A STUDY OF $n$-ARY SUBGROUPS WITH RESPECT TO $t$-CONORM

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Abstract. In this paper, we introduce a notion of fuzzy $n$-ary subgroups with respect to $t$-conorm ($s$-fuzzy $n$-ary subgroups) in an $n$-ary groups $(G, f)$ and have studied their related properties. The main contribution of this paper are studying the properties of $s$-fuzzy $n$-ary subgroups over $s$-level $n$-ary subgroup of $(G, f)$, $n$-ary homomorphism and $ret_a(G, f)$. Moreover some results of the $S$-product of $s$-fuzzy $n$-ary relations in an $n$-ary groups $(G, f)$ are also obtained.

1. Introduction

The theory of fuzzy set was first developed by Zadeh [29] and has been applied to many branches in mathematics. Later fuzzification of the “group” concept into “fuzzy subgroup” was made by Rosenfeld [28]. This work was the first fuzzification of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. The study of $n$-ary systems was initiated by Kasner [26] in 1904, but the important study on $n$-ary groups was done by Dörnte [3]. The theory of $n$-ary systems have many applications. For example, in the theory of automata [23], $n$-ary semigroup and $n$-ary groups are used. The $n$-ary groupoids are applied in the theory of quantum groups [27]. Also the ternary structures in physics are described by Kerner in [25]. The $n$-ary system dealt in detail [4-9,11,12,14-22]. The first fuzzification of $n$-ary system was introduced by Dudek [10]. Moreover Davvaz et. al [2] have studied fuzzy $n$-ary groups as a generalization of Rosenfeld’s fuzzy groups and have investigated their related properties. The notion of intuitionistic fuzzy sets, as a generalization of the notion of fuzzy set. Dudek [13] has introduced intuitionistic fuzzy sets idea’s in $n$-ary systems and has discussed in detail. Triangular norm($t$-norm) and triangular conorm($t$-conorm) are the most general

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families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. Thus, the $t$-norm generalizes the conjunctive (AND) operator and the $t$-conorm generalizes the disjunctive (OR) operator. In application, $t$-norm $T$ and $t$-conorm $S$ are two functions that map the unit square into the unit interval. To study more about $t$-norm and $t$-conorm generalizes the disjunctive (OR) operator. In this paper, we introduce the notion of fuzzy $n$-ary subgroups with respect to $t$-conorm (s-fuzzy $n$-ary subgroup) in $n$-ary group $(G, f)$ and have investigated their related properties.

2. Preliminaries

A non-empty set $G$ together with one $n$-ary operation $f : G^n \rightarrow G$, where $n \geq 2$, is called an $n$-ary groupoid and is denoted by $(G, f)$. According to the general convention used in the theory of $n$-ary groupoids the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by $x^j_i$. In the case $j < i$, it denoted the empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then instead of $x^{i+t}_{i+1}$ and we write $(x)^t$. In this convention

$$f(x_1, \ldots, x_n) = f(x^n_1)$$

and

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{i+t+1}, \ldots, x_n) = f(x^t_1, x_{i+t+1}^{n+1}).$$

An $n$-ary groupoid $(G, f)$ is called an $(i, j)$-associative if

$$f\left(x^{i-1}_1, f(x^{n+i-1}_i, x^{2n-1}_{n+1})\right) = f\left(x^{j-1}_1, f(x^{n+j-1}_j, x^{2n-1}_{n+j})\right)$$

hold for all $x_1, \ldots, x_{2n-1} \in G$. If this identity holds for all $1 \leq i \leq j \leq n$, then we say that the operation $f$ is associative and $(G, f)$ is called an $n$-ary semigroup. It is clear that an $n$-ary groupoid is associative if and only if it is $(1, j)$-associative for all $j = 2, \ldots, n$. In the binary case (i.e. $n = 2$) it is usual semigroup. If for all $x_0, x_1, \ldots, x_n \in G$ and fixed $i \in \{1, \ldots, n\}$ there exists an element $z \in G$ such that

$$f\left(x^{i-1}_1, z, x^n_{i+1}\right) = x_0$$

(1)

then we say that this equation is $i$-solvable or solvable at the place $i$. If the solution is unique, then we say that (1) is uniquely $i$-solvable. An $n$-ary groupoid $(G, f)$ uniquely solvable for all $i = 1, \ldots, n$ is called an $n$-ary quasigroup. An associative $n$-ary quasigroup is called an $n$-ary group.

Fixing an $n$-ary operation $f$, where $n \geq 3$, the elements $a^{n-2}_2$ we obtain the new binary operation $x \circ y = f(x, a^{n-2}_2, y)$. If $(G, f)$ is an $n$-ary group then $(G, \circ)$ is a group. Choosing different elements $a^{n-2}_2$ we obtain different groups. All these groups are isomorphic[8]. So, we can consider only group of the form

$$ret_0(G, f) = (G, \circ), \text{ where } x \circ y = f(x, a^{(n-2)}_2, y).$$

In this group $e = x, x^{-1} = f(x, a^{(n-3)}_3, x, x)$.

In the theory of $n$-ary groups, the following Theorem plays an important role.
Theorem 2.1. For any n-ary group \((G, f)\) there exist a group \((G, \circ)\), its automorphism \(\varphi\) and an element \(b \in G\) such that
\[
f(x_1^n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \ldots \circ \varphi^{n-1}(x_n) \circ b
\] holds for all \(x_i^n \in G\).

In what follows, \(G\) is a non-empty set and \((G, f)\) is an n-ary group unless otherwise specified.

Definition 2.1. By a t-norm, a function \(T : [0,1] \times [0,1] \to [0,1]\) satisfying the following conditions is meant:

\((T1)\) \(T(x, 1) = x;\)
\((T2)\) \(T(x, y) \leq T(x, z)\) if \(y \leq z;\)
\((T3)\) \(T(x, y) = T(y, x);\)
\((T4)\) \(T(x, T(y, z)) = T(T(x, y), z);\)
for all \(x, y, z \in [0,1].\)

Definition 2.2. By a t-conorm, a function \(S : [0,1] \times [0,1] \to [0,1]\) satisfying the following conditions is meant:

\((S1)\) \(S(x, 0) = x;\)
\((S2)\) \(S(x, y) \leq S(x, z)\) if \(y \leq z;\)
\((S3)\) \(S(x, y) = S(y, x);\)
\((S4)\) \(S(x, S(y, z)) = S(S(x, y), z);\)
for all \(x, y, z \in [0,1].\)

Replacing 0 by 1 in condition \((S1)\), we obtain the concept of t-norm \(T.\)

Definition 2.3. Given a t-norm \(T\) and a t-conorm \(S\), \(T\) and \(S\) are dual (with respect to the negation \(\cdot\)) if and only if \(T(x, y)' = S(x', y').\)

Now we generalize the domain of \(S\) to \(\prod_{i=1}^{n} [0,1]\) as follows:

Definition 2.4. The function \(S_n : \prod_{i=1}^{n} [0,1] \to [0,1]\) is defined by:
\[
S_n(\alpha_i^n) = S_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))
\]
for all \(1 \leq i \leq n, n \geq 2.\)

For a t-conorm \(S\) on \(\prod_{i=1}^{n} [0,1]\), it is denoted by
\[
\Delta_\alpha = \{\alpha \in [0,1] | S(\alpha, \alpha, \ldots, \alpha) = \alpha\}.
\]

It is clear that every t-conorm has the following property:
\[
S(\alpha_i^n) \geq \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}
\]
for all \(\alpha_i^n \in [0,1].\)
3. s-fuzzy n-ary subgroups

Definition 3.1. A fuzzy set \( \mu \) in \( G \) is called a s-fuzzy n-ary subgroup of \((G, f)\) if the following axioms holds:
\[(SFnS1) \quad (\forall x_i \in G, (\mu(f(x^n))) \leq S\{\mu(x_1), ..., \mu(x_n)\}),\]
\[(SFnS2) \quad (\forall x \in G, (\mu(x)) \leq \mu(x)).\]

Example 3.1. Let \((\mathbb{Z}_4, f)\) be a 4-ary subgroup derived from additive group \(\mathbb{Z}_4\). Define a fuzzy subset \(\mu\) in \(\mathbb{Z}_4\) as follows:
\[\mu(x) = \begin{cases} 0.1 & \text{if } x = 0, \\ 0.7 & \text{if } x = 1, 2, 3. \end{cases}\]
and let \(S : \prod_{i=1}^{4} [0, 1] \rightarrow [0, 1]\) be a function defined by as follows:
\[S(x^4_1) = \min \{x_1 + x_2 + x_3 + x_4, 1\}\]
for all \(x_i \in [0, 1]\) and a function \(f\) is defined by
\[f(x^4_1) = x_1 + x_2 + x_3 + x_4, \forall x^4_1 \in \mathbb{Z}_4.\]
By routine calculations, we know that \(\mu\) is a s-fuzzy 4-ary subgroup of \((\mathbb{Z}_4, f)\).

Theorem 3.1. If \(\mu_i | i \in I\) is an arbitrary family of s-fuzzy n-ary subgroup of \((G, f)\) then \(\bigcup_{i \in I} \mu_i\) is s-fuzzy n-ary subgroup of \((G, f)\), where \(\bigcup A_i = \bigvee \mu_i\), where \(\bigvee \mu_i(x) = \sup \{\mu_i(x) | x \in G \text{ and } i \in I\}\).

Proof. The proof is trivial.

Theorem 3.2. If \(\mu\) is a fuzzy set in \(G\) is a s-fuzzy n-ary subgroup of \((G, f)\), then so is \(\mu'\), where \(\mu' = 1 - \mu\).

Proof. It is sufficient to show that \(\mu'\) satisfies conditions (SFnS1) and (SFnS2). Let \(x^n_1 \in G\). Then
\[
\mu'(f(x^n_1)) = 1 - \mu(f(x^n_1)) \leq 1 - S\{\mu(x_1), ..., \mu(x_n)\} \leq S\{1 - \mu(x_1), ..., 1 - \mu(x_n)\} = S\{\mu'(x_1), ..., \mu'(x_n)\}.
\]
and
\[
\mu'(x) = 1 - \mu(x) \leq 1 - \mu(x) = \mu'(x).
\]
Hence \(\mu'\) is a s-fuzzy n-ary subgroup of \((G, f)\).

The following Lemma gives the relation between \(T\) and \(S\).

Lemma 3.1. Let \(T\) be a t-norm. Then the t-conorm \(S\) can be defined as
\[S(x^n_1) = 1 - T(1 - x_1, 1 - x_2, ..., 1 - x_n), \forall x^n_1 \in G.\]

Proof. Straightforward.
The following Theorem gives the relation between \( t \)-fuzzy \( n \)-ary subgroup and \( s \)-fuzzy \( n \)-ary subgroup of \( G \).

**Theorem 3.3.** A fuzzy set \( \mu \) of \( G \) is a \( t \)-fuzzy \( n \)-ary subgroup of \((G, f)\) if and only if its complement \( \mu' \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\).

**Proof.** Let \( \mu \) be a \( t \)-fuzzy \( n \)-ary subgroup of \((G, f)\). For all \( x_1^n \in G \), we have

\[
\mu'(f(x_1^n)) = 1 - \mu(f(x_1^n)) \\
\leq 1 - T\{\mu(x_1), \mu(x_2), ..., \mu(x_n)\} \\
= 1 - T\{1 - \mu'(x_1), 1 - \mu'(x_2), ..., 1 - \mu'(x_n)\} \\
= S\{\mu'(x_1), \mu'(x_2), ..., \mu'(x_n)\}.
\]

For all \( x \in G \), we have

\[
\mu'(x) = 1 - \mu(x) \leq 1 - \mu(x) = \mu'(x).
\]

The converse is proved similarly. \( \square \)

**Definition 3.2.** Let \( \mu \) be a fuzzy set in \( G \) and let \( t \in [0, 1] \). Then the set

\[
L(\mu; t) := \{x \in G | \mu(x) \leq t\}
\]

is called anti-level subset \( \mu \) of \( G \).

The following Theorem is a consequence of the Transfer Principle described in [26].

**Theorem 3.4.** A fuzzy set \( \mu \) in \( G \), is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\) if and only if the anti-level subset \( L(\mu; t) \) of \( G \) is an \( n \)-ary subgroup of \((G, f)\) for every \( t \in [0, 1] \), which is called \( s \)-level \( n \)-ary subgroup of \((G, f)\).

**Proof.** Let \( \mu \) be a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\). If \( x_1^n \in G \) and \( t \in [0, 1] \), then \( \mu(x_i) \leq t \) for all \( i = 1, 2, ..., n \). Thus

\[
\mu(f(x_1^n)) \leq S\{\mu(x_1), ..., \mu(x_n)\} \leq t,
\]

which implies \( f(x_1^n) \in L(\mu; t) \). Moreover, for some \( x \in L(\mu; t) \), we have

\[
\mu'(x) \leq \mu(x) \leq t,
\]

which implies \( \exists L(\mu; t) \). Thus \( L(\mu; t) \) is an \( n \)-ary subgroup of \((G, f)\).

Conversely, assume that \( L(\mu; t) \) is an \( n \)-ary subgroup of \((G, f)\). Let us define

\[
t_0 = S\{\mu(x_1), ..., \mu(x_n)\},
\]

for some \( x_1^n \in G \). Then obviously \( x_1^n \in L(\mu; t_0) \), consequently \( f(x_1^n) \in L(\mu; t_0) \). Thus

\[
\mu(f(x_1^n)) \leq \mu'(x_1^n) = S\{\mu(x_1), ..., \mu(x_n)\}.
\]

Now, let \( x \in L(\mu; t) \). Then \( \mu(x) = t_0 \leq t \). Thus \( x \in L(\mu; t_0) \). Since by the assumption, \( \exists L(\mu; t_0) \). Whence \( \mu(x) \leq t_0 = \mu(x) \). This complete the proof. \( \square \)

Using the above theorem, we can prove the following characterization of \( s \)-fuzzy \( n \)-ary subgroups.
Theorem 3.5. A fuzzy set \( \mu \) in \( G \), is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G, f) \) if and only if the anti-level subset \( L(\mu; t) \) of \( G \) is an \( n \)-ary subgroup of \( (G, f) \) for all \( i = 1, 2, ..., n \) and all \( x_i^t \in G \), \( \mu \) satisfies the following conditions:

(i) \( \mu(f(x_i^t)) \leq S(\mu(x_1), ..., \mu(x_n)) \),

(ii) \( \mu(x_i) \leq S(\mu(x_1), ..., \mu(x_{i-1}), \mu(f(x_i^t)), \mu(x_{i+1}), ..., \mu(x_n)) \).

Proof. Assume that \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G, f) \). Similarly as in the proof of Theorem 3.4, we can prove the non-empty level subset \( L(\mu; t) \) under the operation \( f \), that is \( x_i^t \in L(\mu; t) \) implies \( f(x_i^t) \in L(\mu; t) \).

Now let \( x_0, x_{i-1}^t, x_{i+1}^t, \) where \( x_0 = f(x_{i-1}^t), z, x_{i+1}^t \) for some \( i = 1, 2, ..., n \) and \( z \in G \) which implies \( x_0 \in L(\mu; t) \). Then, according to (ii), we have \( \mu(z) \leq t \). So, the the equation (1) has a solution \( z \in \mu(t) \). This mean that level subset \( L(\mu; t) \) is an \( n \)-ary subgroups.

Conversely, assume that level subset \( L(\mu; t) \) is an \( n \)-ary subgroups of \( (G, f) \). Then it is easy to prove the condition (i).

For \( x_i^t \in G \), we define

\[
t_0 = S(\mu(x_1), ..., \mu(x_{i-1}), \mu(f(x_i^t)), \mu(x_{i+1}), ..., \mu(x_n)).
\]

Then \( x_i^t, x_{i+1}^t, f(x_i^t) \in L(\mu, t_0) \). Whence, according to the definition of \( n \)-ary group, we conclude \( x_i \in L(\mu, t_0) \). Thus \( \mu(x_i) \leq t_0 \). This proves the conditions (ii).

Definition 3.3. Let \( (G, f) \) and \( (G', f) \) be an \( n \)-ary groups. A mapping \( g : G \rightarrow G' \) is called an \( n \)-ary homomorphism if \( g(f(x_i^t)) = f(g^n(x_i^t)) \), where \( g^n(x_i^t) = (g(x_1), ..., g(x_n)) \) for all \( x_i^t \in G \).

For any fuzzy set \( \mu \) in \( G' \), we define the preimage of \( \mu \) under \( g \), denoted by \( g^{-1}(\mu) \), is a fuzzy set in \( G \) defined by

\[
g^{-1}(\mu) = \mu_g^{-1}(x) = \mu(g(x)), \forall x \in G.
\]

For any fuzzy set \( \mu \) in \( G \), we define the image of \( \mu \) under \( g \), denoted by \( g(\mu) \), is a fuzzy set in \( G' \) defined by

\[
g(\mu)(y) = \begin{cases} \inf_{x \in g^{-1}(y)} \mu(x), & \text{if } g^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}
\]

for all \( x \in G \) and \( y \in G' \).

Theorem 3.6. Let \( g \) be a \( n \)-ary homomorphism mapping from \( G \) into \( G' \) with \( g(x) = g(x) \) for all \( x \in G \) and \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G', f) \). Then \( g^{-1}(\mu) \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G, f) \).

Proof. Let \( x_i^t \in G \), we have

\[
\mu_{g^{-1}}(f(x_i^t)) = \mu(g(f(x_i^t))) = \mu(f(g^n(x_i^t))) \\
\leq S(\mu(g(x_1), ..., \mu(g(x_n)))) = S(\mu_{g^{-1}}(x_1), ..., \mu_{g^{-1}}(x_n)).
\]
and
\[ \mu_{g^{-1}}(\mathfrak{x}) = \mu(g(\mathfrak{x})) \leq \mu(g(x)) = \mu_{g^{-1}(\mu)}(x). \]

This completes the proof. \( \square \)

If we strengthen the condition of \( g \), then we can construct the converse of Theorem 3.6 as follows.

**Theorem 3.7.** Let \( g \) be a \( n \)-ary homomorphism from \( G \) into \( G' \) and \( g^{-1}(\mu) \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G, f) \). Then \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G', f) \).

**Proof.** For any \( x_1 \in G' \), there exists \( a_1 \in G \) such that \( g(a_1) = x_1 \) and for any \( f(x^n_i) \in (G', f) \), there exists \( f(a^n_i) \in (G, f) \) such that \( g(f(a^n_i)) = f(x^n_i) \). Then
\[
\mu(f(x^n_i)) = \mu(g(f(a^n_i))) = \mu_{g^{-1}}(f(a^n_i)) \leq S\{\mu_{g^{-1}}(a_1), \mu_{g^{-1}}(a_2), \ldots, \mu_{g^{-1}}(a_n)\} = S\{\mu(a_1), \ldots, \mu(a_n)\} = S\{\mu(x_1), \ldots, \mu(x_n)\}.
\]

For any \( \mathfrak{x} \in G' \), there exists \( \mathfrak{a} \in G \) such that \( g(\mathfrak{a}) = \mathfrak{x} \), we have
\[ \mu(\mathfrak{x}) = \mu(g(\mathfrak{a})) = \mu_{g^{-1}}(\mathfrak{a}) \leq \mu_{g^{-1}}(a) = \mu(a) = \mu(x) \]

This completes the proof. \( \square \)

**Theorem 3.8.** Let \( g: G \longrightarrow G' \) be an onto mapping. If \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G, f) \), then \( g(\mu) \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G', f) \).

**Proof.** Let \( g \) be a mapping from \( G \) onto \( G' \) and let \( x^n_i \in G, y^n_i \in G' \). Noticing that
\[
\{x_i(i = 1, 2, \ldots, n) | x_i \in g^{-1}(f(y^n_i)) \} \subseteq \{f(x^n_i) \in G | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), \ldots, x_n \in g^{-1}(y_n)\},
\]
we have
\[
g(\mu)(f(y^n_i)) = \inf \{\mu(x^n_i) | x_i \in g^{-1}(f(y^n_i))\} \leq \inf \{\max\{\mu(x_1), \mu(x_2), \ldots, \mu(x_n)\} | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), \ldots, x_n \in g^{-1}(y_n)\}\]
\[
\leq \max\{\inf\{\mu(x_1) | x_1 \in g^{-1}(y_1)\}, \inf\{\mu(x_2) | x_2 \in g^{-1}(y_2)\}, \ldots, \inf\{\mu(x_n) | x_1 \in g^{-1}(y_n)\}\} \leq S\{g(\mu)(y_1), g(\mu)(y_2), \ldots, g(\mu)(y_n)\}.
\]

and
\[ g(\mu)(\mathfrak{x}) = \inf \{\mu(\mathfrak{a}) | \mathfrak{a} \in g^{-1}(f(\mathfrak{y}))\} \leq \inf \{\mu(x) | x \in g^{-1}(f(y))\} = g(\mu)(x). \]

This completes the proof. \( \square \)
Corollary 3.1. A fuzzy subset \( \mu \) defined on group \((G,.)\) is a \( s \)-fuzzy subgroup if and only if:

1. \( \mu(xy) \leq S\{\mu(x), \mu(y)\} \),
2. \( \mu(x) \leq S\{\mu(y), \mu(xy)\} \),
3. \( \mu(y) \leq S\{\mu(x), \mu(xy)\} \).

These hold for all \( x, y \in G \).

Theorem 3.9. Let \( \mu \) be a \( s \)-fuzzy subgroup of \((G,.)\). If there exists an element \( a \in G \) such that \( \mu(a) \leq \mu(x) \) for every \( x \in G \), then \( \mu \) is a \( s \)-fuzzy subgroup of a group \( ret_a(G, f) \).

Proof. For all \( x, y, a \in G \), let if possible \( \mu \) is not a \( s \)-fuzzy subgroup of a group \( ret_a(G, f) \). Then we have \( \mu(x \circ y) > S\{\mu(x), \mu(y)\} \). That is

\[
S\{\mu(x), \mu(x)\} < \mu(x \circ x) = \mu(f(x, (n-2) a, x)) \leq S\{\mu(x), \mu(x)\} \leq S\{\mu(x), \mu(a)\}.
\]

This holds only if \( \mu(a) > \mu(x) \), which is contradiction to our assumption \( \mu(a) \leq \mu(x) \).

Also, we have \( \mu \) is a \( s \)-fuzzy subgroup of \((G,.)\). Thus \( \mu(x^{-1}) \leq \mu(x) \) is obvious for all \( x \in G \).

which complete the proof. \( \square \)

In Theorem 3.9, the assumption that \( \mu(a) \leq \mu(x) \) cannot be omitted.

Example 3.2. Let \( (\mathbb{Z}_4, f) \) be a 4-ary group from Example 3.1.

Define a fuzzy set \( \mu \) as follows:

\[
\mu(x) = \begin{cases} 
0.4, & \text{if } x = 0, \\
1, & \text{if } x = 1, 2, 3.
\end{cases}
\]

Clearly, \( \mu \) is a \( s \)-fuzzy 4-ary subgroup of \((\mathbb{Z}_4, f)\). For \( ret_1(\mathbb{Z}_4, f) \), define

\[
S(x, y) = \begin{cases} 
\max(x, y), & \text{if } x = y, \\
\min(x + y, 1), & \text{if } x \neq y.
\end{cases}
\]

we have

\[
\mu(0 \circ 0) = \mu(f(0, 1, 1, 0)) = \mu(2) = 1 \geq 0.4 = \mu(0) = S\{\mu(0), \mu(0)\}.
\]

Hence the assumptions \( \mu(a) \leq \mu(x) \) cannot be omitted.

Theorem 3.10. Let \( (G, f) \) be an \( n \)-ary group. If \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of a group \( ret_a(G, f) \) and \( \mu(a) \leq \mu(x) \) for all \( a, x \in G \), then \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\).
Proof. According to Theorem 2.1, any \( n \)-ary group can be represented of the form (2), where \((G, \circ) = ret_a(G, f), \varphi(x) = f(\overline{a}, x, (a^{n-2}) \) and \( b = f(\overline{a}, \ldots, \overline{a}) \). Then we have
\[
\mu(\varphi(x)) = \mu(f(\overline{a}, x, (a^{n-2})) \leq S\{\mu(\overline{a}), \mu(x), \mu(a)\} \leq \mu(x).
\]
\[
\mu(\varphi^2(x)) = \mu(f(\overline{a}, \varphi(x), (a^{n-2})) \leq S\{\mu(\overline{a}), \mu(\varphi(x)), \mu(a)\} \leq S\{\mu(\overline{a}), \mu(x), \mu(a)\} \leq \mu(x).
\]
Consequently, \( \mu(\varphi^k(x)) \leq \mu(x) \). for all \( x \in G \) and \( k \in \mathbb{N} \).

Similarly, for all \( x \in G \) we have
\[
\mu(b) = \mu(f(\overline{a}, \ldots, \overline{a})) \leq \mu(\overline{a}) \leq \mu(x).
\]

Thus
\[
\mu(f(x^n)) = \mu(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \ldots \circ \varphi^{n-2}(x_n) \circ b) \leq S\{\mu(x_1), \mu(\varphi(x_2)), \mu(\varphi^2(x_3)), \ldots, \mu(\varphi^{n-2}(x_n)), \mu(b)\} \leq S\{\mu(x_1), \mu(x_2), \mu(x_3), \ldots, \mu(x_n), \mu(b)\} \leq S\{\mu(x_1), \mu(x_2), \mu(x_3), \ldots, \mu(x_n)\}.
\]

From (4) and (7) of [3], we have
\[
\Sigma = (\mu(\overline{a} \circ \varphi(x) \circ \varphi^2(x) \circ \ldots \circ \varphi^{n-2}(x) \circ b)^{-1}
\]

Thus
\[
\mu(\Sigma) = \mu \left( (\overline{a} \circ \varphi(x) \circ \varphi^2(x) \circ \ldots \circ \varphi^{n-2}(x) \circ b)^{-1} \right) \leq S\{\mu(\overline{a}), \mu(\varphi(x)), \mu(\varphi^2(x)), \ldots, \mu(\varphi^{n-2}(x)), \mu(b)\} \leq S\{\mu(x), \mu(b)\} = \mu(x).
\]

This completes the proof. \( \square \)

Corollary 3.2. If \((G, f)\) is a ternary group, then any \( s \)-fuzzy subgroup of \( ret_a(G, f) \) is a \( s \)-fuzzy ternary subgroup of \((G, f)\).

Proof. Since \( \overline{a} \) is a neutral element of a group \( ret_a(G, f) \) then \( \mu(\overline{a}) \leq \mu(x) \), for all \( x \in G \). Thus \( \mu(\overline{a}) \leq \mu(a) \). But in ternary group \( \overline{a} = a \) for any \( a \in G \), whence \( \mu(a) = \mu(\overline{a}) \leq \mu(\overline{a}) \leq \mu(x) \). So, \( \mu(a) = \mu(\overline{a}) \leq \mu(x) \), for all \( x \in G \). This means that the assumption of Theorem 3.10 is satisfied. Hence \( ret_a(G, f) \) is a \( s \)-fuzzy ternary subgroup of \((G, f)\). This completes the proof. \( \square \)

Example 3.3. Consider the ternary group \((\mathbb{Z}_{12}, f)\), derived from the additive group \( \mathbb{Z}_{12} \). Let \( \mu \) be a \( s \)-fuzzy subgroup of the group of \( ret_1(G, f) \) induced by subgroups \( S_1 = \{11\}, S_2 = \{5, 11\} \) and \( S_3 = \{1, 3, 5, 7, 9, 11\} \). Define a fuzzy set \( \mu \) as follows:

\[
\mu(x) = \begin{cases} 
0.1 & \text{if } x = 11, \\
0.3 & \text{if } x = 5, \\
0.5 & \text{if } x = 1, 3, 7, 9, \\
0.9 & \text{if } x \notin S_3.
\end{cases}
\]
Then
\[
\mu(\overline{5}) = \mu(7) = 0.5 \lesssim 0.3 = \mu(5).
\]

Hence \(\mu\) is not a \(s\)-fuzzy ternary subgroup of \((\mathbb{Z}_{12}, f)\).

**Observations.** From the above Example 3.3 it follows that:

1. There are \(s\)-fuzzy subgroups of \(r\text{et}_a(G, f)\) which are not \(s\)-fuzzy \(n\)-ary subgroups of \((G, f)\).
2. In Theorem 3.10 the assumption \(\mu(a) \leq \mu(x)\) cannot be omitted. In the above example we have \(\mu(1) = 0.5 \lesssim 0.3 = \mu(5)\).
3. The assumption \(\mu(a) \leq \mu(x)\) cannot be replaced by the natural assumption \(\mu(\overline{a}) \leq \mu(x)\). \((\overline{a}\) is the identity of \(r\text{et}_a(G, f))\). In the above example \(\overline{1} = 11\), then \(\mu(11) \leq \mu(x)\) for all \(x \in \mathbb{Z}_{12}\).

**Theorem 3.11.** Let \((G, f)\) be an \(n\)-ary group of \(b\)-derived from the group \((G, o)\). Any fuzzy set \(\mu\) of \((G, o)\) such that \(\mu(b) \leq \mu(x)\) for every \(x \in G\) is a \(s\)-fuzzy \(n\)-ary subgroup of \((G, f)\).

**Proof.** The condition (SFuS1) is obvious. To prove (SFuS2), we have \(n\)-ary group \((G, f)\) \(b\)-derived from the group \((G, o)\), which implies
\[
\overline{x} = (x^{n-2} \circ b)^{-1},
\]
where \(x^{n-2}\) is the power of \(x\) in \((G, o)[4]\).

Thus, for all \(x \in G\)
\[
\mu(\overline{x}) = \mu((x^{n-2} \circ b)^{-1}) \leq \mu(x^{n-2} \circ b)^{-1} \leq S(\mu(x^{n-2}), \mu(b)) = \mu(x).
\]

This complete the proof.

**Corollary 3.3.** Any \(s\)-fuzzy subgroup of a group \((G, o)\) is a \(s\)-fuzzy \(n\)-ary subgroup of an \(n\)-ary group \((G, f)\) derived from \((G, o)\).

**Proof.** If \(n\)-ary group \((G, f)\) is derived from the group \((G, o)\) then \(b = e\). Thus \(\mu(e) \leq \mu(x)\) for all \(x \in G\).

4. **S-product of \(s\)-fuzzy \(n\)-ary relations**

**Definition 4.1.** A fuzzy \(n\)-ary relation on any set \(G\) is a fuzzy set
\[
\mu : G^n = G \times G \times \ldots \times G \text{ (n times)} \to [0, 1].
\]

**Definition 4.2.** Let \(\mu\) be fuzzy \(n\)-ary relation on any set \(G\) and \(\nu\) is a fuzzy set on \(G\). Then \(\mu\) is called \(s\)-fuzzy \(n\)-ary relation on \(\nu\) if
\[
\mu(x^n) \subseteq S(\nu(x_1), \nu(x_2), \ldots, \nu(x_n)),
\]
for all \(x^n \in G\).

**Definition 4.3.** Let \(\mu^n = \mu_1, \mu_2, \ldots, \mu_n\) be a fuzzy sets in \(G\). Then direct \(S\)-product of \(\mu^n\) is defined by
\[
(\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x^n_G) = S(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)), \forall x^n_G \in G.
\]
Lemma 4.1. Let $S$ be a function induced by $t$-conorm and let $\mu^1_0$ be a fuzzy sets in $G$. Then
(i) $\mu_1 \times S \mu_2 \times S \ldots \times S \mu_n$ is an $s$-fuzzy $n$-ary relation on $G$,
(ii) $L(\mu_1 \times \mu_2 \times \ldots \times \mu_n; t) = L(\mu_1; t) \times L(\mu_2; t) \times \ldots \times L(\mu_n; t), \forall t \in [0, 1].$

Proof. The proof is obvious. \hfill \Box

Definition 4.4. Let $S$ be a function induced by $t$-conorm. If $\nu$ is a fuzzy set in $G$, the strongest $s$-fuzzy $n$-ary relation on $G$ that is a $s$-fuzzy $n$-ary relation on $\nu$ is $\mu_\nu$, given by
$$\mu_\nu(x^n_\nu) = S(\nu(x_1), \nu(x_2), \ldots, \nu(x_n)), \forall x^n_\nu \in G.$$

Lemma 4.2. For a given fuzzy set $\nu$ in $G$, let $\mu$ be the strongest $s$-fuzzy $n$-ary relation of $G$. Then for $t \in [0, 1]$, $L(\mu_\nu; t) = L(\nu; t) \times L(\nu; t) \times \ldots \times L(\nu; t).$

Proof. The proof is obvious. \hfill \Box

Proposition 4.1. Let $S$ be a function induced by $t$-conorm and let $\mu_1, \mu_2, \ldots, \mu_n$ be s-fuzzy $n$-ary subgroup of $(G, f)$. Then, $\mu_1 \times \mu_2 \times \ldots \times \mu_n$ is a s-fuzzy $n$-ary subgroup of $(G^n, f)$.

Proof. For $x^n_1 \in G$ and $f(x^n_1) = (f_1(x^n_1), \ldots, f_n(x^n_1)) \in (G^n, f)$, we have
$$(\mu_1 \times \mu_2 \times \ldots \times \mu_n)(f(x^n_1))$$
$$= (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(f_1(x^n_1), \ldots, f_n(x^n_1))$$
$$= S\{\mu_1(f(x^n_1)), \mu_2(f(x^n_1)), \ldots, \mu_n(f(x^n_1))\}$$
$$\leq S\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)\}.$$}

and for all $x = x^n_1, \overline{x} = \overline{x^n_1} \in G^n$, we have
$$(\mu_1 \times \mu_2 \times \ldots \times \mu_n)(\overline{x}) = (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(\overline{x}_1, \ldots, \overline{x}_n)$$
$$= S\{\mu_1(\overline{x}_1), \ldots, \mu_n(\overline{x}_n)\}$$
$$\leq S\{\mu_1(x_1), \ldots, \mu_n(x_n)\}$$
$$= (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x^n_1)$$
$$= (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(x).$$

This completes the proof. \hfill \Box

The following Corollary is the immediate consequence of Proposition 4.1.

Corollary 4.1. Let $S$ be a function induced by $t$-conorm and let $\prod_{i=1}^n (G_i, f)$ be the finite collection of $n$-ary subgroups and $G = \prod_{i=1}^n G_i$ the $S$-product of $G_i$. Let

[Rest of the text continues here]
\[ \mu_i \text{ be a } s\text{-fuzzy } n\text{-ary subgroup of } (G_i, f), \text{ where } 1 \leq i \leq n. \text{ Then } \mu = \prod_{i=1}^{n} \mu_i \text{ defined by} \]

\[ \mu(x^n) = \prod_{i=1}^{n} \mu_i(x^n) = S(\mu(1), \mu(2), ..., \mu(n)). \]

Then \( \mu \) is an \( s\)-fuzzy \( n\)-ary subgroup of \( (G, f) \). \[ \square \]

**Definition 4.5.** Let \( \mu^n \) be fuzzy sets in \( G \). Then, the \( S\)-product of \( \mu^n \), written as \([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_S\), is defined by:

\[ [\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_S(x) = S(\mu_1(x), \mu_2(x), ..., \mu_n(x)) \forall x \in G. \]

**Theorem 4.1.** Let \( \mu^n \) be \( s\)-fuzzy \( n\)-ary subgroup of \((G, f)\). If \( S^* \) is a function induced by \( t\)-conorm dominates \( S \), that is,

\[ S^*(S(x^n), S(y^n), ..., S(z^n)) \leq S(S^*(x_1, y_1, ..., z_1), ..., S^*(x_n, y_n, ..., z_n)) \]

for all \( x^n, y^n, ..., z^n \in [0, 1] \). Then \( S^*\)-product of \( \mu^n \), \([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_{S^*} \), is an \( s\)-fuzzy \( n\)-ary subgroup of \((G, f)\).

**Proof.** Let \( x^n \in G \), we have

\[ [\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_{S^*}(f(x^n)) = S^*(S(\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n))) \]

\[ \leq S^*(S(\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n))) \]

and for all \( x \in G \), we have

\[ [\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_{S^*}(\overline{x}) = S^*(\mu_1(\overline{x}), \mu_2(\overline{x}), ..., \mu_n(\overline{x})) \]

Hence, \([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_{S^*}\) is an \( s\)-fuzzy \( n\)-ary subgroup of \((G, f)\). This completes the proof. \[ \square \]

Let \((G, f)\) and \((G', f)\) be an \( n\)-ary groups. A mapping \( g : G \rightarrow G' \) is an onto homomorphism. Let \( S \) and \( S^* \) be functions induced by \( t\)-conorm such that \( S^* \) dominates \( S \). If \( \mu^n \) are \( s\)-fuzzy \( n\)-ary subgroup of \((G, f)\), then the \( S^*\)-product of \([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_S\), is an \( s\)-fuzzy \( n\)-ary subgroup. Since every onto homomorphic inverse image of an \( s\)-fuzzy \( n\)-ary subgroup, the inverse images \( g^{-1}(\mu_1), g^{-1}(\mu_2), ..., g^{-1}(\mu_n) \) and \( g^{-1}([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_S) \) are \( s\)-fuzzy \( n\)-ary subgroup \((G, f)\).

The following theorem provides the relation between \( g^{-1}([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_S) \) and \( S^*\)-product \(([g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \ldots \cdot g^{-1}(\mu_n)]_{S^*}) \) of \( g^{-1}(\mu_1), g^{-1}(\mu_2) \) and \( g^{-1}(\mu_n) \).
Theorem 4.2. Let $(G, f)$ and $(G, f')$ be an $n$-ary groups. A mapping $g: G \rightarrow G'$ is an onto $n$-ary homomorphism. Let $S^*$ be a function induced by $t$-conorm such that $S^*$ dominates $S$. Let $\mu^*_i$ be $s$-fuzzy $n$-ary subgroups of $(G, f)$. If $[\mu_1, \mu_2, \ldots, \mu_n]_{S^*}$ and is the $S^*$-product of $\mu^*_i$, and $([g^{-1}(\mu_1), g^{-1}(\mu_2), \ldots, g^{-1}(\mu_n)]_{S^*})$ is the $S^*$-product of $g^{-1}(\mu_1), g^{-1}(\mu_2), \ldots, g^{-1}(\mu_n)$ then

$$g^{-1}(\mu_1, \mu_2, \ldots, \mu_n)_{S^*} = [g^{-1}(\mu_1), g^{-1}(\mu_2), \ldots, g^{-1}(\mu_n)]_{S^*}.$$ 

Proof. Let $x \in G$, we have

$$g^{-1}(\mu_1, \mu_2, \ldots, \mu_n)_{S^*}(x) = ([\mu_1, \mu_2, \ldots, \mu_n]_{S^*}(g(x)) = S^*(\mu_1(g(x)), \mu_2(g(x)), \ldots, \mu_n(g(x)))
= S^*(g^{-1}(\mu_1)(x), g^{-1}(\mu_2)(x), \ldots, g^{-1}(\mu_n)(x))
= [g^{-1}(\mu_1), g^{-1}(\mu_2), \ldots, g^{-1}(\mu_n)]_{S^*}.$$ 

This completes the proof. 

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References


