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LIAR'S DOMINATION IN GRAPHS

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Abstract. A set $D \subseteq V(G)$ of a graph $G = (V, E)$ is a liar's dominating set if (1) for all $v \in V(G)$ $|N[v] \cap D| \geq 2$ and (2) for every pair $u, v \in V(G)$ of distinct vertices, $|N[u] \cup N[v] \cap D| \geq 3$. In this paper, we consider the liar's domination number of some middle graphs. Every triple dominating set is a liar's dominating set and every liar's dominating set must be a double dominating set. So, the liar's dominating set lies between double dominating set and triple dominating set.

1. Introduction

Let $G = (V, E)$ be a graph, where V is a vertex set and E is an edge set of a given graph G. For $v \in V(G)$, let $N(v) = \{u|uv \in E(G)\}$ denote the neighborhood of v and $N[v] = \{N(v) \cup v\}$ denote the closed neighborhood of v. A vertex $u \in V(G)$ is said to be dominated by a vertex $v \in V(G)$ if $u \in N[v]$.

A set $D \subseteq V$ is called a dominating set of $G = (V, E)$ if each vertex $v \in V$ is dominated by at least a vertex in D, so $|N[v] \cap D| \geq 1$ for all $v \in V$. A set $D \subseteq V$ is called a k-tuple dominating set if $|N[v] \cap D| \geq k$ for every $x \in V(G)$ and the minimum cardinality of a k-tuple dominating set for G is denoted by $\gamma_{xk}(G)$. A 2-tuple dominating set is also called a double dominating set and denoted by $\gamma_{x2}(G)$ or $dd(G)$. Obviously, a 3-tuple dominating set is also called a triple dominating set [2]. Throughout this paper, we use terminology of P. J. Slater and M. L. Roden, reader may find more details in the references.

Networks are exposed to intrusions which has to be detected. It is assumed here that the possible intrusion points are the vertices of G. A protective device placed at a vertex v can detect the presence of an intrusion in a precise manner when the intrusion point is in $N[v]$. In this paper, we consider liar's dominating set which is assumed that any one protective device in the neighborhood of the intrusion vertex might misreport the location of an intrusion vertex in its closed neighborhood [4].

A dominating set $D \subseteq V(G)$ is a liar's dominating set if any designated vertex $x \in V(G)$ (namely, the intrusion vertex) if all or all but one of the vertices in $N[x] \cup D$ report vertex x, and at most one vertex w in $N[x] \cup D$ either reports

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another vertex than x or it does not report at all, then the vertex x can be correctly identified as the designated vertex.

Every superset of a liar's dominating set is also a liar's dominating set. The minimum cardinality of a liar's dominating set for graph G, is called the liar's domination number of G and denoted by $\gamma_{LR}(G)$.

In this paper, it is assumed that all protective devices can detect correctly and there will be at most one single fault tolerant at the reporting [4]. For any vertex set and denoted by $\gamma_{LR}(G)$. Note that it is assumed that all protection devices can detect correctly and there will be at most a single fault in the reporting. For any vertex set $D \subseteq V(G)$, we will say that vertex x is LR-dominated by D if D can correctly identify x as a designated vertex [5].

2. Preliminaries

Theorem 1. [4] If $D \subseteq V(G)$ is a liar's dominating set then each component of (D) contains at least three vertices.

Theorem 2. [4] For every connected graph G of order $n \geq 3$ we have $\gamma_{x2}(G) \leq \gamma_{LR}(G)$, and, if G has minimum degree $\delta(G) \geq 2$, then $\gamma_{x2}(G) \leq \gamma_{LR}(G) \leq \gamma_{x3}(G)$.

The next theorem gives necessary and sufficient conditions for $D \subseteq V(G)$ to be a liar's dominating set.

Theorem 3. [5] A dominating set $D \subseteq V(G)$ is a liar's dominating set if and only if D double dominates every $v \in V(G)$ and $|N[u] \cup N[v] \cap D| \geq 3$ for every pair u, v of distinct vertices .

In this paper, we consider the liar's domination number for middle graphs of some well-known graphs (The paragraph below...The total graph...). The total graph $T(G)$ of a graph G is a graph such that the vertex set of T corresponds to the vertices and edges of G and two vertices are adjacent in T iff their corresponding elements are either adjacent or incident in G. Total graphs are generalizations of line graphs. The line graph $L(G)$ of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common. The middle graph $M(G)$ is the graph obtained from G by inserting a new vertex into every edge of G and then joining these new vertices by edges, which lie on the adjacent edges of G. In particular, we consider the middle graphs because these graphs are between total graphs and line graphs. In real life problems, every edge corresponds cost, so middle graphs make sense in this situation [1].

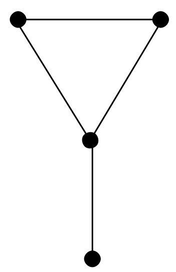

Figure 2.1. G

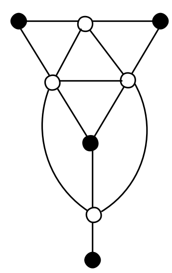

Figure 2.2. $M(G)$

3. Liar's domination in graphs

We give theorems on Liar's domination numbers of middle graphs of some well-known graphs. We will start with the middle graph of a path graph.

Theorem 4. Let $M(P_n)$ be the middle graph of a path graph, where P_n is the path graph of order n, then

$$\gamma_{LR}(M(P_n)) \leq n+1.$$

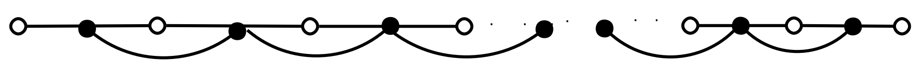

Figure 3.1. $M(P_n)$

Proof. Let the vertices of P_n be $v_1, v_2, ..., v_n$ and the vertices of $V(M(P_n) - P_n)$ be $v_{n+1}, v_{n+2}, ..., v_{2n-1}$. Let $v \in V(M(P_n))$. We investigate all vertices of $M(P_n)$ by four cases to obtain a liar's dominating set. Firstly, we want to obtain double dominating set D.

Case 1. Let v be v_1. $deg(v_1) = deg(v_n) = 1$. So, $N[v_1] = \{v_1, v_{n+1}\}$ from $M(P_n)$ and $|N[v_1]| = 2$. D double dominates every $v \in V(M(P_n))$. Therefore all vertices of $N[v_1]$ must be elements of D, $\{v_1, v_{n+1}\} \in D$. So, $\{v_{2n-1}, v_n\} \in D$ for $v = v_n$. Hence,

$$\{v_1, v_{n+1}, v_{2n-1}, v_n\} \in D$$

Case 2. Let v be an element of $\{v_2, v_3, v_4, ..., v_{n-1}\}$. $deg(v_2) = deg(v_3) = ... = deg(v_{n-1}) = 2$

$$N[v_i] = \{v_i, v_{n+i-1}, v_{n+i}\}$$

for $i = 2, ..., n-1$. Let $N[v_j] \cap N[v_k] = D_{j,k}$, where $k = 2, ..., n-2$ and $j = k+1$. Then

$$\bigcup_{j,k=2}^{n-1} D_{j,k} = \{v_{n+2}, v_{n+3}, v_{n+4}, ..., v_{2n-2}\} \in D.$$

$$\{v_1, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\} \in D$$

and this set double dominates $\forall v_j$, for $j = 1, ..., n$ (one can take different set also with same size). Now, we shall check whether D is triple domination.

We show $|N[u] \cup N[v] \cap D| \geq 3$ for every pair u, v of distinct vertices, for the same vertex set. Hold

$$|N[v_j] \cap D| = 2,\ |N[v_k] \cap D| = 2$$
$$|N[v_j] \cap N[v_k]| \leq 1.$$

So, we have

$$|N[v_j] \cup N[v_k] \cap D| \geq 3, \text{ where } j, k = 1, ..., n.$$

Then, D triple dominates $\forall v_j$, where $j = 1, ..., n$.

Case 3. Let v be v_{n+1}. $deg(v_{n+1}) = 3$, $N[v_{n+1}] = \{v_1, v_2, v_{n+2}, v_{n+1}\}$. We have

$$N[v_{n+1}] \cap D = \{v_1, v_{n+1}, v_{n+2}\}$$

and

$$|N[v_{n+1}] \cap D| = 3.$$

Let v be v_{2n-1}. $deg(v_{2n-1}) = 3$, $N[v_{2n-1}] = \{v_{2n-1}, v_{n-1}, v_n, v_{2n-2}\}$. We get

$$N[v_{2n-1}] \cap D = \{v_{2n-2}, v_{2n-1}, v_n\}$$

and

$$|N[v_{2n-1}] \cap D| = 3.$$

Hence, D triple dominates v_{2n-1}, v_{n+1} by Theorem 3.

Case 4. Let v be an element of $\{v_{n+2}, v_{n+3}, ..., v_{2n-2}\}$ and $deg(v_{n+2}) = deg(v_{n+3}) = ... = deg(v_{2n-2}) = 4$. Then

$$N[v_{n+2}] = \{v_{n+2}, v_{n+1}, v_2, v_3, v_{n+3}\}$$

$$N[v_{n+3}] = \{v_{n+3}, v_{n+2}, v_3, v_4, v_{n+4}\}$$

$$\vdots$$

$$N[v_{2n-2}] = \{v_{2n-2}, v_{2n-3}, v_{n-2}, v_{n-1}, v_{2n-1}\}.$$

We obtain $|N[v_j] \cap D| \geq 3$ where $j = n+2, ..., 2n-2$. Therefore, D triple dominates $\forall v_j$ where $j = n+2, ..., 2n-2$ by Theorem 3.

The liar's dominating set of the middle graph of a path graph consists of end vertices and new vertices of $M(P_n)$ which are obtained from the definition of the middle graph. Consequently,

$$D = \{v_1, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\}$$

is the liar's dominating set of $M(P_n)$ and

$$\gamma_{LR}(M(P_n)) \leq n+1.$$

□

Theorem 5. Let $M(C_n)$ be the middle graph of a cycle graph, where C_n is the cycle graph of order n. Then

$$\gamma_{LR}(M(C_n)) \leq n.$$

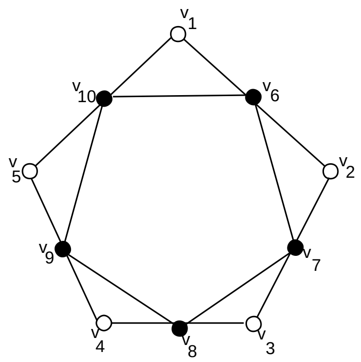

FIGURE 3.2. $M(C_5)$

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of C_n and $v_{n+1}, v_{n+2}, ..., v_{2n}$ be the vertices of $V(M(C_n) - C_n)$. We investigate all vertices of $M(C_n)$ by two cases to obtain a liar's dominating set. Firstly, we want to obtain double dominating set D.

Case 1. $deg(v_i) = 2$ where $i = 1, ..., n$. Let v be an element of $\{v_1, v_2, ..., v_n\}$. D should contain at least 2 vertices of $N[v_i]$ to D double dominates v_i for $i = 1, ..., n$.

$$N[v_1] = \{v_1, v_{n+1}, v_{2n}\}$$

$$N[v_2] = \{v_2, v_{n+1}, v_{n+2}\}$$

$$\vdots$$

$$N[v_{n-1}] = \{v_{n-1}, v_{2n-2}, v_{2n-1}\}$$

$$N[v_n] = \{v_n, v_{2n-1}, v_{2n}\}$$

If $uv \in E(C_n)$, then we take the intersection of close neighborhoods of them.

Let $N[v_j] \cap N[v_k] = D_{j,k}$, where $j, k = 1, ..., n$. So, we have

$$D_1 = \bigcup_{j,k=1}^{n} D_{j,k} = \{v_{n+1}, v_{n+2}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\} \in D.$$

This set double dominates v_i for $i = 1, ..., n$.

Clearly, we have $|N[v_j] \cup N[v_k] \cap D_1| \geq 3$ from $|N[v_j] \cap D_1| = 2$ and $|N[v_j] \cap N[v_k]| = 1$ for the vertices of v_j and v_k, where $v_j v_k \in E(C_n)$, $j, k = 1, ..., n$.

Likewise, we have $|N[v_j] \cup N[v_k] \cap D_1| = 4 \geq 3$ from $|N[v_j] \cap D_1| = 2$ and $|N[v_j] \cap N[v_k]| = 0$ for the vertices of v_j and v_k, where $v_j v_k \notin E(C_n)$, $j, k = 1, ..., n$.

Therefore, D_1 triple dominates v_i for $i = 1, ..., n$.

Case 2. Suppose $deg(v_j) = 4$, where $j = n + 1, ..., 2n$. Let v be an element of $\{v_{n+1}, v_{n+2}, ..., v_{2n}\}$.

$$N[v_{n+1}] = \{v_{n+1}, v_1, v_2, v_{2n}, v_{n+2}\}$$

$$N[v_{n+2}] = \{v_{n+2}, v_2, v_3, v_{n+1}, v_{n+3}\}$$

$$\vdots$$

$$N[v_{2n-1}] = \{v_{2n-1}, v_{n-1}, v_n, v_{2n-2}, v_{2n}\}$$

$$N[v_{2n}] = \{v_{2n}, v_1, v_n, v_{n+1}, v_{2n-1}\}$$

We obtain $|N[v_j] \cap D_1| = 3$ for $j = n + 1, ..., 2n$. Therefore, D_1 triple dominates v_j for $j = n+1, ..., 2n$ by Theorem 3. So, $D = D_1 = \{v_{n+1}, v_{n+2}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\}$ is the liar's dominating set of $M(C_n)$ and

$$\gamma_{LR}(M(C_n)) \leq n.$$

□

Theorem 6. Let $M(W_{1,n})$ be the middle graph of a wheel graph, where $W_{1,n}$ is the wheel graph of order $n + 1$. Then

$$\gamma_{LR}(M(W_{1,n})) \leq n + 1.$$

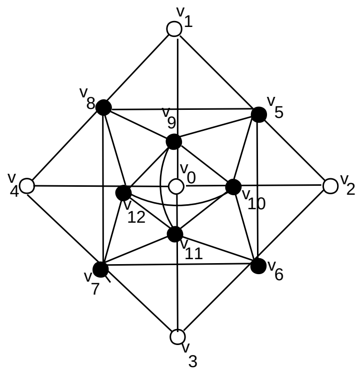

FIGURE 3.3. $M(W_{1,4})$

Proof. Let hub vertex be v_0. Let cycle graph be a clockwise vertex labeled graph from v_1 to v_n and new vertices of cycle graph from the definition of a middle graph clockwise labeled from v_{n+1} to v_{2n}, the other new vertices of $M(W_{1,n})$ from the definition of a middle graph clockwise labeled from v_{2n+1} to v_{3n}.

Now, we show all vertices of $M(W_{1,n})$ by four cases to obtain a liar's dominating set.

Case 1. Let v be an element of $\{v_1, v_2, ..., v_n\}$. $deg(v_1) = deg(v_2) = ... = deg(v_n) = 3$. Then

$$N[v_1] = \{v_1, v_{2n}, v_{2n+1}, v_{n+1}\}$$
$$N[v_2] = \{v_2, v_{n+1}, v_{2n+2}, v_{n+2}\}$$
$$\vdots$$
$$N[v_i] = \{v_i, v_{n+i-1}, v_{2n+i}, v_{n+i}\}$$
$$\vdots$$
$$N[v_n] = \{v_n, v_{2n}, v_{3n}, v_{2n-1}\}$$

for $i = 3, ..., n-1$.

Let $N[v_j] \cap N[v_k] = D_{j,k}$ and $D_{j,k} \in D$, where $j, k = 1, ..., n$. We get

$$D_1 = \bigcup_{j,k=1}^{n} D_{j,k} = \{v_{n+1}, v_{n+2}, v_{n+3}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\}.$$

Case 2. Now, we consider v_0. $deg(v_0) = n$, $N[v_0] = \{v_0, v_{2n+1}, v_{2n+2}, ..., v_{3n}\}$. Let u, v be sequential vertices of $N(v_0)$. For instance, $u = v_{2n+1}$ and $v = v_{2n+2}$. $D_2 = \{v_{2n+1}, v_{2n+2}\}$. Now, we obtain $|N[v_0] \cap D_2| = 2$ and this set double dominates v_0. If we take $N[v_j] \cap D_2$ where $j = 1, ..., n$, then we obtain $v_{2n+1} \in N[v_1]$ and $v_{2n+2} \in N[v_2]$. Let D_3 be $D_1 \cup D_2$. We obtain $|N[v_1] \cap D_3| = 3$, $|N[v_2] \cap D_3| = 3$ and $|N[v_j] \cap D_3| = 2$, where $j = 3, ..., n$. The vertex $v_{n+1} \in N[v_1] \cap N[v_2]$ must remove from D_3 to reduce the domination number. So, we have $|N[v_j] \cap D_3| = 2$, for $j = 0, ..., n$. Therefore, $D_3 = \{v_{2n+1}, v_{2n+2}, v_{n+2}, v_{n+3}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\}$ double dominates v_j, where $j = 0, ..., n$ and $|D_3| = n+1$. Now, u, v be non-sequential vertices of $N(v_0)$. For instance, $u = v_{2n+1}$ and $v = v_{2n+3}$. $D_4 = \{v_{2n+1}, v_{2n+3}\}$.

This set double dominates v_0. If we take $N[v_j] \cap D_4$, where $j = 1, ..., n$, then we obtain $v_{2n+1} \in N[v_1]$ and $v_{2n+3} \in N[v_3]$. Let $D_5 = D_1 \cup D_4$. We obtain $|N[v_1] \cap D_5| = 3$, $|N[v_3] \cap D_5| = 3$ and $|N[v_j] \cap D_5| = 2$, where $j = 2, 4, 5, ..., n$ and also $|N[v_1] \cap N[v_3]| = 0$. So, $D_5 = \{v_{2n+1}, v_{2n+3}, v_{n+1}, v_{n+2}, v_{n+3}, ..., v_{2n-2}, v_{2n-1}, v_{2n}\}$ double dominates v_j, where $j = 0, ..., n$ and $|D_5| = n+2$. Therefore, D_3 has to be D instead of D_5. Now, we take this set as D to check triple domination.

We show $|N[u] \cup N[v] \cap D| \geq 3$ for every pair u, v of distinct vertices, for the same vertex set. Hold

$$|N[v_j] \cap D| \geq 2,\ |N[v_k] \cap D| \geq 2 \text{ and } |N[v_j] \cap N[v_k]| \leq 1.$$

So, we get

$$|N[v_j] \cup N[v_k] \cap D| \geq 3, \text{ where } j, k = 0, ..., n.$$

Then, D triple dominates $\forall v_i$, where $i = 0, ..., n$.

Case 3. Let v be an element of $\{v_{n+1}, v_{n+2}, ..., v_{2n}\}$. $deg(v_{n+1}) = deg(v_{n+2}) = , ..., = deg(v_{2n}) = 6$.

$$\begin{gathered} N[v_{n+1}] = \{v_{n+1}, v_1, v_n, v_{n+2}, v_{2n}, v_{2n+1}, v_{3n}\} \\ N[v_{n+2}] = \{v_{n+2}, v_1, v_2, v_{n+1}, v_{n+3}, v_{2n+1}, v_{2n+2}\} \\ \vdots \\ N[v_{n+i}] = \{v_{n+i}, v_{i-1}, v_i, v_{n+i-1}, v_{n+i-1}, v_{2n+i-1}, v_{2n+i}\} \\ \vdots \\ N[v_{2n}] = \{v_{2n}, v_{n-1}, v_n, v_{2n-1}, v_{n+1}, v_{3n-1}, v_{3n}\} \end{gathered}$$

for $i = 3, ..., n-1$. We obtain $|N[v_j] \cap D| \geq 3$, where $j = n+1, ..., 2n$. Therefore, D triple dominates $\forall v_j$, where $j = n+1, ..., 2n$ by theorem3.

Case 4. Let v be an element of $\{v_{2n+1}, v_{2n+2}, ..., v_{3n}\}$. $deg(v_{2n+1}) = deg(v_{2n+2}) = , ..., = deg(v_{3n}) = n+3$.

$$\begin{gathered} N[v_{2n+1}] = \{v_0, v_1, v_{n+1}, v_{2n}, v_{2n+1}, v_{2n+2}, ..., v_{3n}\} \\ N[v_{2n+2}] = \{v_0, v_2, v_{n+1}, v_{n+2}, v_{2n+1}, v_{2n+2}, ..., v_{3n}\} \\ \vdots \\ N[v_{2n+i}] = \{v_0, v_i, v_{n+i-1}, v_{n+i}, v_{2n+1}, v_{2n+2}, ..., v_{3n}\} \\ \vdots \\ N[v_{3n}] = \{v_0, v_n, v_{2n-1}, v_{2n}, v_{2n+1}, v_{2n+2}, ..., v_{3n}\} \end{gathered}$$

for $i = 3, ..., n-1$. We have $|N[v_j] \cap D| \geq 3$, where $j = 2n+1, ..., 3n$. Hence, D triple dominates $\forall v_j$, where $j = 2n+1, ..., 3n$ by Theorem 3.

Consequently, D, obtained from Case1 and Case2, is a Liar's dominating set of $M(W_{1,n})$. Thus,

$$\gamma_{LR}(M(W_{1,n})) \leq n+1.$$

□

Theorem 7. $GP(5,2)$ is Petersen Graph,

$$\gamma_{LR}(M(GP(5,2))) \leq 10.$$

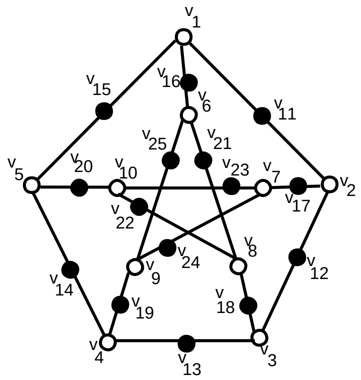

FIGURE 3.4. Vertices of $M(GP(5,2))$.
Let the vertices of $M(GP(5,2))$ labeled by v_i, for $i = 1, ..., 25$.

Proof. In Figure 3.4 one can find $V(M(GP(5,2)))$ not $E(M(GP(5,2)))$ to avoid complicity.

Let $N[v_j] \cap N[v_k] = D_{j,k}$ with $D_{j,k} \subseteq V(M(GP(5,2)))$ for $j,k = 1, ..., 5$. D must contain one of the vertices each of $D_{j,k}$.

$$N[v_1] = \{v_1, v_{11}, v_{15}, v_{16}\}$$
$$N[v_2] = \{v_2, v_{11}, v_{12}, v_{17}\}$$
$$N[v_3] = \{v_3, v_{12}, v_{13}, v_{18}\}$$
$$N[v_4] = \{v_4, v_{13}, v_{14}, v_{19}\}$$
$$N[v_5] = \{v_5, v_{14}, v_{15}, v_{20}\}$$

We obtain $\bigcup\limits_{j,k=1}^{5} D_{j,k} = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\} \in D$. Likewise, for $j,k = 6, ..., 10$ clearly we have $N[v_j] \cap N[v_k] = D_{j,k}$ with $D_{j,k} \subseteq V(M(GP(5,2)))$. Thus, we get

$$N[v_6] = \{v_6, v_{16}, v_{21}, v_{25}\}$$
$$N[v_7] = \{v_7, v_{17}, v_{23}, v_{24}\}$$
$$N[v_8] = \{v_8, v_{18}, v_{21}, v_{22}\}$$
$$N[v_9] = \{v_9, v_{19}, v_{24}, v_{25}\}$$
$$N[v_{10}] = \{v_{10}, v_{20}, v_{22}, v_{23}\}$$

So, $\bigcup\limits_{j,k=6}^{10} D_{j,k} = \{v_{21}, v_{22}, v_{23}, v_{24}, v_{25}\} \in D$. Hence, from these two situation

$$D = \bigcup_{j,k=1}^{10} D_{j,k} = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}\}.$$

This set double dominates $\forall v \in V(M(GP(5,2)))$.

Now, we show D triple dominates $\forall v_j \in V(M(GP(5,2)))$, where $j = 1, ..., 10$. We show $|N[u] \cup N[v] \cap D| \geq 3$ for every pair u, v of distinct vertices, for the same vertex set. $|N[v_j] \cap D| = 2$, $|N[v_k] \cap D| = 2$ and $|N[v_j] \cap N[v_k]| \leq 1$. So, $|N[v_j] \cup N[v_k] \cap D| \geq 3$, where $j,k = 1, ..., 10$. Then, D triple dominates $\forall v_j$, where $j = 1, ..., 10$. We have $deg(v_j) = 6$ for $j = 11, ..., 25$. It is easy to see that $|N[v_j] \cap$

$D| \geq 3$. Clearly, we obtain $|N[v_j] \cup N[v_k] \cap D| \geq 3$ for $j, k = 1, ..., 25$. Consequently, $D = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}\}$ is the liar's dominating set and then $\gamma_{LR}(M(GP(5, 2))) \leq 10$. □

References

[1] D. Dogan. Average Lower Domination Number For Some Middle Graphs, *International J. Math. Combin.*, **4**(2012), 58–67.

[2] F. Harary and T. W. Haynes. Double Domination in Graphs, *Ars Combinatoria*, **55**(2000), 201–213.

[3] B.S. Panda and S. Paul. Liar's domination in graphs: Complexity and algorithm, *Discrete Applied Mathematics*, **161**(7–8)(2013), 1085–1092.

[4] M.L. Roden and P. J. Slater. Liar's domination in graphs, *Discrete Mathematics*, **309**(19)(2009), 5884–5890.

[5] P. J. Slater. Liar's domination, *Networks,* **54**(2)(2009), 70–74.

[6] P.J. Slater. Domination and location in acyclic graphs, *Networks*, **17**(1987), 55–64.

[7] P.J. Slater. Dominating and reference sets in a graph, *J. Math. Phys. Sci.*, **22** (1988), 445–455.

[8] P.J. Slater. Locating dominating sets and locating-dominating sets, In Y. Alavi, A. Schwenk (Eds.) *Graph theory, combinatorics, and algorithms : proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs* (Vol. 2, pp. 1073–1079), Wiley-Intersci, New York, 1995.

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