A STUDY OF \( n \)-ARY SUBGROUPS
WITH RESPECT TO \( t \)-CONORM

D.R. Prince Williams

Abstract. In this paper, we introduce a notion of fuzzy \( n \)-ary subgroups with respect to \( t \)-conorm (\( s \)-fuzzy \( n \)-ary subgroups) in an \( n \)-ary groups \((G, f)\) and have studied their related properties. The main contribution of this paper are studying the properties of \( s \)-fuzzy \( n \)-ary subgroups over \( s \)-level \( n \)-ary subgroup of \((G, f)\), \( n \)-ary homomorphism and \( ret_a(G, f)\). Moreover some results of the \( S \)-product of \( s \)-fuzzy \( n \)-ary relations in an \( n \)-ary groups \((G, f)\) are also obtained.

1. Introduction

The theory of fuzzy set was first developed by Zadeh [29] and has been applied to many branches in mathematics. Later fuzzification of the “group” concept into “fuzzy subgroup” was made by Rosenfeld [28]. This work was the first fuzzification of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. The study of \( n \)-ary systems was initiated by Kasner [26] in 1904, but the important study on \( n \)-ary groups was done by Dörnte [3]. The theory of \( n \)-ary systems have many applications. For example, in the theory of automata [23], \( n \)-ary semigroup and \( n \)-ary groups are used. The \( n \)-ary groupoids are applied in the theory of quantum groups [27]. Also the ternary structures in physics are described by Kerner in [25]. The \( n \)-ary system dealt in detail [4-9,11,12,14-22]. The first fuzzification of \( n \)-ary system was introduced by Dudek [10]. Moreover Davvaz et. al [2] have studied fuzzy \( n \)-ary groups as a generalization of Rosenfeld’s fuzzy groups and have investigated their related properties. The notion of intuitionistic fuzzy sets, as a generalization of the notion of fuzzy set. Dudek [13] has introduced intuitionistic fuzzy sets idea’s in \( n \)-ary systems and has discussed in detail. Triangular norm (\( t \)-norm) and triangular conorm (\( t \)-conorm) are the most general

2010 Mathematics Subject Classification. 04A72,08A72, 03E72,20N25.

Key words and phrases. fuzzy subgroup, anti-level set, \( s \)-fuzzy \( n \)-ary subgroup, \( s \)-fuzzy \( n \)-ary relation.
families of binary operations that satisfy the requirement of the conjunction and
disjunction operators, respectively. Thus, the \( t \)-norm generalizes the conjunctive
(AND) operator and the \( t \)-conorm generalizes the disjunctive (OR) operator. In
application, \( t \)-norm \( T \) and \( t \)-conorm \( S \) are two functions that map the unit square
into the unit interval. To study more about \( t \)-conorm see \cite{24}. In this paper, we
introduce the notion of fuzzy \( n \)-ary subgroups with respect to \( t \)-conorm (\( s \)-fuzzy \( n \)-
ary subgroup) in \( n \)-ary group \( (G, f) \) and have investigated their related properties.

2. Preliminaries

A non-empty set \( G \) together with one \( n \)-ary operation \( f : G^n \rightarrow G \), where
\( n \geq 2 \), is called an \( n \)-ary groupoid and is denoted by \( (G, f) \). According to the
general convention used in the theory of \( n \)-ary groupoids the sequence of elements
\( x_i, x_{i+1}, \ldots, x_j \) is denoted by \( x_i^j \). In the case \( j < i \), it denoted the empty symbol.
If \( x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x \), then instead of \( x_i^{i+t} \) and we write \( (x_i^t) \).
In this convention
\[
f(x_1, \ldots, x_n) = f(x_1^n)
\]

and
\[
f(x_1, \ldots, x_i, \ldots, x_{i+t+1}, \ldots, x_n) = f(x_1, (x_i^t), x_{i+t+1}^{n+1}).
\]

An \( n \)-ary groupoid \( (G, f) \) is called an \((i,j)\)-associative if
\[
f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})
\]
hold for all \( x_1, \ldots, x_{2n-1} \in G \). If this identity holds for all \( 1 \leq i \leq j \leq n \),
then we say that the operation \( f \) is associative and \( (G, f) \) is called an \( n \)-ary semigroup.
It is clear that an \( n \)-ary groupoid is associative if and only if it is \((1,1)\)-associative
for all \( j = 2, \ldots, n \). In the binary case (i.e. \( n=2 \)) it is usual semigroup. If for all
\( x_0, x_1, \ldots, x_n \in G \) and fixed \( i \in \{1, \ldots, n\} \) there exists an element \( z \in G \) such that
\[
f(x_1^{i-1}, z, x_i^n) = x_0 \tag{1}
\]
then we say that this equation is \( i \)-solvable or solvable at the place \( i \). If the solution
is unique, then we say that (1) is uniquely \( i \)-solvable. An \( n \)-ary groupoid \( (G, f) \)
uniquely solvable for all \( i = 1, \ldots, n \) is called an \( n \)-ary quasigroup . An associative
\( n \)-ary quasigroup is called an \( n \)-ary group .

Fixing an \( n \)-ary operation \( f \), where \( n \geq 3 \), the elements \( a_{0}^{n-2} \) we obtain the new
binary operation \( x \circ y = f(x, a_{0}^{n-2}, y) \). If \( (G, f) \) is an \( n \)-ary group then \( (G, \circ) \) is
a group. Choosing different elements \( a_{0}^{n-2} \) we obtain different groups. All these
groups are isomorphic\cite{8}. So, we can consider only group of the form
\[
ret_a(G, f) = (G, \circ), \text{ where } x \circ y = f(x, a^{(n-2)}, y).
\]
In this group \( e = \text{, } x^{-1} = f(y, a^{(n-3)}, x, y) \).

In the theory of \( n \)-ary groups, the following Theorem plays an important role.
A STUDY OF $n$-ARY SUBGROUPS WITH RESPECT TO $t$-CONORM

Theorem 2.1. For any $n$-ary group $(G, f)$ there exist a group $(G, \circ)$, its automorphism $\varphi$ and an element $b \in G$ such that
\[
f(x^n_1) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \ldots \circ \varphi^{n-1}(x_n) \circ b
\]
holds for all $x^n_1 \in G$.

In what follows, $G$ is a non-empty set and $(G, f)$ is an $n$-ary group unless otherwise specified.

Definition 2.1. By a $t$-norm, a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions is meant:
\begin{enumerate}
  \item[(T1)] $T(x, 1) = x$;
  \item[(T2)] $T(x, y) \leq T(x, z)$ if $y \leq z$;
  \item[(T3)] $T(x, y) = T(y, x)$;
  \item[(T4)] $T(x, T(y, z)) = T(T(x, y), z)$;
\end{enumerate}
for all $x, y, z \in [0, 1]$.

Definition 2.2. By a $t$-conorm, a function $S : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions is meant:
\begin{enumerate}
  \item[(S1)] $S(x, 0) = x$;
  \item[(S2)] $S(x, y) \leq S(x, z)$ if $y \leq z$;
  \item[(S3)] $S(x, y) = S(y, x)$;
  \item[(S4)] $S(x, S(y, z)) = S(S(x, y), z)$;
\end{enumerate}
for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition (S1), we obtain the concept of $t$-norm $T$.

Definition 2.3. Given a $t$-norm $T$ and a $t$-conorm $S$, $T$ and $S$ are dual (with respect to the negation $\neg$) if and only if $(T(x, y))' = S(x', y')$.

Now we generalize the domain of $S$ to $\prod_{i=1}^n [0, 1]$ as follows:

Definition 2.4. The function $S_n : \prod_{i=1}^n [0, 1] \to [0, 1]$ is defined by:
\[
S_n(\alpha^n_1) = S_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n))
\]
for all $1 \leq i \leq n, n \geq 2$.

For a $t$-conorm $S$ on $\prod_{i=1}^n [0, 1]$, it is denoted by
\[
\Delta_S = \{ \alpha \in [0, 1] \mid S(\alpha, \ldots, \alpha) = \alpha \}.
\]
It is clear that every $t$-conorm has the following property:
\[
S(\alpha^n_1) \geq \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}
\]
for all $\alpha^n_1 \in [0, 1]$. 
3. s-fuzzy n-ary subgroups

**Definition 3.1.** A fuzzy set \( \mu \) in \( G \) is called a s-fuzzy n-ary subgroup of \((G, f)\) if the following axioms holds:

\[
\begin{align*}
(SF_1) & \quad (\forall x^n \in G), (\mu(f(x^n)) \leq S\{\mu(x_1), \ldots, \mu(x_n)\}), \\
(SF_{2}) & \quad (\forall x \in G), (\mu(\overline{x}) \leq \mu(x)).
\end{align*}
\]

**Example 3.1.** Let \((\mathbb{Z}_4, f)\) be a 4-ary subgroup derived from additive group \( \mathbb{Z}_4 \). Define a fuzzy subset \( \mu \) in \( \mathbb{Z}_4 \) as follows:

\[
\mu(x) = \begin{cases} 
0.1 & \text{if } x = 0, \\
0.7 & \text{if } x = 1, 2, 3.
\end{cases}
\]

and let \( S : \prod_{i=1}^{4} [0, 1] \rightarrow [0, 1] \) be a function defined by as follows:

\[
S(x^n_1) = \min\{x_1 + x_2 + x_3 + x_4, 1\}
\]

for all \( x^n_1 \in [0, 1] \) and a function \( f \) is defined by

\[
f(x^n_1) = x_1 + 4x_2 + 4x_3 + 4x_4, \forall x^n_1 \in \mathbb{Z}_4.
\]

By routine calculations, we know that \( \mu \) is a s-fuzzy 4-ary subgroup of \((\mathbb{Z}_4, f)\).

**Theorem 3.1.** If \( \{\mu_i\}_{i \in I} \) is an arbitrary family of s-fuzzy n-ary subgroup of \((G, f)\) then \( \bigcup_{i \in I} \mu_i \) is s-fuzzy n-ary subgroup of \((G, f)\), where \( \bigcup A_i = \bigvee \mu_i \), where \( \bigvee \mu_i(x) = \text{sup}\{\mu_i(x) | x \in G \text{ and } i \in I\} \).

**Proof.** The proof is trivial. \( \square \)

**Theorem 3.2.** If \( \mu \) is a fuzzy set in \( G \) is a s-fuzzy n-ary subgroup of \((G, f)\), then so is \( \mu' \), where \( \mu' = 1 - \mu \).

**Proof.** It is sufficient to show that \( \mu' \) satisfies conditions (SF1) and (SF2).

Let \( x^n \in G \). Then

\[
\mu'(f(x^n)) = 1 - \mu(f(x^n)) 
\leq 1 - S\{\mu(x_1), \ldots, \mu(x_n)\} 
\leq S\{1 - \mu(x_1), \ldots, 1 - \mu(x_n)\} 
= S\{\mu'(x_1), \ldots, \mu'(x_n)\}.
\]

and

\[
\mu'(\overline{x}) = 1 - \mu(\overline{x}) \leq 1 - \mu(x) = \mu'(x).
\]

Hence \( \mu' \) is a s-fuzzy n-ary subgroup of \((G, f)\). \( \square \)

The following Lemma gives the relation between \( T \) and \( S \).

**Lemma 3.1.** Let \( T \) be a t-norm. Then the t-conorm \( S \) can be defined as

\[
S(x^n_1) = 1 - T(1 - x_1, 1 - x_2, \ldots, 1 - x_n), \forall x^n_1 \in G.
\]

**Proof.** Straightforward. \( \square \)
The following Theorem gives the relation between \( t \)-fuzzy \( n \)-ary subgroup and \( s \)-fuzzy \( n \)-ary subgroup of \( G \).

**Theorem 3.3.** A fuzzy set \( \mu \) of \( G \) is a \( t \)-fuzzy \( n \)-ary subgroup of \((G, f)\) if and only if its complement \( \mu' \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\).

**Proof.** Let \( \mu \) be a \( t \)-fuzzy \( n \)-ary subgroup of \((G, f)\). For all \( x_i^n \in G \), we have
\[
\mu'(f(x_i^n)) = 1 - \mu(f(x_i^n)) \\
\leq 1 - T\{\mu(x_1), \mu(x_2), ..., \mu(x_n)\} \\
= 1 - T\{1 - \mu'(x_1), 1 - \mu'(x_2), ..., 1 - \mu'(x_n)\} \\
= S\{\mu'(x_1), \mu'(x_2), ..., \mu'(x_n)\}.
\]
For all \( x \in G \), we have
\[
\mu'(\pi) = 1 - \mu(\pi) \leq 1 - \mu(x) = \mu'(x).
\]
The converse is proved similarly. \( \square \)

**Definition 3.2.** Let \( \mu \) be a fuzzy set in \( G \) and let \( t \in [0,1] \). Then the set
\[
L(\mu; t) := \{ x \in G | \mu(x) \leq t \}
\]
is called anti-level subset \( \mu \) of \( G \).

The following Theorem is a consequence of the Transfer Principle described in [26].

**Theorem 3.4.** A fuzzy set \( \mu \) in \( G \), is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\) if and only if the anti-level subset \( L(\mu; t) \) of \( G \) is an \( n \)-ary subgroup of \((G, f)\) for every \( t \in [0,1] \), which is called \( s \)-level \( n \)-ary subgroup of \((G, f)\).

**Proof.** Let \( \mu \) be a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\). If \( x_i^n \in G \) and \( t \in [0,1] \), then \( \mu(x_i) \leq t \) for all \( i = 1, 2, ..., n \). Thus
\[
\mu(f(x_i^n)) \leq S\{\mu(x_1), ..., \mu(x_n)\} \leq t,
\]
which implies \( f(x_i^n) \in L(\mu; t) \). Moreover, for some \( x \in L(\mu; t) \), we have
\[
\mu(\pi(x)) \leq \mu(x) \leq t,
\]
which implies \( \pi(x) \in L(\mu; t) \). Thus \( L(\mu; t) \) is an \( n \)-ary subgroup of \((G, f)\).

Conversely, assume that \( L(\mu; t) \) is an \( n \)-ary subgroup of \((G, f)\). Let us define
\[
t_0 = S\{\mu(x_1), ..., \mu(x_n)\}
\]
for some \( x_i^n \in G \). Then obviously \( x_i^n \in L(\mu; t_0) \), consequently \( f(x_i^n) \in L(\mu; t_0) \). Thus
\[
\mu(f(x_i^n)) \leq t_0 = S\{\mu(x_1), ..., \mu(x_n)\}.
\]
Now, let \( x \in L(\mu; t) \). Then \( \mu(x) = t_0 \leq t \). Thus \( x \in L(\mu; t_0) \). Since, by the assumption, \( \pi(x) \in L(\mu; t_0) \). Whence \( \mu(\pi(x)) \leq t_0 = \mu(x) \). This complete the proof. \( \square \)

Using the above theorem, we can prove the following characterization of \( s \)-fuzzy \( n \)-ary subgroups.
Theorem 3.5. A fuzzy set \( \mu \) in \( G \), is a s-fuzzy \( n \)-ary subgroup of \( (G, f) \) if and only if the anti-level subset \( L(\mu; t) \) of \( G \) is an \( n \)-ary subgroup of \( (G, f) \) for all \( i = 1, 2, \ldots, n \) and all \( x_1^n \in G \), \( \mu \) satisfies the following conditions:

(i) \( \mu(f(x_1^n)) \leq S(\mu(x_1), \ldots, \mu(x_n)) \),

(ii) \( \mu(x_i) \leq S(\mu(x_1), \ldots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i+1}), \ldots, \mu(x_n)) \).

Proof. Assume that \( \mu \) is a s-fuzzy \( n \)-ary subgroup of \( (G, f) \). Similarly as in the proof of Theorem 3.4, we can prove the non-empty level subset \( L(\mu; t) \) under the operation \( f \), that is \( x_1^n \in L(\mu; t) \) implies \( f(x_1^n) \in L(\mu; t) \).

Now let \( x_0, x_1^{n-1}, x_{n+1} \), where \( x_0 = f(x_1^{n-1}, z, x_{n+1}) \) for some \( i = 1, 2, \ldots, n \) and \( z \in G \) which implies \( x_0 \in L(\mu; t) \). Then, according to (ii), we have \( \mu(z) \leq t \). So, the theorem (1) has a solution \( z \in \mu(t) \). This means that level subset \( L(\mu; t) \) is an \( n \)-ary subgroup.

Conversely, assume that level subset \( L(\mu; t) \) is an \( n \)-ary subgroups of \( (G, f) \). Then it is easy to prove the condition (i).

For \( x_1^n \in G \), we define

\[
    t_0 = S(\mu(x_1), \ldots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i+1}), \ldots, \mu(x_n)).
\]

Then \( x_1^{n-1}, x_{n+1}, f(x_1^n) \in L(\mu; t_0) \). Whence, according to the definition of \( n \)-ary group, we conclude \( x_i \in L(\mu; t_0) \). Thus \( \mu(x_i) \leq t_0 \). This proves the conditions (ii).

Definition 3.3. Let \((G, f)\) and \((G', f)\) be an \( n \)-ary groups. A mapping \( g : G \to G' \) is called an \( n \)-ary homomorphism if \( g(f(x_1^n)) = f(g^n(x_1^n)) \), where \( g^n(x_1^n) = (g(x_1), \ldots, g(x_n)) \) for all \( x_1^n \in G \).

For any fuzzy set \( \mu \) in \( G' \), we define the preimage of \( \mu \) under \( g \), denoted by \( g^{-1}(\mu) \), is a fuzzy set in \( G \) defined by

\[
    g^{-1}(\mu) = \mu_{g^{-1}}(x) = \mu(g(x)), \forall x \in G.
\]

For any fuzzy set \( \mu \) in \( G \), we define the image of \( \mu \) under \( g \), denoted by \( g(\mu) \), is a fuzzy set in \( G' \) defined by

\[
    g(\mu)(y) = \begin{cases} 
    \inf_{x \in g^{-1}(y)} \mu(x), & \text{if } g^{-1}(y) \neq \emptyset, \\
    0, & \text{otherwise.}
    \end{cases}
\]

for all \( x \in G \) and \( y \in G' \).

Theorem 3.6. Let \( g \) be a \( n \)-ary homomorphism mapping from \( G \) into \( G' \) with \( g(\overline{x}) = g(x) \) for all \( x \in G \) and \( \mu \) is a s-fuzzy \( n \)-ary subgroup of \((G', f)\). Then \( g^{-1}(\mu) \) is a s-fuzzy \( n \)-ary subgroup of \((G, f)\).

Proof. Let \( x_1^n \in G \), we have

\[
    \mu_{g^{-1}}(f(x_1^n)) = \mu(g(f(x_1^n))) = \mu(f(g^n(x_1^n))) \leq S(\mu(g(x_1)), \ldots, \mu(g(x_n))) = S(\mu_{g^{-1}}(x_1), \ldots, \mu_{g^{-1}}(x_n)).
\]
and
\[ \mu_{g^{-1}}(x) = \mu(g(x)) \leq \mu(g(x)) = \mu_{g^{-1}(\mu)}(x). \]

This completes the proof. \( \Box \)

If we strengthen the condition of \( g \), then we can construct the converse of Theorem 3.6 as follows.

**Theorem 3.7.** Let \( g \) be a \( n \)-ary homomorphism from \( G \) into \( G' \) and \( g^{-1}(\mu) \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\). Then \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G', f)\).

**Proof.** For any \( x_1 \in G' \), there exists \( a_1 \in G \) such that \( g(a_1) = x_1 \) and for any \( f(x^n) \in (G', f) \), there exists \( f(a^n) \in (G, f) \) such that \( g(f(a^n)) = f(x^n) \). Then
\[
\mu(f(x^n)) = \mu(g(f(a^n))) = \mu_{g^{-1}(f(a^n))} \\
\leq S\{\mu_{g^{-1}(a_1)}, \mu_{g^{-1}(a_2)}, ..., \mu_{g^{-1}(a_n)}\} \\
= S\{\mu(a_1), ..., \mu(a_n)\} \\
= S\{\mu(x_1), ..., \mu(x_n)\}.
\]

For any \( x \in G' \), there exists \( x \in G \) such that \( g(x) = x \), we have
\[
\mu(x) = \mu(g(x)) = \mu_{g^{-1}(x)}(x) = \mu(x) = \mu(x).
\]

This completes the proof. \( \Box \)

**Theorem 3.8.** Let \( g : G \longrightarrow G' \) be an onto mapping. If \( \mu \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\), then \( g(\mu) \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G', f)\).

**Proof.** Let \( g \) be a mapping from \( G \) onto \( G' \) and let \( x^n \in G, y^n \in G' \). Noticing that
\[
\{x_i(i = 1, 2, ..., n)|x_i \in g^{-1}(f(y^n))\} \subseteq \\
\{f(x^n) \in G|x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n)\}.
\]

we have
\[
g(\mu)(f(y^n)) \\
= \inf\{\mu(x^n)|x_i \in g^{-1}(f(y^n))\} \\
\leq \inf\{\max\{\mu(x_1), \mu(x_2), ..., \mu(x_n)\}|x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n)\} \\
= \max\{\inf\{\mu(x_1)|x_1 \in g^{-1}(y_1)\}, \inf\{\mu(x_2)|x_1 \in g^{-1}(y_2)\}, ..., \inf\{\mu(x_n)|x_1 \in g^{-1}(y_n)\}\} \\
\leq S\{g(\mu)(y_1), g(\mu)(y_2), ..., g(\mu)(y_n)\}.
\]

and
\[
g(\mu)(x) = \inf\{\mu(x)|x \in g^{-1}(f(y))\} \leq \inf\{\mu(x)|x \in g^{-1}(f(y))\} = g(\mu)(x).
\]

This completes the proof. \( \Box \)
Corollary 3.1. A fuzzy subset $\mu$ defined on group $(G, \cdot)$ is a $s$-fuzzy subgroup if and only if
\begin{enumerate}
\item $\mu(xy) \leq S\{\mu(x), \mu(y)\}$,
\item $\mu(x) \leq S\{\mu(y), \mu(xy)\}$,
\item $\mu(y) \leq S\{\mu(x), \mu(xy)\}$.
\end{enumerate}
holds for all $x, y \in G$.

Theorem 3.9. Let $\mu$ be a $s$-fuzzy subgroup of $(G, \cdot)$. If there exists an element $a \in G$ such that $\mu(a) < \mu(x)$ for every $x \in G$, then $\mu$ is a $s$-fuzzy subgroup of a group $\text{ret}_a(G, f)$.

Proof. For all $x, y, a \in G$, let if possible $\mu$ is not a $s$-fuzzy subgroup of a group $\text{ret}_a(G, f)$. Then we have $\mu(x \circ y) > S\{\mu(x), \mu(y)\}$. That is
\begin{align*}
S\{\mu(x), \mu(x)\} &< \mu(x \circ x) \\
&= \mu(f(x, \frac{n-2}{a}, x)) \\
&= \mu(\mu(x), \mu(y)) \\
S\{\mu(x), \mu(x)\} &< S\{\mu(x), \mu(a)\}.
\end{align*}

This holds only if $\mu(a) > \mu(x)$, which is contradiction to our assumption $\mu(a) \leq \mu(x)$.

Also, we have $\mu$ is a $s$-fuzzy subgroup of $(G, \cdot)$. Thus $\mu(x^{-1}) \leq \mu(x)$ is obvious for all $x \in G$.

which complete the proof. $\square$

In Theorem 3.9, the assumption that $\mu(a) \leq \mu(x)$ cannot be omitted.

Example 3.2. Let $(Z_4, f)$ be a 4-ary group from Example 3.1. Define a fuzzy set $\mu$ as follows:
\[
\mu(x) = \begin{cases} 
0.4, & \text{if } x = 0, \\
1, & \text{if } x = 1, 2, 3.
\end{cases}
\]
Clearly, $\mu$ is a $s$-fuzzy 4-ary subgroup of $(Z_4, f)$. For $\text{ret}_1(Z_4, f)$, define
\[
S(x, y) = \begin{cases} 
\max(x, y), & \text{if } x = y, \\
\min(x + y, 1), & \text{if } x \neq y.
\end{cases}
\]
we have
\[
\mu(0 \circ 0) = \mu((f(0, 1, 1, 0))) = \mu(2) = 1 \geq 0.4 = \mu(0) = S\{\mu(0), \mu(0)\}.
\]
Hence the assumptions $\mu(a) \leq \mu(x)$ cannot be omitted.

Theorem 3.10. Let $(G, f)$ be an $n$-ary group. If $\mu$ is a $s$-fuzzy $n$-ary subgroup of a group $\text{ret}_a(G, f)$ and $\mu(a) \leq \mu(x)$ for all $a, x \in G$, then $\mu$ is a $s$-fuzzy $n$-ary subgroup of $(G, f)$.
Thus the assumption of Theorem 3.10 is satisfied. Hence we have
\[ \mu(\varphi(x)) = \mu(f(\overline{\pi}, x, (a^{-1})) \leq S\{\mu(\overline{\pi}), \mu(x), \mu(a)\} \leq \mu(x). \]

Similarly, for all \( x \in G \) we have
\[ \mu(b) = \mu(f(\overline{\pi}, ..., \overline{\alpha})) \leq \mu(\overline{\pi}) \leq \mu(x). \]

Thus
\[ \mu(f(x^n)) = \mu(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ ... \circ \varphi^{n-2}(x_n) \circ b) \]
\[ \leq S\{\mu(x_1), \mu(\varphi(x_2)), \mu(\varphi^2(x_3)), ..., \mu(\varphi^{n-2}(x_n)), \mu(b)\} \]
\[ \leq S\{\mu(x_1), \mu(x_2), \mu(x_3), ..., \mu(x_n), \mu(b)\} \]
\[ \leq S\{\mu(x_1), \mu(x_2), \mu(x_3), ..., \mu(x_n)\}. \]

From (4) and (7) of [3], we have
\[ \overline{\pi} = (\mu(\varphi(x) \varphi^2(x) \varphi^{n-2}(x) \circ b)^{-1} \]

Thus
\[ \mu(\overline{\pi}) = \mu\left((\varphi(x) \varphi^2(x) \varphi^{n-2}(x) \circ b)^{-1}\right) \]
\[ \leq S\{\mu(\varphi(x)), \mu(\varphi^2(x)), ..., \mu(\varphi^{n-2}(x)), \mu(b)\} \]
\[ \leq S\{\mu(x), \mu(b)\} = \mu(x). \]

This completes the proof. \( \square \)

**Corollary 3.2.** If \((G, f)\) is a ternary group, then any \(s\)-fuzzy subgroup of \(\text{ret}_a(G, f)\) is an \(s\)-fuzzy ternary subgroup of \((G, f)\).

**Proof.** Since \(\overline{\pi}\) is a neutral element of a group \(\text{ret}_a(G, f)\) then \(\mu(\overline{\pi}) \leq \mu(x)\), for all \(x \in G\). Thus \(\mu(\overline{\pi}) \leq \mu(a)\). But in ternary group \(\overline{\pi} = a\) for any \(a \in G\), whence \(\mu(a) = \mu(\overline{\pi}) \leq \mu(\overline{\pi}) \leq \mu(x)\). So, \(\mu(a) = \mu(\overline{\pi}) \leq \mu(x)\), for all \(x \in G\). This means that the assumption of Theorem 3.10 is satisfied. Hence \(\text{ret}_a(G, f)\) is an \(s\)-fuzzy ternary subgroup of \((G, f)\). This completes the proof. \( \square \)

**Example 3.3.** Consider the ternary group \((\mathbb{Z}_{12}, f)\), derived from the additive group \(\mathbb{Z}_{12}\). Let \(\mu\) be a \(s\)-fuzzy subgroup of the group of \(\text{ret}_1(G, f)\) induced by subgroups \(S_1 = \{11\}, S_2 = \{5, 11\}\) and \(S_3 = \{1, 3, 5, 7, 9, 11\}\). Define a fuzzy set \(\mu\) as follows:
\[ \mu(x) = \begin{cases} 
0.1 & \text{if } x = 11, \\
0.3 & \text{if } x = 5, \\
0.5 & \text{if } x = 1, 3, 7, 9, \\
0.9 & \text{if } x \notin S_3.
\end{cases} \]
Then
\[ \mu(5) = \mu(7) = 0.5 \leq 0.3 = \mu(5). \]

Hence \( \mu \) is not a \( s \)-fuzzy ternary subgroup of \( (\mathbb{Z}_{12}, f) \).

**Observations.** From the above Example 3.3 it follows that:

1. There are \( s \)-fuzzy subgroups of \( \text{ret}_a(G, f) \) which are not \( s \)-fuzzy \( n \)-ary subgroups of \((G, f)\).
2. In Theorem 3.10 the assumption \( \mu(a) \leq \mu(x) \) cannot be replaced by the natural assumption \( \mu(\pi) \leq \mu(x) \). (\( \pi \) is the identity of \( \text{ret}_a(G, f) \)). In the above example \( \pi = 11 \), then \( \mu(11) \leq \mu(x) \) for all \( x \in \mathbb{Z}_{12} \).

**Theorem 3.11.** Let \((G, f)\) be an \( n \)-ary group of \( b \)-derived from the group \((G, \circ)\). Any fuzzy set \( \mu \) of \((G, \circ)\) such that \( \mu(b) \leq \mu(x) \) for every \( x \in G \) is a \( s \)-fuzzy \( n \)-ary subgroup of \((G, f)\).

**Proof.** The condition (SFuS1) is obvious. To prove (SFuS2), we have \( n \)-ary group \((G, f)\) \( b \)-derived from the group \((G, \circ)\), which implies
\[ x = (x^{n-2} \circ b)^{-1}, \]
where \( x^{n-2} \) is the power of \( x \) in \((G, \circ)[4]\).

Thus, for all \( x \in G \)
\[ \mu(x) = \mu((x^{n-2} \circ b)^{-1}) \leq \mu(x^{n-2} \circ b)^{-1} \leq S\{\mu(x^{n-2}), \mu(b)\} = \mu(x). \]

This complete the proof. \( \square \)

**Corollary 3.3.** Any \( s \)-fuzzy subgroup of a group \((G, \circ)\) is a \( s \)-fuzzy \( n \)-ary subgroup of an \( n \)-ary group \((G, f)\) derived from \((G, \circ)\).

**Proof.** If \( n \)-ary group \((G, f)\) is derived from the group \((G, \circ)\) then \( b = e \). Thus \( \mu(e) \leq \mu(x) \) for all \( x \in G \). \( \square \)

4. **S-product of \( s \)-fuzzy \( n \)-ary relations**

**Definition 4.1.** A fuzzy \( n \)-ary relation on any set \( G \) is a fuzzy set
\[ \mu : G^n = G \times G \times \cdots \times G \ (n \times) \rightarrow [0, 1]. \]

**Definition 4.2.** Let \( \mu \) be fuzzy \( n \)-ary relation on any set \( G \) and \( \nu \) is a fuzzy set on \( G \). Then \( \mu \) is called \( s \)-fuzzy \( n \)-ary relation on \( \nu \) if
\[ \mu(x^n_1) \leq S(\nu(x_1), \nu(x_2), \ldots, \nu(x_n)), \]
for all \( x^n_1 \in G \).

**Definition 4.3.** Let \( \mu^n_1, \mu^n_2, \ldots, \mu^n_n \) be a fuzzy sets in \( G \). Then direct \( S \)-product of \( \mu^n_1 \) is defined by
\[ (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x^n_1) = S(\mu_1(x_1), \mu_2(x_2), \ldots, \mu_n(x_n)), \forall x^n_1 \in G. \]
Lemma 4.1. Let \( S \) be a function induced by \( t \)-conorm and let \( \mu_i^\nu \) be a fuzzy sets in \( G \). Then
(i) \( \mu_1 \times \mu_2 \times \cdots \mu_n \) is a \( s \)-fuzzy \( n \)-ary relation on \( G \),
(ii) \( L(\mu_1 \times \mu_2 \times \cdots \mu_n; t) = L(\mu_1; t) \times L(\mu_2; t) \times \cdots \times L(\mu_n; t), \forall t \in [0,1] \).

Proof. The proof is obvious. \( \square \)

Definition 4.4. Let \( S \) be a function induced by \( t \)-conorm. If \( \nu \) is a fuzzy set in \( G \), the strongest \( s \)-fuzzy \( n \)-ary relation on \( G \) that is a \( s \)-fuzzy \( n \)-ary relation on \( \nu \) is \( \mu_\nu \), given by
\[
\mu_\nu(x_\nu^n) = S(\nu(x_1^n), \nu(x_2^n), \ldots, \nu(x_n^n)), \forall x_\nu^n \in G.
\]

Lemma 4.2. For a given fuzzy set \( \nu \) in \( G \), let \( \mu \) be the strongest \( s \)-fuzzy \( n \)-ary relation of \( G \). Then for \( t \in [0,1] \), \( L(\mu; t) = L(\nu; t) \times L(\nu; t) \times \cdots \times L(\nu; t) \).

Proof. The proof is obvious. \( \square \)

Proposition 4.1. Let \( S \) be a function induced by \( t \)-conorm and let \( \mu_1, \mu_2, \ldots, \mu_n \) be \( s \)-fuzzy \( n \)-ary subgroup of \( (G,f) \). Then, \( \mu_1 \times \mu_2 \times \cdots \times \mu_n \) is a \( s \)-fuzzy \( n \)-ary subgroup of \( (G^n,f) \).

Proof. For \( x_1^n \in G \) and \( f(x_1^n) = (f_1(x_1^n), \ldots, f_n(x_1^n)) \in (G^n,f) \), we have
\[
(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(f(x_1^n)) = (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(f_1(x_1^n), \ldots, f_n(x_1^n)) = S(\mu_1(f_1(x_1^n)), \mu_2(f_1(x_1^n)), \ldots, \mu_n(f_1(x_1^n))) \leq S \{S(\mu_1(x_1), \mu_1(x_2), \ldots, \mu_1(x_n)), \ldots, S(\mu_n(x_1), \mu_n(x_2), \ldots, \mu_n(x_n))\} = S(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x_1, \ldots, x_n) = S(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x_1^n).
\]
and for all \( x = x_1^n, a = a_1^n \in G^n \), we have
\[
(\mu_1 \times \mu_2 \times \cdots \times \mu_n)(a) = (\mu_1 \times \mu_2 \times \cdots \mu_n)(a_1^n) = S(\mu_1(a_1^n), \ldots, \mu_n(a_1^n)) \leq S(\mu_1(a_1^n), \ldots, \mu_n(a_1^n)) = (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x_1^n) = (\mu_1 \times \mu_2 \times \cdots \times \mu_n)(x).
\]
This completes the proof. \( \square \)

The following Corollary is the immediate consequence of Proposition 4.1.

Corollary 4.1. Let \( S \) be a function induced by \( t \)-conorm and let \( \prod_{i=1}^n (G_i,f) \) be the finite collection of \( n \)-ary subgroups and \( G = \prod_{i=1}^n G_i \) the \( S \)-product of \( G_i \). Let
μᵢ be a s-fuzzy n-ary subgroup of \((G, f)\), where \(1 \leq i \leq n\). Then, \(μ = \prod_{i=1}^{n} μᵢ\) defined by
\[
μ(x^n) = \prod_{i=1}^{n} μᵢ(xᵢ^n) = S(μ₁(x₁), μ₂(x₂), ..., μₙ(xₙ)).
\]
Then μ is a s-fuzzy n-ary subgroup of \((G, f)\).

**Definition 4.5.** Let \(μᵢ^n\) be fuzzy sets in \(G\). Then, the \(S\)-product of \(μᵢ^n\), written as \([μ₁ · μ₂ · ... · μₙ]_S\), is defined by:
\[
[μ₁ · μ₂ · ... · μₙ]_S(x) = S(μ₁(x), μ₂(x), ..., μₙ(x)) \quad ∀ x ∈ G.
\]

**Theorem 4.1.** Let \(μᵢ^n\) be s-fuzzy n-ary subgroup of \((G, f)\). If \(S^*\) is a function induced by t-conorm dominates \(S\), that is,
\[
S^*(S(x^n), S(y^n), ..., S(z^n)) \leq S(S^*(x₁, y₁, ..., z₁), ..., S^*(xₙ, yₙ, ..., zₙ))
\]
for all \(x^n, y^n, ..., z^n \in [0, 1]\). Then \(S^*\)-product of \(μᵢ^n\), \([μ₁ · μ₂ · ... · μₙ]_S^*\), is a s-fuzzy n-ary subgroup of \((G, f)\).

**Proof.** Let \(x^n_i \in G\), we have
\[
[μ₁ · μ₂ · ... · μₙ]_S^*(f(x^n)) = S^*(μ₁(f(x^n₁)), μ₂(f(x^n₂)), ..., μₙ(f(x^nₙ))) ≤ S^*(S(μ₁(x₁), μ₂(x₂), ..., μₙ(xₙ)), S(μ₁(x₁), μ₂(x₂), ..., μₙ(xₙ))) ≤ S(S^*(μ₁(x₁), μ₂(x₂), ..., μₙ(xₙ)), S(μ₁(x₁), μ₂(x₂), ..., μₙ(xₙ))) = [μ₁ · μ₂ · ... · μₙ]_S^*(S^*(x₁), ..., S^*(μ₁, μ₂, ..., μₙ)_S^*(xₙ))
\]
and for all \(x ∈ G\), we have
\[
[μ₁ · μ₂ · ... · μₙ]_S^*(x) = S^*(μ₁(x), μ₂(x), ..., μₙ(x)) \leq S(μ₁(x), μ₂(x), ..., μₙ(x)) = [μ₁ · μ₂ · ... · μₙ]_S^*(x).
\]
Hence, \([μ₁ · μ₂ · ... · μₙ]_S^*\) is a s-fuzzy n-ary subgroup of \((G, f)\). This completes the proof.

Let \((G, f)\) and \((G', f')\) be an n-ary groups. A mapping \(g : G \to G'\) is an onto homomorphism. Let \(S\) and \(S^*\) be functions induced by t-conorm such that \(S^*\) dominates \(S\). If \(μᵢ^n\) are s-fuzzy n-ary subgroup of \((G, f)\), then the \(S^*\)-product of \(μᵢ^n\), \([μᵢ · μⱼ · ... · μₙ]_S^*\), is a s-fuzzy n-ary subgroup. Since every onto homomorphic inverse image of a s-fuzzy n-ary subgroup, the inverse images \(g^{-1}(μ₁), g^{-1}(μ₂), ..., g^{-1}(μₙ)\) and \(g^{-1}([μ₁ · μ₂ · ... · μₙ]_S^*)\) are s-fuzzy n-ary subgroup \((G, f)\).

The following theorem provides the relation between \(g^{-1}([μ₁ · μ₂ · ... · μₙ]_S^*)\) and \(S^*\)-product \(([g^{-1}(μ₁) · g^{-1}(μ₂) · ... · g^{-1}(μₙ)]_S^*)\) of \(g^{-1}(μ₁), g^{-1}(μ₂)\) and \(g^{-1}(μₙ)\).
THEOREM 4.2. Let \((G, f)\) and \((G', f')\) be an \(n\)-ary groups. A mapping \(g : G \rightarrow G'\) is an onto \(n\)-ary homomorphism. Let \(S^*\) be a function induced by \(t\)-conorm such that \(S^*\) dominates \(S\). Let \(\mu^*_1\) be \(s\)-fuzzy \(n\)-ary subgroups of \((G, f)\). If \([\mu_1, \mu_2, \ldots, \mu_n]_{S^*}\) and is the \(S^*\)-product of \(\mu^*_1\), and \((g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \ldots \cdot g^{-1}(\mu_n))_{S^*}\) is the \(S^*\)-product of \(g^{-1}(\mu_1), g^{-1}(\mu_2), \ldots, g^{-1}(\mu_n)\) then

\[
g^{-1}(\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n)_{S^*} = [g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \ldots \cdot g^{-1}(\mu_n)]_{S^*}.
\]

Proof. Let \(x \in G\), we have

\[
g^{-1}(\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n)_{S^*}(x) = ([\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_n]_{S^*}(g(x))
\]

\[
= [S^*([\mu_1(g(x)) \cdot \mu_2(g(x)) \cdot \ldots \cdot \mu_n(g(x)))]
\]

\[
= S^*[g^{-1}(\mu_1)(x) \cdot g^{-1}(\mu_2)(x) \cdot \ldots \cdot g^{-1}(\mu_n)(x)]
\]

\[
= [g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \ldots \cdot g^{-1}(\mu_n)]_{S^*}.
\]

This completes the proof. \(\Box\)

Acknowledgement: The author is extremely grateful to the learned referee’s for their valuable comments and suggestions which helped me a lot for improving the standard of this paper.

References


Received by editors 02.09.2016; Revised version 02.10.2016; Available online 17.10.2016.