ON BANHATTI AND ZAGREB INDICES

I. Gutman, V. R. Kulli, B. Chaluvaraju, and H. S. Boregowda

Abstract. Let $G = (V, E)$ be a connected graph. The Zagreb indices were introduced as early as in 1972. They are defined as $M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$ and $M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v)$, where $d_G(u)$ denotes the degree of a vertex $u$. The $K$ Banhatti indices were introduced by Kulli in 2016. They are defined as $B_1(G) = \sum_{ue} [d_G(u) + d_G(e)]$ and $B_2(G) = \sum_{ue} d_G(u)d_G(e)$, where $ue$ means that the vertex $u$ and edge $e$ are incident and $d_G(e)$ denotes the degree of the edge $e$ in $G$. These two types of indices are closely related. In this paper, we obtain some relations between them. We also provide lower and upper bounds for $B_1(G)$ and $B_2(G)$ of a connected graph in terms of Zagreb indices.

1. Introduction

The graphs considered here are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree $d_G(v)$ of a vertex $v$ is the number of vertices adjacent to $v$. The edge connecting the vertices $u$ and $v$ will be denoted by $uv$. Let $d_G(e)$ denote the degree of an edge $e = uv$ in $G$, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The vertices and edges of a graph are said to be its elements. For additional definitions and notations, the reader may refer to [11].

A molecular graph is a graph in which the vertices correspond to the atoms and the edges to the bonds of a molecule. A single number that can be computed from the molecular graph, and used to characterize some property of the underlying molecule is said to be a topological index or molecular structure descriptor. Numerous such descriptors have been considered in theoretical chemistry, and have found some applications, especially in QSPR/QSAR research, see [6, 9, 17].

2010 Mathematics Subject Classification. 05C05; 05C07; 05C35.

Key words and phrases. Zagreb index, hyper-Zagreb index, K Banhatti index, K hyper-Banhatti index.
In [12], Kulli introduced the first and second K Banhatti indices, intending to take into account the contributions of pairs of incident elements. The first K Banhatti index $B_1(G)$ and the second K Banhatti index $B_2(G)$ of a graph $G$ are defined as

$$B_1(G) = \sum_{ue} [d_G(u) + d_G(e)]$$

and

$$B_2(G) = \sum_{ue} d_G(u) d_G(e)$$

where $ue$ means that the vertex $u$ and edge $e$ are incident in $G$.

The first and second K hyper–Banhatti indices of a graph $G$ are defined as

$$HB_1(G) = \sum_{ue} [d_G(u) + d_G(e)]^2$$

and

$$HB_2(G) = \sum_{ue} [d_G(u) d_G(e)]^2.$$  

The K hyper–Banhatti indices were introduced by Kulli in [13].

The degree–based graph invariants $M_1(G)$ and $M_2(G)$, called Zagreb indices, were introduced long time ago [10] and have been extensively studied. For their history, applications, and mathematical properties, see [2, 6, 7, 8, 15] and the references cited therein.

The first and second Zagreb indices take into account the contributions of pairs of adjacent vertices. The first and second Zagreb indices of a graph $G$ are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2$$

or

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

In [14], Milicic et al., reformulated the first Zagreb index in terms of edge-degrees instead of vertex-degrees and defined the respective topological index as

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2.$$  

Followed by the first Zagreb index of a graph $G$, Furtula and one of the present authors [5] introduced the so-called forgotten topological index $F$, defined as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in V(G)} [d_G(u)^2 + d_G(v)^2].$$

In [16], Shirdel et al., introduced the first hyper–Zagreb index of $G$ and defined it as

$$HM_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2.$$  

2. Comparison of Banhatti and Zagreb–type indices

Theorem 2.1. For any graph $G$, the first Banhatti index is related to the first Zagreb index as $B_1(G) = 3M_1(G) - 4m$.  

Proof. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then

$$B_1(G) = \sum_{uv} [d_G(u) + d_G(v)]$$

$$= \sum_{uv \in E(G)} [d_G(u) + d_G(uv)] + \sum_{uv \in E(G)} [d_G(v) + d_G(uv)]$$

$$= \sum_{uv \in E(G)} [d_G(u) + d_G(u) + d_G(v) - 2]$$

$$+ \sum_{uv \in E(G)} [d_G(v) + d_G(u) + d_G(v) - 2]$$

$$= \sum_{uv \in E(G)} [3d_G(u) + 3d_G(v) - 4] = 3M_1(G) - 4m. \qed$$

Theorem 2.2. For any graph $G$, the second Banhatti index is related to the first Zagreb and hyper-Zagreb indices as $B_2(G) = HM_1(G) - 2M_1(G)$.

Proof. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then

$$B_2(G) = \sum_{uv} d_G(u) d_G(v)$$

$$= \sum_{uv \in E(G)} d_G(u) d_G(uv) + \sum_{uv \in E(G)} d_G(v) d_G(uv)$$

$$= \sum_{uv \in E(G)} d_G(u) [d_G(u) + d_G(v) - 2]$$

$$+ \sum_{uv \in E(G)} d_G(v) [d_G(u) + d_G(v) - 2]$$

$$= \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2 - 2[d_G(u) + d_G(v)]$$

$$= HM_1(G) - 2M_1(G). \qed$$

Theorem 2.3. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $EM_1(G) = HM_1(G) - 4M_1(G) + 4m$.

Proof. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v) - 2]^2$$

$$= \sum_{uv \in E(G)} \left([d_G(u) + d_G(v)]^2 - 4[d_G(u) + d_G(v)] + 4\right)$$

$$= HM_1(G) - 4M_1(G) + 4m. \qed$$
Theorem 2.4. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $B_1(G) = HM_1(G) - EM_1(G) - M_1(G)$.

Proof.
\[
EM_1(G) = HM_1(G) - 4M_1(G) + 4m \\
= HM_1(G) - M_1(G) - [3M_1(G) - 4m] \\
= HM_1(G) - M_1(G) - B_1(G).
\]

Theorem 2.5. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $B_2(G) = EM_1(G) + 2M_1(G) - 4m$.

Proof.
\[
EM_1(G) = HM_1(G) - 4M_1(G) + 4m \\
= B_2(G) - 2M_1(G) + 4m.
\]

Corollary 2.1. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $B_1(G) + B_2(G) = HM_1(G) + M_1(G) - 4m$.

Theorem 2.6. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $HB_1(G) = 2HM_1(G) - 4M_1(G) + 24m$.

Proof.
\[
HB_1(G) = \sum_{uv} [d_G(u) + d_G(v)]^2 \\
= \sum_{uv \in E(G)} [d_G(u) + d_G(uv)]^2 + \sum_{uv \in E(G)} [d_G(v) + d_G(uv)]^2 \\
= \sum_{uv \in E(G)} [d_G(u) + d_G(u) + d_G(v) - 2]^2 \\
+ \sum_{uv \in E(G)} [d_G(v) + d_G(u) + d_G(v) - 2]^2 \\
= \sum_{uv \in E(G)} [2(d_G(u) + d_G(v))^2 - 4(d_G(u) + d_G(v)) + 24].
\]

Theorem 2.6 follows now from the definitions of the hyper–Zagreb and first Zagreb indices, and the fact that $E(G)$ has $m$ elements.

Corollary 2.2. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $B_2(G) = \frac{1}{2}HB_1(G) - 12m$.

Proof.
\[
HB_1(G) = 2[HM_1(G) - 2M_1(G)] + 24m = 2B_2(G) + 24m.
\]
Corollary 2.3. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then

$$B_1(G) = \frac{1}{2} HB_1(G) - EM_1(G) + M_1(G) - 12m.$$ 

Proof.

$$HB_1(G) = 2[HM_1(G) - M_1(G)] - 2M_1(G) + 24m$$
$$= 2[B_1(G) + EM_1(G)] - 2M_1(G) + 24m$$
$$= 2B_1(G) + 2EM_1(G) - 2M_1(G) + 24m.$$ 

\[\square\]

Theorem 2.7. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then

$$HB_1(G) = 5F(G) + 8M_2(G) - 12M_1(G) + 8m.$$ 

Proof.

$$HB_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2$$
$$= \sum_{uv \in E(G)} [d_G(u) + d_G(uv)]^2 + \sum_{uv \in E(G)} [d_G(v) + d_G(uv)]^2$$
$$= \sum_{uv \in E(G)} [d_G(u) + d_G(u) + d_G(v) - 2]^2$$
$$+ \sum_{uv \in E(G)} [d_G(v) + d_G(u) + d_G(v) - 2]^2$$
$$= \sum_{uv \in E(G)} [5[d_G(u)]^2 + d_G(v)] + 8d_G(u)d_G(v)$$
$$- 12[d_G(u) + d_G(v)] + 8$$
$$= 5F(G) + 8M_2(G) - 12M_1(G) + 8m.$$ 

\[\square\]

In order to prove our next result, we use the earlier established:

Theorem 2.8. [19] Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then

$$EM_1(G) = F(G) + 2M_2(G) - 4M_1(G) + 4m.$$ 

Corollary 2.4. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $B_1(G) = F(G) + 2M_2(G) - M_1(G) - EM_1(G).$

Proof. From Theorem 2.8, we have

$$EM_1(G) = F(G) + 2M_2(G) - M_1(G) - (3M_1(G) - 4m)$$
$$= F(G) + 2M_2(G) - M_1(G) - B_1(G).$$ 

\[\square\]
Corollary 2.5. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then $B_2(G) = F(G) + 2M_2(G) - 2M_1(G)$.

Proof. From Theorem 2.5, we have

$$B_2(G) = EM_1(G) + 2M_1(G) - 4m$$
$$= F(G) + 2M_2(G) - 4M_1(G) + 4m + 2M_1(G) - 4m$$
$$= F(G) + 2M_2(G) - 2M_1(G).$$

3. Bounds on Banhatti and Zagreb-type indices

Theorem 3.1. For any graph $G$,

$$M_1(G) \leq B_1(G).$$

Equality is attained if and only if $G$ is totally disconnected or $G \cong mK_2$.

Proof. Let $G$ be a simple graph with $n$ vertices and $m$ edges. Then by Theorem 2.1, we have $B_1(G) = 3M_1(G) - 4m$. Clearly $M_1(G) \leq B_1(G)$ follows. Now we prove the second part.

The graph $G$ satisfied the given condition

$$\Leftrightarrow B_1(G) = M_1(G)$$
$$\Leftrightarrow 3M_1(G) - 4m = M_1(G)$$
$$\Leftrightarrow M_1(G) = 2m.$$

Since $\sum d_G(u)^2 = 2m = \sum d_G(u)$, and $\sum(d_G(u)^2 - d_G(u)) = 0$, because $d_G(u)^2 - d_G(u) \geq 0$.

$$\Leftrightarrow d_G(u)^2 = d_G(u)$$
$$\Leftrightarrow d_G(u) = 0 \text{ or } d_G(u) = 1.$$

Thus the result follows.

Here, we use the following existing results of the Zagreb and K Banhatti indices of regular graph.

Theorem 3.2. [15] Let $G$ be an $r$-regular graph. Then

$$M_1(G) = nr^2 \quad \text{and} \quad M_2(G) = \frac{1}{2}nr^3.$$

Theorem 3.3. [12] Let $G$ be an $r$-regular graph. Then

$$B_1(G) = nr(3r - 2) \quad \text{and} \quad B_2(G) = 2nr^2(r - 1).$$

Theorem 3.4. For any connected graph $G$,

$$B_2(G) \geq 4M_2(G) - 2M_1(G).$$

Equality is attained if and only if $G$ is a regular graph.
Proof.

\[ B_2(G) = \sum_{ue} d_G(u) d_G(e) \]

\[ = \sum_{uv \in E(G)} d_G(u) [d_G(u) + d_G(v) - 2] \]

\[ + \sum_{uv \in E(G)} d_G(v) [(d_G(u) + d_G(v) - 2] \]

\[ = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2 + 2d_G(u)d_G(v)] - 2M_1(G) \]

\[ \geq \sum_{uv \in E(G)} 4d_G(u)d_G(v) - 2M_1(G). \]

Since

\[ d_G(u)^2 + d_G(v)^2 \geq 2d_G(u)d_G(v) \]

and

\[ \sum_{uv \in E(G)} d_G(u)^2 + d_G(v)^2 \geq \sum_{uv \in E(G)} 2d_G(u)d_G(v), \]

the result follows.

The equality case attains directly from Theorems 2.1, 2.2, 3.2, and 3.3. \( \square \)

Now, we use the following existing results to prove our next result.

**Theorem 3.5.** [19] Let \( G \) be a simple graph with \( n \geq 3 \) vertices and \( m \) edges. Then

\[ M_1(G) \geq \frac{4m^2}{n} \quad \text{and} \quad M_2(G) \geq \frac{4m^3}{n^2}. \]

**Theorem 3.6.** For any connected graph \( G \) with \( n \geq 3 \) vertices and \( m \) edges,

\[ B_2(G) \geq \frac{8m^2(2m - n)}{n^2}. \]

Further, equality is attained if and only if \( G \) is a regular graph.

**Proof.** From Theorems 3.3-3.5, the desired result follows. \( \square \)

**Theorem 3.7.** For any connected graph \( G \) with \( n \geq 3 \) vertices and \( m \) edges,

\[ \frac{4m(3m - n)}{n} \leq B_1(G) \leq 3m^2 - m. \]

The lower bound becomes equality if and only if \( G \) is regular. Equality in the upper bound is attained if and only if \( G \cong K_{1,n-1} \) or \( G \cong K_3 \).

**Proof.** From Theorems 2.1 and 3.5, bearing in mind that of \( M_1(G) \leq m(m + 1) \), the lower and upper bounds on \( B_1(G) \) follow.

The second part is obvious. \( \square \)

We now obtain lower and upper bounds on \( B_1(G) \) in terms of the minimum degree \( \delta(G) \) and the maximum degree \( \Delta(G) \) of \( G \).
Theorem 3.8. For any graph $G$ with $n \geq 3$ vertices and $m$ edges,
$$2m \left[ 3\delta(G) - 2 \right] \leq B_1(G) \leq 2m \left[ 3\Delta(G) - 2 \right].$$
Further, equality in both lower and upper bounds is attained if and only if $G$ is regular.

Proof. Let $G$ be a graph with $n \geq 3$ vertices and $m$ edges. Then
$$B_1(G) = \sum_{e \in E(G)} [d_G(u) + d_G(v)]$$
$$= \sum_{e \in E(G)} [d_G(u) + (d_G(u) + d_G(v) - 2)]$$
$$+ \sum_{e \in E(G)} [d_G(v) + (d_G(u) + d_G(v) - 2)]$$
$$= \sum_{e \in E(G)} 3(d_G(u) + d_G(v)) - 4m.$$
But $2\delta(G) \leq d_G(u) + d_G(v) \leq 2\Delta(G)$. Bearing this in mind,
$$6\delta(G) \leq 3[d_G(u) + d_G(v)] \leq 6\Delta(G)$$
$$6\delta(G) - 4 \leq 3[d_G(u) + d_G(v)] - 4 \leq 6\Delta(G) - 4$$
$$2m \left[ 3\delta(G) - 2 \right] \leq B_1(G) \leq 2m \left[ 3\Delta(G) - 2 \right].$$
Further, equality in both lower and upper bounds holds if and only if $d_G(u) + d_G(v) = 2\delta(G) = 2\Delta(G)$, for each $uv \in E(G)$, which implies that $G$ is a regular graph.

The following two existing results of hyper-Zagreb index to prove our next two results in terms of $\delta(G)$ and $\Delta(G)$ of $G$.

Theorem 3.9. [4] For any simple graph $G$ with $n \geq 3$ vertices and $m$ edges,
$$HM_1(G) \leq \frac{[\delta(G) + \Delta(G)]^2}{4m\delta(G)\Delta(G)} M_1(G)^2.$$

Theorem 3.10. [4] For any graph $G$ with $n \geq 3$ vertices and $m$ edges,
$$\delta(G)M_1(G) + 2M_2(G) \leq HM_1(G) \leq \Delta(G)M_1(G) + 2M_2(G),$$
with equality if and only if $G$ is a regular graph.

Theorem 3.11. For any connected graph $G$ with $n \geq 3$ vertices and $m$ edges,
$$B_2(G) \leq \frac{[\delta(G) + \Delta(G)]^2}{4m\delta(G)\Delta(G)} M_1(G)^2 - 2M_1(G).$$

Proof. From Theorem 3.9, we have
$$HM_1(G) - 2M_1(G) \leq \frac{[\delta(G) + \Delta(G)]^2}{4m\delta(G)\Delta(G)} M_1(G)^2 - 2M_1(G).$$
whereas from Theorem 2.2,

\[ B_2(G) \leq \frac{[\delta(G) + \Delta(G)]^2}{4m\delta(G)\Delta(G)} M_1(G)^2 - 2M_1(G). \]

\[ \square \]

**Theorem 3.12.** For any connected graph \( G \) with \( n \geq 3 \) vertices,

\[ [\delta(G) - 2]M_1(G) + 2M_2(G) \leq B_2(G) \leq [\Delta(G) - 2]M_1(G) + 2M_2(G). \]

Further, equality in both lower and upper bounds hold if and only if \( G \) is regular.

**Proof.** From Theorem 3.10, we have

\[ \delta(G)M_1(G) + 2M_2(G) - 2M_1(G) \leq HM_1(G) - 2M_1(G) \leq \Delta(G)M_1(G) + 2M_2(G) - 2M_1(G). \]

Then from Theorem 2.2, we get the desired result.

Further, equality in both lower and upper bounds will hold if and only if \( d_G(u) + d_G(v) = 2\delta(G) = 2\Delta(G) \), for each \( uv \in E(G) \), which implies that \( G \) is a regular graph. \( \square \)

Now, we use the following existing results to prove our next result of \( B_1(T) \).

**Theorem 3.13.** [7] For any tree \( T \) with \( n \geq 3 \) vertices and \( m \) edges,

\[ 4n - 6 \leq M_1(T) \leq n(n-1). \]

**Theorem 3.14.** For any tree \( T \) with \( n \geq 3 \) vertices and \( m \) edges,

\[ 8n - 14 \leq B_1(T) \leq (n-1)(3n-4). \]

Further, equality in the lower bound is attained if and only if \( T \cong P_n \) and in the upper bound if and only if \( T \cong K_{1,n-1} \).

**Proof.** From Theorems 2.1 and 3.13, we have

\[ 4n - 6 \leq \frac{1}{3}[B_1(T) + 4m] \leq n(n-1) \]

\[ 12n - 18 - 4m \leq B_1(T) \leq 3n(n-1) - 4m. \]

Since for any tree \( T, m = n - 1 \), the result follows.

Further, the equality in the lower bound is attained if and only if \( T \cong P_n \) because \( B_1(P_n) = 8n - 14 \). Equality in the upper bound is attained if and only if \( T \cong K_{1,n-1} \) because \( B_1(K_{1,n-1}) = (n-1)(3n-4) \). \( \square \)

In order to prove our next result (upper bound) of \( B_1(G) \) via \( M_1(G) \), we apply of the Biernacki–Pidek–Ryll–Nardzewski inequality [1].
Theorem 3.15. [1] Let \( a \) and \( b \) be \( n \)-tuples such that \( x \leq a_i \leq X \) and \( y \leq b_i \leq Y \) for \( i = 1, 2, \ldots, n \). Then
\[
\left[ \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \right] \leq \frac{1}{4} (X - x)(Y - y),
\]
with \( \lfloor \cdot \rfloor \) being the greatest integer function. Equality occurs when \( n \) is even.

Theorem 3.16. For any connected graph \( G \) with \( n \geq 3 \) vertices and \( m \) edges,
\[
B_1(G) \leq \frac{3n}{4} [\Delta(G) - \delta(G)]^2 + \frac{4m}{n} (3m - n).
\]

Proof. Let \( a_i = b_i = d_G(u_i) \) for \( i = 1, 2, \ldots, n \) with \( x = \delta(G) = y \) and \( X = \Delta(G) = Y \). Then
\[
\left[ \frac{1}{n} \sum_{i=1}^{n} d_G(u_i)^2 - \frac{1}{n^2} \left( \sum_{i=1}^{n} d_G(u_i) \right)^2 \right] \leq \frac{1}{4} [\Delta(G) - \delta(G)]^2
\]
\[
\left[ \frac{1}{n} M_1(G) - \frac{1}{n^2} (2m)^2 \right] \leq \frac{1}{4} [\Delta(G) - \delta(G)]^2
\]
\[
\frac{1}{n} M_1(G) - \frac{4m^2}{n^2} \leq \frac{1}{4} [\Delta(G) - \delta(G)]^2.
\]
Since
\[
M_1(G) \geq \frac{4m^2}{n} \Rightarrow \frac{1}{n} M_1(G) \geq \frac{4m^2}{n^2},
\]
we have
\[
M_1(G) - \frac{4m^2}{n} \leq \frac{n}{4} [\Delta(G) - \delta(G)]^2
\]
\[
\frac{1}{3} \left( B_1(G) + 4m \right) - \frac{4m^2}{n} \leq \frac{n}{4} [\Delta(G) - \delta(G)]^2
\]
\[
B_1(G) + 4m - \frac{12m^2}{n} \leq \frac{3n}{4} [\Delta(G) - \delta(G)]^2.
\]
Hence the upper bound follows. \( \square \)

In order to prove our next result (lower bound) of \( B_1(G) \) in terms of the minimum degree \( \delta(G) \), the maximum degree \( \Delta(G) \) and the forgotten topological index \( F(G) \), we use of the well known Cassel’s inequality [18].

Theorem 3.17. [18] Let \( (a_1, a_2, \ldots, a_n) \) and \( (b_1, b_2, \ldots, b_n) \) be positive real numbers, satisfying the condition \( 0 < \ell \leq \frac{x}{4L} \leq L < \infty \) for each \( k \in \{1, 2, \ldots, n\} \), where \( \ell \) and \( L \) are some constants. Let \( (w_1, w_2, \ldots, w_n) \) be positive weights. Then
\[
\left( \sum_{i=1}^{n} w_i a_i^2 \right) \left( \sum_{i=1}^{n} w_i b_i^2 \right) \leq \frac{(L + \ell)^2}{4L \ell} \left( \sum_{i=1}^{n} w_k a_i \right)^2.
\]
Theorem 3.18. For any connected graph \( G \) with \( n \geq 3 \) vertices and \( m \) edges,
\[
B_1(G) \geq \frac{2m\delta(G)\Delta(G)}{2\delta(G)\Delta(G)} F(G) - 4m.
\]

Proof. Let \( a_i = d_G(u_i)^{3/2} \) and \( b_i = d_G(u_i)^{1/2} \) with \( \ell = \delta(G) \), \( L = \Delta(G) \) and \( w_1 = 1 \) for all \( 1 \leq i \leq n \). By Theorem 3.17 (Cassel’s inequality),
\[
\sum_{i=1}^{n} d_G(u_i)^3 \sum_{i=1}^{n} d_G(u_i) \leq \frac{(\delta(G) + \Delta(G))^2}{8\delta(G)\Delta(G)} d_G(u_i)^2 \]
\[
F(G) 2m \leq \frac{(\delta(G) + \Delta(G))^2}{2\delta(G)\Delta(G)} M_1(G) \]
\[
F(G) \leq \left( \frac{(\delta(G) + \Delta(G))^2}{8m\delta(G)\Delta(G)} \right) \frac{1}{3} [B_1(G) + 4m].
\]
Thus the result follows.

Now, we obtain lower and upper bounds on \( EM_1(G), B_1(G) \), and \( B_2(G) \) in terms of \( \delta(G), \Delta(G) \), and \( M_1(G) \), using Abel’s inequality as follows.

Theorem 3.19. [3] Let \( \{a_1, a_2, \ldots, a_n\} \) and \( \{b_1, b_2, \ldots, b_n\} \) with
\[
b_1 \geq b_2 \geq \cdots \geq b_n \geq 0
\]
be two sequences of real numbers and \( S_k = a_1 + a_2 + \cdots + a_k \) for \( k = 1, 2, \ldots, n \). If
\[
\omega = \min_{1 \leq k \leq n} S_k \quad \text{and} \quad \Omega = \max_{1 \leq k \leq n} S_k,
\]
then
\[
\omega b_1 \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq \Omega b_1.
\]

In order to prove our next result we make use of the following definition:
The line graph \( L(G) \) of the graph \( G \) is the graph whose vertices correspond to the edges of \( G \) and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) are adjacent (that is, are incident with a common vertex).

Theorem 3.20. For any connected graph \( G \) with \( n \geq 3 \) vertices and \( m \) edges,
\[
4(\delta(G) - 1)^2 \leq EM_1(G) \leq 2[M_1(G) - 2m](\Delta(G) - 1)
\]
\[
HM_1(G) - M_1(G)(2\Delta(G) - 1) + 4m(\Delta(G) - 1) \leq B_1(G) \leq HM_1(G) - M_1(G) - 4(\delta(G) - 1)^2
\]
\[
4(\delta(G) - 1)^2 + 2M_1(G) - 4m \leq B_2(G) \leq [2M_1(G) - 4m] \Delta(G).
\]

Proof. Inequality (3.1): Let \( a_i = d_G(e_i) \) with \( e_i = u_iv_i \) for \( i \neq j \) and \( b_1 \geq b_2 \geq \cdots \geq b_n \geq 0 \). Clearly, \( b_1 = \max d_G(e_i) \) and \( 2\delta(G) - 2 \leq b_1 \leq 2\Delta(G) - 2 \), where \( S_k = a_1 + a_2 + \cdots + a_k \) for \( k = 1, 2, \ldots, n \).
Therefore $\omega = \min_{1 \leq k \leq n} S_k = \min_{1 \leq i \leq n} d_G(e_i) \Rightarrow \omega \geq 2(\delta(G) - 1)$ and

$$\Omega = \max_{1 \leq k \leq n} S_k = \max_{1 \leq i \leq n} d_G(e_i) = S_n$$

$$= \sum_{i=1}^{n} d_G(e_i) = 2|E(L(G))| = 2 \left[ \frac{1}{2} \sum_{i=1}^{n} d_G(u_i)^2 - m \right]$$

$$= 2 \left[ \frac{1}{2} M_1(G) - m \right] = M_1(G) - 2m.$$

By Theorem 3.19 (Abel’s inequality), we get

$$\omega b_1 \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq \Omega b_1$$

$$(2\delta(G) - 2)b_1 \leq a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq (2\Delta(G) - 2)b_1$$

$$4(\delta(G) - 1)^2 \leq \sum_{i=1}^{n} d_G(e_i)^2 \leq [M_1(G) - 2m](2\Delta(G) - 2)$$

$$4(\delta(G) - 1)^2 \leq E M_1(G) \leq 2[M_1(G) - 2m](\Delta(G) - 1).$$

Inequality (3.2): From (3.1) and Theorem 2.4, we get

$$HM_1(G) - M_1(G)(2\Delta(G) - 1) + 4m(\Delta(G) - 1) \leq B_1(G) \leq$$

$$HM_1(G) - M_1(G) - 4(\delta(G) - 1)^2.$$

Inequality (3.3): From (3.1) and Theorem 2.5, we get

$$4(\delta(G) - 1)^2 + 2M_1(G) - 4m \leq B_2(G) \leq$$

$$(2M_1(G) - 4m)\Delta(G).$$

Finally, we obtain the lower and upper bounds on $B_1(G)$ and $B_2(G)$ in terms of the number of pendent vertices and minimal non-pendent vertices of $G$.

**Theorem 3.21.** For any $(n, m)$-graph $G$ with $\eta$ pendent vertices and minimal non-pendent vertex degree $\delta_1(G)$,

$$6\delta_1(G)(m - \eta) + 3\eta(1 + \delta_1(G)) - 4m \leq B_1(G) \leq$$

$$6\Delta(G)(m - \eta) + 3\eta(1 + \Delta(G)) - 4m$$

(3.4)

$$4\delta_1(G)(\delta_1(G) - 1)(m - \eta) + (\delta_1(G)^2 - 1)\eta \leq B_2(G) \leq$$

$$4\Delta(G)(\Delta(G) - 1)(m - \eta) + (\Delta(G)^2 - 1)\eta.$$
Proof. Inequality (3.4):

\[
B_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(e)]
\]

\[
= \sum_{uv \in E(G)} [d_G(u) + (d_G(u) + d_G(v) - 2)] + \sum_{uv \in E(G)} [d_G(v) + (d_G(u) + d_G(v) - 2)]
\]

\[
= \sum_{uv \in E(G)} 3[d_G(u) + d_G(v)] - 4
\]

\[
= \sum_{uv \in E(G); d_G(u), d_G(v) \neq 1} 3[d_G(u) + d_G(v)] + \sum_{uv \in E(G); d_G(u) = 1} 3[1 + d_G(v)] - \sum_{uv \in E(G)} 4
\]

\[
\leq 6 \Delta(G)(m - \eta) + 3 \eta (1 + \Delta(G)) - 4m.
\]

Thus the upper bound follows.

Similarly,

\[
B_1(G) \geq \sum_{uv \in E(G); d_G(u), d_G(v) \neq 1} 6 \delta_1(G) + \sum_{uv \in E(G); d_G(u) = 1} 3 \eta (1 + \delta_1(G)) - \sum_{uv \in E(G)} 4
\]

\[
= 6 \delta_1(G)(m - \eta) + 3 \eta (1 + \delta_1(G)) - 4m.
\]

Hence the lower bound follows.

Inequality (3.5):

\[
B_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(e)
\]

\[
= \sum_{uv \in E(G)} d_G(u) [d_G(u) + d_G(v) - 2]
\]

\[
= \sum_{uv \in E(G); d_G(u), d_G(v) \neq 1} d_G(u) [d_G(u) + d_G(v) - 2] + \sum_{uv \in E(G); d_G(u) = 1} d_G(v) [d_G(u) + d_G(v) - 2]
\]

\[
+ \sum_{uv \in E(G); d_G(u) = 1} 1[d_G(v) - 1] + \sum_{uv \in E(G); d_G(u) = 1} d_G(v) [d_G(v) - 1]
\]

\[
\leq \sum_{uv \in E(G); d_G(u), d_G(v) \neq 1} [\Delta(G)(2\Delta(G) - 2) + \Delta(G)(2\Delta(G) - 2)] + \sum_{uv \in E(G); d_G(u) = 1} [\Delta(G) - 1] + \sum_{uv \in E(G); d_G(u) = 1} [\Delta(G) - 1].
\]
Thus the upper bound follows.

Similarly,

\[ B_2(G) \geq \sum_{uv \in E(G): d_G(u), d_G(v) \neq 1} 2\delta_1(G) \left[ 2\delta_1(G) - 2 \right] \]

\[ + \sum_{uv \in E(G): d_G(u) = 1} \left[ \delta_1(G) - 1 \right] + \sum_{uv \in E(G): d_G(u) = 1} \delta_1(G) \left[ \delta_1(G) - 1 \right] \]

\[ = 6\delta_1(G)(m - \eta) + 3\eta(1 + \delta_1(G)) - 4m. \]

Hence the lower bound follows.

**Remark 3.1.** In the inequalities (3.4) and (3.5), equality is attained if and only if \( d_G(u) = d_G(v) = \Delta(G) = \delta_1(G) \) for each \( uv \in E(G) \) with \( d_G(u), d_G(v) \neq 1 \) and \( d_G(v) = \Delta(G) = \delta_1(G) \) for each \( uv \in E(G) \) with \( d_G(u) = 1 \).

**References**


Received by editors 01.02.2017; Available online 06.02.2017.

**I. Gutman**: Faculty of Science, University of Kragujevac, Kragujevac, Serbia  
E-mail address: gutman@kg.ac.rs

**V. R. Kulli**: Department of Mathematics, Gulbarga University, Gulbarga – 585 106, India  
E-mail address: vrkulli@gmail.com

**B. Chaluvaraju**: Department of Mathematics Bangalore University, Jnana Bharathi Campus, Bangalore – 560 056, India  
E-mail address: bchaluvvaraju@gmail.com

**H. S. Boregowda**: Department of Studies and Research in Mathematics, Tumkur University, University Constituent College Campus, Tumkur – 571 103, India  
E-mail address: bgamarasa@gmail.com