let G be a fuzzy graph. Then $S^f(G)$ denotes the Subdivision of fuzzy graph G and is obtained from G by subdividing each edge of G once. A fuzzy graph H is said to be a Subdivision of G, if it is obtained from G by subdividing each edge of G at most once.*

Theorem 3.3. *If a fuzzy graph G is not excellent, then there is a Subdivision of fuzzy graph H of G which is excellent.*

Proof. Let G be a fuzzy graph which is not excellent. Let A be the set of all good vertices of G and let $B = V(G) - A$. Since G is not excellent, $B \neq \emptyset$. Then fix one $x \in B$. Among all the γ^f-sets of G, select one γ^f-set D_1 such that $|N^f(x) \cap D_1|$ is maximum. Let $V_1 = N^f(x) \cap D_1 \subseteq A$. For each $y \in N^f(x) \cap D_1$, subdivide the edge xy. Let w_y be the vertex introduced while subdividing the edge xy, such that $\mu(xy) \leqslant \sigma(x) \wedge \sigma(y)$. Let the resulting graph be H_1. Also, $V(H_1) = V(G) \cup \{w_y/y \in N^f(x) \cap D_1 \text{ in } G\}$. As $D_1 \cup \{x\}$ is dominating set for H_1, $\gamma^f(H_1) \leqslant \gamma^f(G)+1$. Now we have to prove that $\gamma^f(H_1) = \gamma^f(G)+1$. Assume that $\gamma^f(H_1) = \gamma^f(G)$ and let S be a γ^f-set of H_1. If $x \notin S$ and $x_y \notin S$, $\forall y \in V_1$, then $V_1 \subseteq S$. Therefore, S must contain at least one vertex of $N^f(x) \cap (V(G) - V_1)$. As $|S| = \gamma^f(G)$ and $w_y \notin S$, $\forall y \in S$,S is a γ^f-set for G also. Hence, $S \cap (N^f(x) \cap (V(G) - V_1) \subseteq A$, $|S \cap N^f(x)| > |D_1 \cap N(x_1)|$ which is contradiction to the selection of D_1. Thus S must contain either x or at least one w_y. If $x \in S$, then take $S_1 = (S \cup \{y/w_y \in S\}) - \{w_y/w_y \in S\}$. Then S_1 is a γ^f-set for G and as $x \in S$, $x \in A$ which is a contradiction. Hence $x \notin S$ and $w_y \in S$ for some y. Fix one y_0 such that $y_0 \in S$. Then $S_2 = (S \cup \{y/y \neq y_0, w_y \in S\}) - \{w_y/y \neq y_0, w_y \in S\}$ is also a dominating set for H_1. Note that $x \notin S_2$, $w_y \notin S_2$ for every $y \neq y_0$ and $w_y \in S_2$. Thus $S_2 \cup \{x\} - \{wy_0\}$ is a γ^f-set for G which is a contradiction as $x \notin A$. Then $\gamma^f(H_1) \neq \gamma^f(G)$. Hence $\gamma^f(H_1) = \gamma^f(G)+1$. For each $y \in V_1, D_1 \cup \{w_y\}$ and $D_1 \cup \{x\}$ are γ^f-sets of H_1. Let $a \in A$ and D^* be a γ^f-set of G such that $a \subseteq D^*$. Then $D^* \cup \{x\}$ is a γ^f-set of H_1 containing $'a'$. In H_1, the set of all vertices which are good and contains $A \cup \{x, w_y/y \in V_1\}$ and hence the set of all bad vertices in H_1, is a proper subset of B. Thus we see that

(i) H_1 is a subdivision of G.
(ii) The set of all bad vertices of H_1, is a proper subset of the set of all bad vertices of G.
(iii) If x_0 is a bad vertex of H_1 and D_2 is a γ^f-set of H_1 such that, $|N^f(x_0) \cap D_2|$ is maximum, then obtain a subdivision H_2 of H_1 by subdividing the edges x_0y, $y \in N^f(x_0) \cap D_2$, $N^f(x_0) \subset V(G)$. As $N^f(x_0)$ is contained in $V(G)$, the subdivision H_2 of H_1 is a subdivision of G. That is, the edges H_1 which are subdivides to obtain H_2 are edges in G and they are not subdivided while obtaining H_2.
Therefore, the number of bad vertices in $H_2 <$ the number of bad vertices of $H_1 <$ the number of bad vertices of G.

Proceeding like this, we obtain a finite sequence $H_1, H_2, ..., H_k$ of subdivision of G with the following property:

(i) Each H_{i+1} is a subdivision of H_i.

(ii) The number of bad vertices of $H_{i+1} <$ the number of bad vertices of H_i.

Hence for some $k_1, (\leqslant |B|)$, we obtain an excellent graph H_k. $\square$

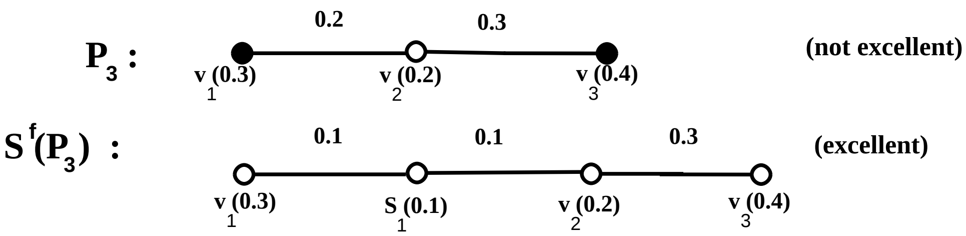
Definition 3.4. (Fuzzy excellent subdivision number) *For a given fuzzy graph G, if $S^f(G)$ is a subdivision of G, $\left|V(S^f(G)) - V(G)\right|$ is denoted by $P(S^f(G))$, then*

$min\{P(S^f(G)) : S^f(G)$ *is a subdivision of* G *and* $S^f(G)$ *is fuzzy excellent* $\}$

is called the fuzzy excellent subdivision number of G and is denoted by $ES^f d_n(G)$, We note that

(i) *If G itself is fuzzy excellent, then $ES^f d_n(G) = 0$.*
(ii) *For $G = K_{1,n}(n \geqslant 2), ES^f d_n(G) = n - 1$.*
(iii) *$ES^f d_n(G) = 1$ if $n \equiv 0(mod\,3)$. If $n \equiv 0(mod\,3)$ and $n \geqslant 6$, P_n has exactly $\frac{n}{3}$ good vertices and $\frac{2n}{3}$ bad vertices, while subdividing any one of the edges of P_n, it becomes an excellent graph. Thus $ES^f d_n(G) = 1$ while $|B(P_n)| = \frac{2n}{3}$.*

Example 3.2.



Also P_3 contains one good vertex and two bad vertex and $ES^f d_n(P_3) = 1$.

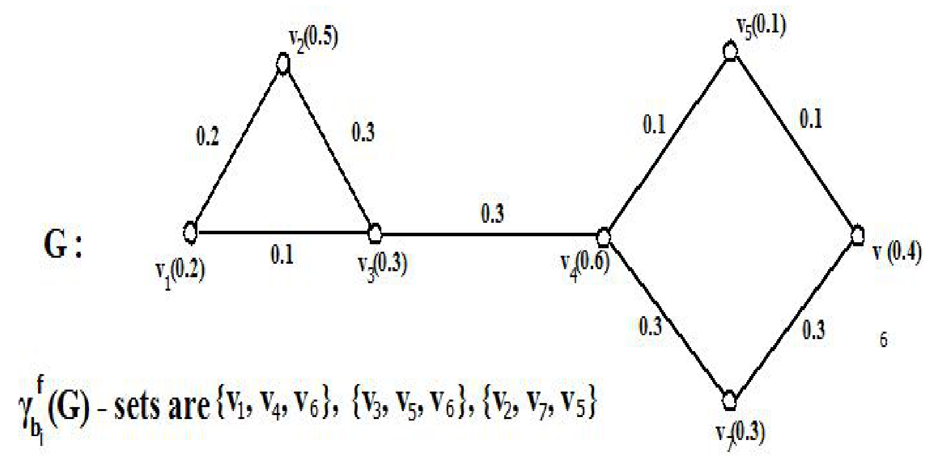
4. Fuzzy Bridge independent graph

Definition 4.1. (Fuzzy Bridge) *A subset S of G is said to be fuzzy bridge independent dominating set of G if every $u \in S$, there exists a vertex $v \in V - S$ such that $uv \in E$ and $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$ does not increase the component of G.*

Definition 4.2. (Fuzzy Bridge dominating set) *A dominating set D of G is said to be an fuzzy bridge independent dominating set, if $\langle D \rangle$ contains no bridge of G. The minimum cardinality of a fuzzy bridge independent dominating set of G is said to be the fuzzy bridge independent domination numner and is denoted by $\gamma^f_{b_i}(G)$.*

Definition 4.3. *A fuzzy graph G is said to be $\gamma^f_{b_i}$-excellent if each vertex of G is in some $\gamma^f_{b_i}$-set of G.*

Example 4.1.



Therefore every vertex of G belongs to $\gamma^f_{b_i}(G)$. Hence G is $\gamma^f_{b_i}$-fuzzy excellent.

Definition 4.4. *A fuzzy graph G is said to be γ^f-flexible, if given any vertex $u \in G$, there is a γ^f-set S of G not containing u.*

Proosition 4.1. *Every connected fuzzy graph G of order P is an fuzzy induced subgraph of a γ^f-excellent, $\gamma^f_{b_i}$-excellent, γ^f-flexible graph H of order $P+\gamma^f(G)+1$ and further $\gamma^f(G) \leqslant \gamma^f(H) \leqslant \gamma^f_{b_i}(H) \leqslant \gamma^f(G)+1$.*

Proof. Let G be a connected fuzzy graph of order P. Let $D = \{d_1, d_2, ..., d_m\}$ be a γ^f-set of G. Let us construct a fuzzy graph H as follows

$$V(H) = V(G) \cup \{d'_1, d'_2, ..., d'_m, w\}$$
$$E(H) = E(G) \cup \{d_i d'_i / i = 1, 2, ..., m \quad \text{and} \quad \mu(d_i d'_i) \leqslant \sigma(d_i) \wedge \sigma(d'_i)\}$$
$$\cup \{dw / d \subset V(G) - D \;\&\; \mu(dw) \leqslant \sigma(d) \wedge \sigma(w)\}.$$

Clearly $\gamma^f(H) = \gamma^f(G) + 1$. Now let $S = \{d_i, w / i = 1, 2, ..., m\}$. To each i, let $S_i = \{d_i, d'_i, w / j \neq i\}$. To each $u \in V(G) - D$, let $S_u = \{v, d_i / i = 1, 2, ..., m\}$ and $S_0 = \{d'_i, w / i = 1, 2, ..., m\}$. The sets S, $S_i (1 \leqslant i \leqslant m)$, $S_u (u \notin D)$ and S_0 are γ^f-sets of H. It follows that H is γ^f-excellent, $\gamma^f_{b_i}$-excellent, γ^f_i-excellent, γ^f-flexible and $\gamma^f(G) \leqslant \gamma^f(H) \leqslant \gamma^f_{b_i}(H) \leqslant \gamma^f(G) + 1$. $\square$

Lemma 4.1. *Let X_1 and X_2 be two γ^f-sets for G. Let uv be an edge in X_1 such that $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$, which is a fuzzy bridge in G. Let G_1 and G_2 be the components of $G - uv$. Let $A_i = X_1 \cap G_i$ and $B_i = X_2 \cap G_i$ for $i = 1, 2$. Then $A_i \cup B_j$ is a γ^f-set for G, where $\{i, j\} = \{1, 2\}$.*

Proof. If A_i is a γ^f-set for G_i, then A_i is a fuzzy dominating set for G_i. Suppose A_i is not a γ^f-set for G_i. Let C be a γ^f-set for G_i. Clearly $|C| < |A_i|$ and $C \cup A_j, j \neq i$ is a dominating set for G. Thus

$$|C \cup A_j| = |C| + |A_j| < |A_i| + |A_j| = |X_1| = \gamma^f,$$

which is contradiction. Therefore, A_i is γ^f-set of G_i. Let A_i dominates $G_i \cup \{uv\}$ and B_i dominates $G_i - \{uv\}$. Therefore $D_1 = A_1 \cup B_2$ and $D_3 = A_2 \cup B_1$ are fuzzy dominating sets for G.

$$2\gamma^f(G) \leqslant |D_1| + |D_2| = |A_1| + |B_2| + |A_2| + |B_1| = |X_1| + |X_2| = 2\gamma^f(G).$$

Hence $|D_1| = |D_2| = \gamma^f(G)$. Thus $A_i \cup B_j, i \neq j$ is a γ^f-set for G. □

Lemma 4.2. *If G is γ^f-excellent and γ^f-flexible, then G is $\gamma^f_{b_i}$-excellent. Further $\gamma^f_{b_i}(G) = \gamma^f(G)$.*

Proof. Let us assume that G be connected. Let u be a vertex in G. Since G is γ^f-excellent, there is a γ^f-set X_1 containing u. Let $e = ab$ be bridge in G and let $d^f(e) = min\ \{d^f(u,a), d^f(u,b)\}$. Label the bridges of G as $e_1, e_2, ..., e_k$ such that, $i < j$ and $d^f(e_i) \leqslant d^f(e_j)$. This is possible only when each e_i is an bridge. If $\langle X_1 \rangle$ contains no bridge e_i, then X_1 is a $\gamma^f_{b_i}$-set containing u. Suppose that $\langle X_1 \rangle$ contains some e_i. Assume that $e_1, e_2, ...e_{i-1} \notin \langle X_1 \rangle$ and $e_i \in \langle X_1 \rangle$. Let $e_i = ab$, then both $a, b \in X_1$. Let G_1 be the component of $G - e_i$ which contains u and a & the component G_2 contains b. Since G is flexible, there is γ^f-set X_2 of G not containing b. By the above lemma, $D = (X_1 \cap G_1) \cup (X_2 \cap G_2)$ is a γ^f-set of G. By using labelling procedure, $e_1, e_2, ...e_{i-1} \in G_1$ as $e_1, e_2, ...e_{i-1} \notin \langle X_1 \rangle$ they are not in $\langle D \rangle$ also. Thus $e_1, e_2, ...e_{i-1}, e_i \notin \langle D \rangle$. Therefore, we get a γ^f-set D of G containing u and the edges $e_1, e_2, ...e_i \notin \langle D \rangle$. Proceeding like this, we get γ^f-set D' of G containing u and $e_j \notin \langle D' \rangle$, for every j. This dominating set D' is a $\gamma^f_{b_i}$-set of G containing u. Hence, as u is arbitrary, G is $\gamma^f_{b_i}$-excellent and $\gamma^f_{b_i}(G) = \gamma^f(G)$. □

Definition 4.5. (Fuzzy distance) *For any two points u, v of a fuzzy graph we define fuzzy distance between u and v by*

$$d^f(u,v) = \{\textit{the sum of the edges weights of the edges in the shortest } u{-}v \textit{ path such that } \mu(uv) \leqslant \sigma(u) \wedge \sigma(v),\ \forall\ u, v \in P\}.$$

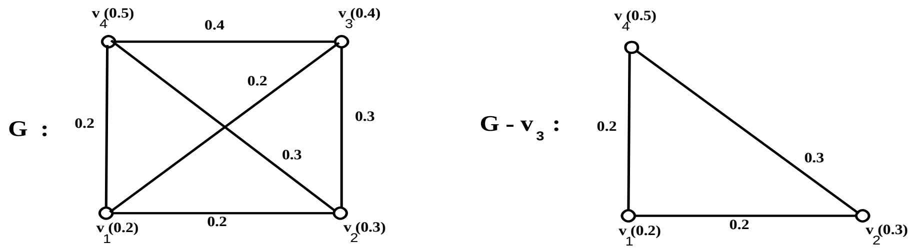
Corollary 4.1. *Let G be a γ^f-excellent graph. Let $u \in V(G)$ such that u is in every γ^f-set of G. Let $v_1, v_2, ..., v_k \in V(G)$ such that uv_i is a bridge in G, $\forall i = 1, 2, ..., k$ and $\mu(uv_i) \leqslant \sigma(u) \wedge \sigma(v_i)$. Then there is a γ^f-set S of G containing all v_i's.*

Corollary 4.2. *Let G be a γ^f-excellent graph. Let u be a vertex which belongs to every γ^f-set of G. Then there is at least one edge uw such that $\mu(uw) \leqslant \sigma(u) \wedge \sigma(w)$, which is not a bridge.*

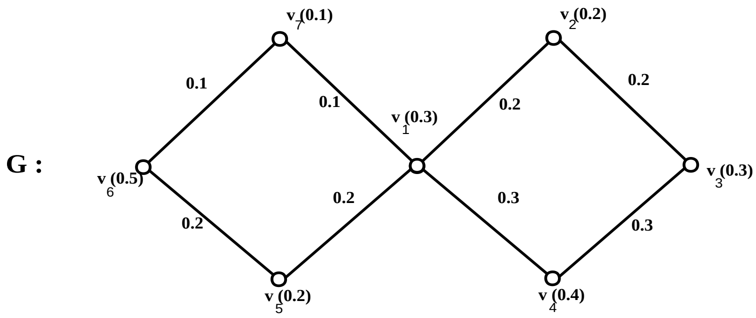
Definition 4.6. *Let G be a fuzzy graph and for each $u \neq v \in V(G)$ there exist $uv \in E(G)$ such that $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$. Then $\gamma^{uf}(G, u) = min\{|D| : D \subseteq V;\ D$ dominates $G - u$, $N[v] \cap S \neq \emptyset\}$.*

Definition 4.7. *(i) If $\gamma^{uf}(G, u) = \gamma^f(G)$ then u is fuzzy level vertex of G.*
(ii) If $\gamma^{uf}(G, u) = \gamma^f(G) - 1$ then u is fuzzy non-level vertex of G.

Example 4.2. 1).



Here $\gamma^f(G) = 1$ and $\gamma^{v_3 f}(G, v_3) = 1$. v_3 is level vertex of G.
2).



Here $\gamma^f(G) = 3$. v_6 is non-level vertex of G.

Lemma 4.3. *For every* $u \in V(G)$, $\gamma^{uf}(G, u) \leqslant \gamma^f(G) \leqslant \gamma^{uf}(G, u) + 1$.

Proof. As every γ^f-set of G dominates $G - u$, we have $\gamma^{uf}(G, u) \leqslant \gamma^f(G)$. Let D be a $\gamma^{uf}(G, u)$-set of G. If $N[u] \cap S \neq \emptyset$, then D dominates G, so in this case $\gamma^f(G) \leqslant |D| = \gamma^{uf}(G, u) \leqslant \gamma^f(G)$ and $\gamma^{uf}(G, u) = \gamma^f(G)$. If $N[u] \cap S = \emptyset$, then $D \cup \{u\}$ is a dominating set of G and hence $\gamma^f(G) \leqslant |D \cup \{u\}| = |D| + 1 \leqslant \gamma^{uf}(G, u) + 1$. □

5. Applications

The fuzzy relations are wide spread and important in the field of Clustering analysis, Computer networks and Pattern recognition. The earliest ideas of dominating sets data back, to the origin of game of Chess in India. In this game, one studies of chess pieces which cover various opposing pieces or various squares of the board.

6. Conclusion

In this paper we define new concept called Excellent Domination in fuzzy graphs and fuzzy bridge independent domination graph. Further, We can extend this concept to various types of Excellent fuzzy graphs.

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