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MULTI-VALUED FIXED POINTS RESULTS VIA RATIONAL TYPE CONTRACTIVE CONDITIONS IN COMPLEX VALUED METRIC SPACES

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Abstract. We establish some common fixed point theorems for multi-valued mappings in complex valued metric spaces satisfying rational type contractive conditions. Our results unify, extend and improve many recent results of the literature.

1. Introduction

Banach contraction principle [**4**] plays a vital role in the develovepment of metric fixed point theory. In 1969, Nadler [**14**] introduced the concept of multi-valued contraction mappings by combining the ideas of set-valued mapping and Lipchitz mapping and obtained some fixed point results. Several researchers studied, extended and generalized the result in [**4**] in many directions, see [**2, 6, 7, 12, 15, 16, 19, 20**].

On the other hand, many authors have extended the idea of metric space to many of it's generalizations and obtained some interesting extensions of Banach contraction principle [**4**]. For example, see [**5, 8, 13, 18**].

The study and extension of fixed point results of rational type contractive conditions is worrisome in cone metric spaces, this motivated Azam *et al.* [**3**] to introduced a generalized metric space called complex valued metric space, and obtained a sufficient conditions for the existence of common fixed points for a pair of mappings satisfying contractive conditions. In 2012, Rouzkard and Imdad [**17**] presented some common fixed point theorems satisfying rational type expressions which generalized the results of Azam *et al.* [**3**]. Afterwards, Sintunavarat and

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Kumam [**21**] established common fixed point theorems which are more general than the results in [**3**] by replacing the constant of contraction with some control functions and also gave some results for weakly compatible mappings in complex valued metric space. Sitthikul and Saejung [**22**] continue the study of fixed point theorems in this direction and obtained results which generalized the results proved in [**17, 21**]. Later in 2014, Kumar and Hussain [**11**] generalized the results in [**22**].

Recently, Kumam *et al.* [**10**] investigated some fixed point results by using the idea of contractive conditions of control functions in the context of complex valued metric spaces, their results generalized the results of [**11, 17, 21, 22**]. On the other hand, the notion of multi-valued contractive mappings in complex valued metric space was introduced by Ahmad *et al.* [**1**], in which they improved and extended the results of [**3**].

In this paper, we adopt the method in [**1**] to extend and improve the results of Kumam *et al.* [**10**] to multi-valued mappings in complex valued metric spaces without using the notion of continuity.

2. Preliminaries

To begin with, we recall some basic definitions and results which will be useful in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$, we define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } \Re(z_1) \leqslant \Re(z_2), \Im(z_1) \leqslant \Im(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

(i) $\Re(z_1) = \Re(z_2), \Im(z_1) < \Im(z_2)$;
(ii) $\Re(z_1) < \Re(z_2), \Im(z_1) = \Im(z_2)$;
(iii) $\Re(z_1) < \Re(z_2), \Im(z_1) < \Im(z_2)$;
(iv) $\Re(z_1) = \Re(z_2), \Im(z_1) = \Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$, if $z_1 \neq z_2$ and one of (i),(ii),(iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied.

Definition 2.1. [**3**]. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \longrightarrow \mathbb{C}$ satisfies the following conditions:*

D_1. $0 \precsim d(z, w)$ *for all* $z, w \in X$;
D_2. $d(z, w) = 0 \Longleftrightarrow z = w$ *for all* $z, w \in X$;
D_3. $d(z, w) = d(w, z)$ *for all* $z, w \in X$;
D_4. $d(z, w) \precsim d(z, u) + d(u, w)$ *for all* $z, u, w \in X$.

Then, d is called a complex valued metric on $\mathbb{C}$, and the pair (X, d) is called a complex valued metric space.

Example 2.1. Let $X = [0, \infty)$ and $x, y \in X$. Define $d : X \times X \longrightarrow \mathbb{C}$ by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{i}{2}, & \text{if } x \neq y. \end{cases}$$

Then, d is a complex valued metric and hence (X,d) is a complex valued metric space.

Definition 2.2. *Let (X,d) be a complex valued metric space.*

i. A point $x \in X$ is called an interior point of a set $A \subset X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x,r) = \{y : d(x,y) \prec r\} \subset A$;

ii. A point $x \in X$ is called a limit point of a set $A \subset X$ whenever for every $0 \prec r \in \mathbb{C}$, $B(x,r) \bigcap (A \backslash \{x\}) \neq \phi$;

iii. A subset $A \subset X$ is called open whenever each element of A is an interior point of A;

iv. A subset $B \subset X$ is called closed whenever each limit point of B belongs to B;

v. The family $F = \{B(x,r) : x \in X, 0 \prec r\}$ is a sub-basis for a Hausdorff topology $\tau_{\mathbb{C}}$ on X.

Definition 2.3. [**3**]. *Let $\{z_n\}$ be a sequence in a complex valued metric space X and $z \in X$.*

i. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(z_n, z) \prec c$ for all $n > n_0$, then $\{z_n\}$ is said to be convergent, $\{z_n\}$ converges to z and z is called the limit point of $\{z_n\}$. We denote this by $\lim_{n\to\infty} z_n = z$ or $z_n \to z$ as $n \to \infty$;

ii. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(z_n, z_{n+m}) \prec c$ for all $n > n_0$, then $\{z_n\}$ is called a Cauchy sequence in (X,d);

iii. If every Cauchy sequence in (X,d) is convergent in (X,d), then (X,d) is called a complete complex valued metric space.

Lemma 2.1. [**3**]. *Let (X,d) be a complex valued metric space and $\{z_n\}$ be a sequence in (X,d). Then, $\{z_n\}$ converges to z if and only if $|d(z_n,z)| \to 0$ as $n \to \infty$.*

Lemma 2.2. [**3**]. *Let (X,d) be a complex valued metric space and $\{z_n\}$ be a sequence in (X,d). Then, $\{z_n\}$ is a Cauchy sequence if and only if $|d(z_n, z_{n+m})| \to 0$ as $n \to \infty$.*

Remark 2.1. *For every $z_1, z_2, z_3 \in \mathbb{C}$ and $a, b \in [0,\infty)$. If*

i. $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$;

ii. $z_1 \precsim z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$;

iii. $a \leqslant b \Longrightarrow az_1 \precsim bz_1$ for all $z_1 \in \mathbb{C}$.

Let (X,d) be a complex valued metric space. Let denote the nonempty(resp. bounded, closed and bounded) subset of X by $N(X)(resp.B(X), CB(X))$.

For $a \in X$, $z \in \mathbb{C}$ and $B \in N(X)$, define $s(z) = \{w \in \mathbb{C} : z \preceq w\}$, and $s(a,B) = \bigcup_{b\in B} s(d(a,b)) = \bigcup_{b\in B}\{w \in \mathbb{C} : d(a,b) \preceq w\}$.

For $A, B \in B(X)$, denote

$$s(A,B) = \Big(\bigcap_{a\in A} s(a,B)\Big) \bigcap \Big(\bigcap_{b\in B} s(b,A)\Big).$$

Remark 2.2. *Let (X, d) be a complex valued metric space. If $\mathbb{C} = \mathbb{R}$, then (X, d) is a metric space. Moreover, for $A, B \in CB(X), H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by complex valued metric d.*

Let (X, d) be a complex valued metric space, and $T : X \longrightarrow CB(X)$ be a multi-valued mapping. For $x \in X$ and $A \in CB(X)$, define $W_x(A) = \{d(x, a) : a \in A\}$.

Thus, for $x, y \in X$ we define, $W_x(Ty) = \{d(x, a) : a \in Ty\}$.

Definition 2.4. *Let (X, d) be a complex valued metric space. A nonempty set $A \subset X$ is called bounded from below if there exists some $z \in \mathbb{C}$, such that $z \preceq a$ for all $a \in A$.*

Definition 2.5. *Let (X, d) be a complex valued metric space. A multi-valued mapping $K : X \longrightarrow 2^{\mathbb{C}}$ is called bounded from below if for $x \in X$ there exists $z_x \in \mathbb{C}$, such that $z_x \preceq a$ for all $a \in Tx$.*

Definition 2.6. *Let (X, d) be a complex valued metric space. A multi-valued mapping $T : X \longrightarrow CB(X)$ is said to have lower bounded property (l.b property) on (X, d) if for any $x \in X$ the multi-valued mapping $K_x : X \longrightarrow 2^{\mathbb{C}}$ defined by $K_x(Ty) = W_x(Ty)$ is bounded from below. That is, for $x, y \in X$ there exists an element $l_x(Ty) \in \mathbb{C}$ such that $l_x(Ty) \preceq a$ for all $a \in W_x(Ty)$, where $l_x(Ty)$ is called lower bound of T associated with (x, y).*

Definition 2.7. *Let (X, d) be a complex valued metric space. A multi-valued mapping $T : X \longrightarrow CB(X)$ is said to have greatest lower bounded property (g.l.b property) on (X, d) if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{C}$ for all $x, y \in X$. We denote the g.l.b of $W_x(Ty)$ by $d(x, Ty)$ and define it as; $d(x, Ty) = \inf\{d(x, a) : a \in Ty\}$.*

Definition 2.8. *Let (X, d) be a complex valued metric space and $S, T : X \longrightarrow CB(X)$. Then*

i. A point $z \in X$ is said to be a fixed point of T if $z \in Tz$;
ii. A point $z \in X$ is said to be a common fixed point of S and T if $z \in Sz$ and $z \in Tz$.

3. Main Results

In this section we prove our main results, before that we give the following well known results which are essential.

Proposition 3.1. [**22**]. *Let (X, d) be a complete complex valued metric space and $K, L : X \longrightarrow X$. Let $u_0 \in X$ and define the sequence $\{u_n\}$ by*

$$u_{2n+1} = Ku_{2n} \text{ and } u_{2n+2} = Lu_{2n+1} \text{ for all } n = 0, 1, 2, \ldots.$$

Assume that there exists a mapping $\lambda : X \times X \longrightarrow [0, 1)$ satisfying

$$\lambda(LKu, v) \leqslant \lambda(u, v) \text{ and } \lambda(u, KLv) \leqslant \lambda(u, v), \text{ for all } u, v \in X.$$

Then,

$$\lambda(u_{2n}, v) \leqslant \lambda(u_0, v) \text{ and } \lambda(u, u_{2n+1}) \leqslant \lambda(u, u_1),$$

for all $u, v \in X$ and $n = 0, 1, 2, \ldots$.

The following lemma can be derived easily from Proposition 3.1 above.

LEMMA 3.1. *Let (X, d) be a complete complex valued metric space and $L : X \longrightarrow X$. Let $z_0 \in X$ and define the sequence $\{z_n\}$ by*

$$z_{n+1} = Lz_n, \text{ for all } n = 0, 1, 2, \ldots.$$

Suppose that there exists a mapping $\lambda : X \times X \longrightarrow [0, 1)$ satisfying

$$\lambda(Lz, z^*) \leqslant \lambda(z, z^*) \text{ and } \lambda(z, Lz^*) \leqslant \lambda(z, z^*), \text{ for all } z, z^* \in X.$$

Then,

$$\lambda(z_n, z^*) \leqslant \lambda(z_0, z^*) \text{ and } \lambda(z, z_{n+1}) \leqslant \lambda(z, z_1),$$

for all $z, z^* \in X$ and $n = 0, 1, 2, \ldots$.

LEMMA 3.2. [**10**]. *Let (X, d) be a complete complex valued metric space and $K, L : X \longrightarrow X$. Let $w_0 \in X$ and define the sequence $\{w_n\}$ by*

$$w_{2n+1} = Kw_{2n} \text{ and } w_{2n+2} = Lw_{2n+1} \text{ for all } n = 0, 1, 2, \ldots.$$

Assume that there exists a mapping $\lambda : X \longrightarrow [0, 1)$ satisfying

$$\lambda(LKw) \leqslant \lambda(w) \text{ and } \lambda(KLw) \leqslant \lambda(w), \text{ for all } w \in X.$$

Then,

$$\lambda(w_{2n}) \leqslant \lambda(w_0) \text{ and } \lambda(w_{2n+1}) \leqslant \lambda(w_1),$$

for all $w \in X$ and $n = 0, 1, 2, \ldots$.

THEOREM 3.1. *Let (X, d) be a complete complex valued metric space, $\lambda_i : X \times X \longrightarrow [0, 1)$ for $i = 1, 2, 3, ..., 8$ and $T : X \longrightarrow CB(X)$ be a multi-valued mapping with g.l.b property such that for all $x, y \in X$ the following conditions holds:*

i. $\lambda_i(Tx, y) \leqslant \lambda_i(x, y)$ *and* $\lambda_i(x, Ty) \leqslant \lambda_i(x, y)$;

ii.

$$\begin{aligned}
&\lambda_1(x, y)d(x, y) + \lambda_2(x, y)d(x, Ty) + \lambda_3(x, y)d(y, Tx) + \lambda_4(x, y)d(x, Tx) \\
&\quad + \lambda_5(x, y)d(y, Ty) + \lambda_6(x, y)\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \\
&\quad + \lambda_7(x, y)\frac{d(x, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \\
&\quad + \lambda_8(x, y)\frac{d(x, Ty)[1 + d(y, Tx)]}{1 + d(x, y) + d(y, Tx)} \in s(Tx, Ty),
\end{aligned} \tag{3.1}$$

where

$$\lambda_1(x, y) + \lambda_3(x, y) + \lambda_4(x, y) + \lambda_5(x, y) + \lambda_6(x, y) + 2[\lambda_2(x, y) + \lambda_7(x, y) + \lambda_8(x, y)] < 1.$$

Then, T has a unique fixed point.

Proof. (Existence:) Let $x_0 \in X$ (arbitrary) and $x_1 \in Tx_0$. From (3.1), we get

$$\begin{aligned}\lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)} \in s(Tx_0,Tx_1).\end{aligned}$$

Which implies that

$$\begin{aligned}\lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)} \in \left(\bigcap_{x\in Tx_0} s(x,Tx_1)\right).\end{aligned}$$

$$\begin{aligned}\lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)} \in s(x,Tx_1), \forall x \in Tx_0.\end{aligned}$$

Since $x_1 \in Tx_0$, we get

$$\begin{aligned}\lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)} \in s(x_1,Tx_1).\end{aligned}$$

$$\begin{aligned}\lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)} \in \Bigg(\bigcup_{y\in Tx_1} s(d(x_1,y))\Bigg).\end{aligned}$$

Thus, there exists some $x_2 \in Tx_1$ such that

$$\begin{aligned}\lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)} \in s(d(x_1,x_2)).\end{aligned}$$

By definition, we get

$$\begin{aligned}d(x_1,x_2) \preceq \lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,Tx_1) + \lambda_3(x_0,x_1)d(x_1,Tx_0)\\ &+ \lambda_4(x_0,x_1)d(x_0,Tx_0) + \lambda_5(x_0,x_1)d(x_1,Tx_1)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_0,Tx_0)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,Tx_1)[1+d(x_1,Tx_0)]}{1+d(x_0,x_1)+d(x_1,Tx_0)}.\end{aligned}$$

Using the g.l.b property of T, we get

$$\begin{aligned}d(x_1,x_2) \preceq \lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)d(x_0,x_2) + \lambda_3(x_0,x_1)d(x_1,x_1)\\ &+ \lambda_4(x_0,x_1)d(x_0,x_1) + \lambda_5(x_0,x_1)d(x_1,x_2)\\ &+ \lambda_6(x_0,x_1)\frac{d(x_1,x_2)[1+d(x_0,x_1)]}{1+d(x_0,x_1)}\\ &+ \lambda_7(x_0,x_1)\frac{d(x_0,x_2)[1+d(x_0,x_1)]}{1+d(x_0,x_1)}\\ &+ \lambda_8(x_0,x_1)\frac{d(x_0,x_2)[1+d(x_1,x_1)]}{1+d(x_0,x_1)+d(x_1,x_1)}.\end{aligned}$$

Which implies

$$\begin{aligned}|d(x_1,x_2)| \leqslant{}& \lambda_1(x_0,x_1)|d(x_0,x_1)| + \lambda_2(x_0,x_1)|d(x_0,x_2)| + \lambda_4(x_0,x_1)|d(x_0,x_1)| \\ &+ \lambda_5(x_0,x_1)|d(x_1,x_2)| + \lambda_6(x_0,x_1)|d(x_1,x_2)| + \lambda_7(x_0,x_1)|d(x_0,x_2)| \\ &+ \lambda_8(x_0,x_1)|d(x_0,x_2)|.\end{aligned}$$

$$\begin{aligned}|d(x_1,x_2)| \leqslant{}& \lambda_1(x_0,x_1)|d(x_0,x_1)| + \lambda_2(x_0,x_1)|d(x_0,x_1)| + \lambda_2(x_0,x_1)|d(x_1,x_2)| \\ &+ \lambda_4(x_0,x_1)|d(x_0,x_1)| + \lambda_5(x_0,x_1)|d(x_1,x_2)| + \lambda_6(x_0,x_1)|d(x_1,x_2)| \\ &+ \lambda_7(x_0,x_1)|d(x_0,x_1)| + \lambda_7(x_0,x_1)|d(x_1,x_2)| + \lambda_8(x_0,x_1)|d(x_0,x_1)| \\ &+ \lambda_8(x_0,x_1)|d(x_1,x_2)|.\end{aligned}$$

Thus,

$$|d(x_1,x_2)| \leqslant h|d(x_0,x_1)|,$$

where

$$h = \frac{\lambda_1(x_0,x_1) + \lambda_2(x_0,x_1) + \lambda_4(x_0,x_1) + \lambda_7(x_0,x_1) + \lambda_8(x_0,x_1)}{1-(\lambda_2(x_0,x_1) + \lambda_5(x_0,x_1) + \lambda_6(x_0,x_1) + \lambda_7(x_0,x_1) + \lambda_8(x_0,x_1)} < 1.$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that for $n = 0, 1, 2, \dots$ we have

$$|d(x_n,x_{n+1})| \leqslant h^n|d(x_0,x_1)|, \qquad x_{n+1} \in Tx_n.$$

So for $m > n$, we have

$$\begin{aligned}|d(x_n,x_m)| &\leqslant |d(x_n,x_{n+1})| + |d(x_{n+1},x_{n+2})| + \cdots + |d(x_{m-1},x_m)| \\ &\leqslant [h^n + h^{n+1} + \cdots + h^{m-1}]|d(x_0,x_1)| \\ &\leqslant \left[\frac{h^n}{1-h}\right]|d(x_0,x_1)|.\end{aligned}$$

Therefore,

$$|d(x_n,x_m)| \leqslant \left[\frac{h^n}{1-h}\right]|d(x_0,x_1)| \longrightarrow 0 \text{ as } n,m \longrightarrow \infty.$$

Thus $\{x_n\}$ is a Cauchy sequence in X. It follows from the completeness of X that there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.
We now show that $z \in Tz$. From (3.1), we have

$$\begin{aligned}\lambda_1(x_n,z)d(x_n,z) &+ \lambda_2(x_n,z)d(x_n,Tz) + \lambda_3(x_n,z)d(z,Tx_n) + \lambda_4(x_n,z)d(x_n,Tx_n) \\ &+ \lambda_5(x_n,z)d(z,Tz) + \lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\ &+ \lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\ &+ \lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)} \in s(Tx_n,Tz).\end{aligned}$$

It implies that

$$\begin{aligned}
\lambda_1(x_n,z)d(x_n,z) &+ \lambda_2(x_n,z)d(x_n,Tz) + \lambda_3(x_n,z)d(z,Tx_n) + \lambda_4(x_n,z)d(x_n,Tx_n) \\
&+ \lambda_5(x_n,z)d(z,Tz) + \lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)} \in \left(\bigcap_{u\in Tx_n} s(u,Tz)\right).
\end{aligned}$$

$$\begin{aligned}
\lambda_1(x_n,z)d(x_n,z) &+ \lambda_2(x_n,z)d(x_n,Tz) + \lambda_3(x_n,z)d(z,Tx_n) + \lambda_4(x_n,z)d(x_n,Tx_n) \\
&+ \lambda_5(x_n,z)d(z,Tz) + \lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)} \in s(u,Tz), \forall u \in T_n.
\end{aligned}$$

Since $x_{n+1} \in Tx_n$, we get

$$\begin{aligned}
\lambda_1(x_n,z)d(x_n,z) &+ \lambda_2(x_n,z)d(x_n,Tz) + \lambda_3(x_n,z)d(z,Tx_n) + \lambda_4(x_n,z)d(x_n,Tx_n) \\
&+ \lambda_5(x_n,z)d(z,Tz) + \lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)} \in s(x_{n+1},Tz).
\end{aligned}$$

$$\begin{aligned}
\lambda_1(x_n,z)d(x_n,z) &+ \lambda_2(x_n,z)d(x_n,Tz) + \lambda_3(x_n,z)d(z,Tx_n) + \lambda_4(x_n,z)d(x_n,Tx_n) \\
&+ \lambda_5(x_n,z)d(z,Tz) + \lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)} \\
&+ \lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)} \in \left(\bigcup_{u^*\in Tz} s(d(x_{n+1},u^*))\right).
\end{aligned}$$

So, there exists some $z_n \in Tz$ such that

$$\begin{aligned}\lambda_1(x_n,z)d(x_n,z)&+\lambda_2(x_n,z)d(x_n,Tz)+\lambda_3(x_n,z)d(z,Tx_n)+\lambda_4(x_n,z)d(x_n,Tx_n)\\&+\lambda_5(x_n,z)d(z,Tz)+\lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)}\\&+\lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)}\\&+\lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)}\in s(d(x_{n+1},z_n)).\end{aligned}$$

Therefore, by definition we have

$$\begin{aligned}d(x_{n+1},z_n)\preceq{}&\lambda_1(x_n,z)d(x_n,z)+\lambda_2(x_n,z)d(x_n,Tz)+\lambda_3(x_n,z)d(z,Tx_n)\\&+\lambda_4(x_n,z)d(x_n,Tx_n)+\lambda_5(x_n,z)d(z,Tz)\\&+\lambda_6(x_n,z)\frac{d(z,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)}\\&+\lambda_7(x_n,z)\frac{d(x_n,Tz)[1+d(x_n,Tx_n)]}{1+d(x_n,z)}\\&+\lambda_8(x_n,z)\frac{d(x_n,Tz)[1+d(z,Tx_n)]}{1+d(x_n,z)+d(z,Tx_n)}.\end{aligned}$$

Using the g.l.b property of T and Lemma 3.1, we get

$$\begin{aligned}d(x_{n+1},z_n)\preceq{}&\lambda_1(x_0,z)d(x_n,z)+\lambda_2(x_0,z)d(x_n,z_n)+\lambda_3(x_0,z)d(z,x_{n+1})\\&+\lambda_4(x_0,z)d(x_n,x_{n+1})+\lambda_5(x_0,z)d(z,z_n)\\&+\lambda_6(x_0,z)\frac{d(z,z_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,z)}\\&+\lambda_7(x_0,z)\frac{d(x_n,z_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,z)}\\&+\lambda_8(x_0,z)\frac{d(x_n,z_n)[1+d(z,x_{n+1})]}{1+d(x_n,z)+d(z,x_{n+1})}.\end{aligned}$$

By (D_4) of Definition 2.1, we have

$$d(z,z_n)\preceq d(z,x_{n+1})+d(x_{n+1},z_n)$$

Thus,

$$\begin{aligned}d(z,z_n)\preceq{}&d(z,x_{n+1})+\lambda_1(x_0,z)d(x_n,z)+\lambda_2(x_0,z)d(x_n,z_n)+\lambda_3(x_0,z)d(z,x_{n+1})\\&+\lambda_4(x_0,z)d(x_n,x_{n+1})+\lambda_5(x_0,z)d(z,z_n)\\&+\lambda_6(x_0,z)\frac{d(z,z_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,z)}\\&+\lambda_7(x_0,z)\frac{d(x_n,z_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,z)}\\&+\lambda_8(x_0,z)\frac{d(x_n,z_n)[1+d(z,x_{n+1})]}{1+d(x_n,z)+d(z,x_{n+1})}.\end{aligned}$$

Which implies

$$
\begin{aligned}
|d(z,z_n)| \leqslant d(z,x_{n+1}) &+ \lambda_1(x_0,z)|d(x_n,z)| + \lambda_2(x_0,z)|d(x_n,z_n)| \\
&+ \lambda_3(x_0,z)|d(z,x_{n+1})| + \lambda_4(x_0,z)|d(x_n,x_{n+1})| + \lambda_5(x_0,z)|d(z,z_n)| \\
&+ \lambda_6(x_0,z)\left|\frac{d(z,z_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,z)}\right| \\
&+ \lambda_7(x_0,z)\left|\frac{d(x_n,z_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,z)}\right| \\
&+ \lambda_8(x_0,z)\left|\frac{d(x_n,z_n)[1+d(z,x_{n+1})]}{1+d(x_n,z)+d(z,x_{n+1})}\right|.
\end{aligned}
$$

Taking limit as $n \longrightarrow \infty$, we get

$$
\begin{aligned}
|d(z,z_n)| &\leqslant \lambda_2(x_0,z)|d(z,z_n)| + \lambda_5(x_0,z)|d(z,z_n)| + \lambda_6(x_0,z)|d(z,z_n)| \\
&\qquad\qquad + \lambda_7(x_0,z)|d(z,z_n)| + \lambda_8(x_0,z)|d(z,z_n)| \\
&\leqslant (\lambda_2(x_0,z) + \lambda_5(x_0,z) + \lambda_6(x_0,z) + \lambda_7(x_0,z) + \lambda_8(x_0,z))|d(z,z_n)| \\
&\leqslant \theta|d(z,z_n)|,
\end{aligned}
$$

where

$$
\theta = \lambda_2(x_0,z) + \lambda_5(x_0,z) + \lambda_6(x_0,z) + \lambda_7(x_0,z) + \lambda_8(x_0,z) < 1.
$$

By Lemma 2.1 it implies that $z_n \to z$ as $n \to \infty$. Hence $z \in Tz$ since Tz is closed. Thus, T has a fixed point.
(Uniqueness:) Let $w \neq z$ be another fixed point of T. Then by (3.1), we get

$$
\begin{aligned}
\lambda_1(z,w)d(z,w) + \lambda_2(z,w)d(z,Tw) &+ \lambda_3(z,w)d(w,Tz) + \lambda_4(z,w)d(z,Tz) \\
&+ \lambda_5(z,w)d(w,Tw) + \lambda_6(z,w)\frac{d(w,Tw)[1+d(z,Tz)]}{1+d(z,w)} \\
&+ \lambda_7(z,w)\frac{d(z,Tw)[1+d(z,Tz)]}{1+d(z,w)} \\
&+ \lambda_8(z,w)\frac{d(z,Tw)[1+d(w,Tz)]}{1+d(z,w)+d(w,Tz)} \in s(Tz,Tw).
\end{aligned}
$$

Which implies

$$
\begin{aligned}
d(z,w) \preceq{} &\lambda_1(z,w)d(z,w) + \lambda_2(z,w)d(z,Tw) + \lambda_3(z,w)d(w,Tz) \\
&+ \lambda_4(z,w)d(z,Tz) + \lambda_5(z,w)d(w,Tw) + \lambda_6(z,w)\frac{d(w,Tw)[1+d(z,Tz)]}{1+d(z,w)} \\
&+ \lambda_7(z,w)\frac{d(z,Tw)[1+d(z,Tz)]}{1+d(z,w)} + \lambda_8(z,w)\frac{d(z,Tw)[1+d(w,Tz)]}{1+d(z,w)+d(w,Tz)}.
\end{aligned}
$$

Thus,

$$\begin{aligned}|d(z,w)| \leqslant&\lambda_1(z,w)|d(z,w)| + \lambda_2(z,w)|d(z,w)| + \lambda_3(z,w)|d(w,z)| \\ &+ \lambda_4(z,w)|d(z,z)| + \lambda_5(z,w)|d(w,w)| + \lambda_6(z,w)\left|\frac{d(w,w)[1+d(z,z)]}{1+d(z,w)}\right| \\ &+ \lambda_7(z,w)\left|\frac{d(z,w)[1+d(z,z)]}{1+d(z,w)}\right| + \lambda_8(z,w)\left|\frac{d(z,w)[1+d(w,z)]}{1+d(z,w)+d(w,z)}\right|.\end{aligned}$$

$$\begin{aligned}|d(z,w)| &\leqslant \lambda_1(z,w)|d(z,w)| + \lambda_2(z,w)|d(z,w)| + \lambda_3(z,w)|d(w,z)| \\ &\quad + \lambda_7(z,w)\left|\frac{d(z,w)}{1+d(z,w)}\right| + \lambda_8(z,w)\left|\frac{d(z,w)[1+d(w,z)]}{1+d(z,w)+d(w,z)}\right| \\ &\leqslant [\lambda_1(z,w) + \lambda_2(z,w) + \lambda_3(z,w) + \lambda_7(z,w) + \lambda_8(z,w)]|d(z,w)|.\end{aligned}$$

A contradiction, since

$$\lambda_1(z,w) + \lambda_2(z,w) + \lambda_3(z,w) + \lambda_7(z,w) + \lambda_8(z,w) < 1.$$

Hence z is a unique fixed point of T in X. □

We deduce the following results from Theorem 3.1 above.

Corollary 3.1. *If $\lambda_8 = 0$ and all the other conditions of Theorem 3.1 hold, then T has a unique fixed point.*

Corollary 3.2. *If $\lambda_i = 0$ for $i = 7, 8$ and all the other conditions of Theorem 3.1 hold, then T has a unique fixed point.*

Corollary 3.3. *If $\lambda_i = 0$ for $i = 6, 7, 8$ and all the other conditions of Theorem 3.1 hold, then T has a unique fixed point.*

Corollary 3.4. *If $\lambda_i = 0$ for $i = 2, 3, 4, 5$ and all the other conditions of Theorem 3.1 hold, then T has a unique fixed point.*

Remark 3.1. *In Theorem 3.1 above, if $\lambda_i = 0$ for some $i \in \{1, 2, 3, \dots, 8\}$, then Theorem 3.1 will reduce to many and different results in the literature (see* [**4, 10, 22**]*).*

Theorem 3.2. *Let (X, d) be a complete complex valued metric space, $\lambda_i : X \times X \longrightarrow [0, 1)$ for $i = 1, 2, 3, ..., 9$ and $S, T : X \longrightarrow CB(X)$ be a multi-valued mappings with g.l.b property such that for all $x, y \in X$ the following conditions holds:*

i. $\lambda_i(TSx, y) \leqslant \lambda_i(x, y)$ *and* $\lambda_i(x, STy) \leqslant \lambda_i(x, y)$;

ii.

$$
\begin{aligned}
&\lambda_1(x,y)d(x,y) + \lambda_2(x,y)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \lambda_3(x,y)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} \\
&\quad + \lambda_4(x,y)\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + \lambda_5(x,y)\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)} \\
&\quad + \lambda_6(x,y)\frac{d(y,Ty)[d(x,Sx)+d(y,Sx)]}{1+d(x,y)+d(Sx,Ty)} \\
&\quad + \lambda_7(x,y)\frac{d(x,Ty)[d(x,Sx)+d(y,Sx)]}{1+d(x,y)+d(Sx,Ty)} \\
&\quad + \lambda_8(x,y)\frac{d(x,Sx)[d(x,Ty)+d(y,Ty)]}{1+d(x,y)+d(Sx,Ty)} \\
&\quad + \lambda_9(x,y)\frac{d(y,Sx)[d(x,Ty)+d(y,Ty)]}{1+d(x,y)+d(Sx,Ty)} \in s(Sx,Ty),
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
\lambda_1(x,y) + \lambda_2(x,y) + \lambda_3(x,y) + 2[\lambda_4(x,y) + \lambda_5(x,y) \\
+ \lambda_7(x,y) + \lambda_9(x,y)] + 3[\lambda_6(x,y) + \lambda_8(x,y)] < 1.
\end{aligned}
$$

Then, S and T have a unique common fixed point.

Proof. (Existence:) Let $x_0 \in X$ (arbitrary) and $x_1 \in Sx_0$. From (3.2), we get

$$
\begin{aligned}
&\lambda_1(x_0,x_1)d(x_0,x_1) + \lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} + \lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \in s(Sx_0,Tx_1).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
&\lambda_1(x_0,x_1)d(x_0,x_1)+\lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&\quad+\lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}+\lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}\\
&\quad+\lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&\quad+\lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\quad+\lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\quad+\lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\quad+\lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\in\left(\bigcap_{a\in Sx_0}s(a,Tx_1)\right).
\end{aligned}
$$

$$
\begin{aligned}
&\lambda_1(x_0,x_1)d(x_0,x_1)+\lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&\quad+\lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}+\lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}\\
&\quad+\lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&\quad+\lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\quad+\lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\quad+\lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\quad+\lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\in s(a,Tx_1),\forall a\in Sx_0.
\end{aligned}
$$

Since $x_1 \in Sx_0$, we get

$$\begin{aligned}
&\lambda_1(x_0,x_1)d(x_0,x_1) + \lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} + \lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \in s(x_1,Tx_1).
\end{aligned}$$

$$\begin{aligned}
&\lambda_1(x_0,x_1)d(x_0,x_1) + \lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} + \lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)} \\
&\quad + \lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&\quad + \lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \in \left(\bigcup_{b\in Tx_1} s(d(x_1,b))\right).
\end{aligned}$$

Therefore, there exists some $x_2 \in Tx_1$ such that

$$\begin{aligned}
&\lambda_1(x_0,x_1)d(x_0,x_1)+\lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&\qquad+\lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}+\lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}\\
&\qquad+\lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&\qquad+\lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\qquad+\lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\qquad+\lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&\qquad+\lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\in s(d(x_1,x_2)).
\end{aligned}$$

Using definition of $s(.)$ and g.l.b property of S and T, we have

$$\begin{aligned}
d(x_1,x_2)\preceq{}&\lambda_1(x_0,x_1)d(x_0,x_1)+\lambda_2(x_0,x_1)\frac{d(x_0,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&+\lambda_3(x_0,x_1)\frac{d(x_1,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}\\
&+\lambda_4(x_0,x_1)\frac{d(x_0,Sx_0)d(x_0,Tx_1)}{1+d(x_0,x_1)}\\
&+\lambda_5(x_0,x_1)\frac{d(x_1,Sx_0)d(x_1,Tx_1)}{1+d(x_0,x_1)}\\
&+\lambda_6(x_0,x_1)\frac{d(x_1,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&+\lambda_7(x_0,x_1)\frac{d(x_0,Tx_1)[d(x_0,Sx_0)+d(x_1,Sx_0)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&+\lambda_8(x_0,x_1)\frac{d(x_0,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}\\
&+\lambda_9(x_0,x_1)\frac{d(x_1,Sx_0)[d(x_0,Tx_1)+d(x_1,Tx_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}.
\end{aligned}$$

$$
\begin{aligned}
d(x_1,x_2) \preceq \lambda_1(x_0,x_1)d(x_0,x_1) &+ \lambda_2(x_0,x_1)\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)} \\
&+ \lambda_3(x_0,x_1)\frac{d(x_1,x_1)d(x_0,x_2)}{1+d(x_0,x_1)} + \lambda_4(x_0,x_1)\frac{d(x_0,x_1)d(x_0,x_2)}{1+d(x_0,x_1)} \\
&+ \lambda_5(x_0,x_1)\frac{d(x_1,x_1)d(x_1,x_2)}{1+d(x_0,x_1)} \\
&+ \lambda_6(x_0,x_1)\frac{d(x_1,x_2)[d(x_0,x_1)+d(x_1,x_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&+ \lambda_7(x_0,x_1)\frac{d(x_0,x_2)[d(x_0,x_1)+d(x_1,x_1)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&+ \lambda_8(x_0,x_1)\frac{d(x_0,x_1)[d(x_0,x_2)+d(x_1,x_2)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)} \\
&+ \lambda_9(x_0,x_1)\frac{d(x_1,x_1)[d(x_0,x_2)+d(x_1,x_2)]}{1+d(x_0,x_1)+d(Sx_0,Tx_1)}.
\end{aligned}
$$

Which implies

$$
\begin{aligned}
|d(x_1,x_2)| \leqslant \lambda_1(x_0,x_1)|d(x_0,x_1)| &+ \lambda_2(x_0,x_1)\left|\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}\right| \\
&+ \lambda_3(x_0,x_1)\left|\frac{d(x_1,x_1)d(x_0,x_2)}{1+d(x_0,x_1)}\right| + \lambda_4(x_0,x_1)\left|\frac{d(x_0,x_1)d(x_0,x_2)}{1+d(x_0,x_1)}\right| \\
&+ \lambda_5(x_0,x_1)\left|\frac{d(x_1,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}\right| \\
&+ \lambda_6(x_0,x_1)\left|\frac{d(x_1,x_2)[d(x_0,x_1)+d(x_1,x_1)]}{1+d(x_0,x_1)+d(x_1,x_2)}\right| \\
&+ \lambda_7(x_0,x_1)\left|\frac{d(x_0,x_2)[d(x_0,x_1)+d(x_1,x_1)]}{1+d(x_0,x_1)+d(x_1,x_2)}\right| \\
&+ \lambda_8(x_0,x_1)\left|\frac{d(x_0,x_1)[d(x_0,x_2)+d(x_1,x_2)]}{1+d(x_0,x_1)+d(x_1,x_2)}\right| \\
&+ \lambda_9(x_0,x_1)\left|\frac{d(x_1,x_1)[d(x_0,x_2)+d(x_1,x_2)]}{1+d(x_0,x_1)+d(x_1,x_2)}\right|.
\end{aligned}
$$

$$\begin{aligned}
|d(x_1,x_2)| &\leqslant \lambda_1(x_0,x_1)|d(x_0,x_1)| + \lambda_2(x_0,x_1)\left|\frac{d(x_0,x_1)d(x_1,x_2)}{1+d(x_0,x_1)}\right| \\
&\quad + \lambda_4(x_0,x_1)\left|\frac{d(x_0,x_1)d(x_0,x_2)}{1+d(x_0,x_1)}\right| + \lambda_6(x_0,x_1)\left|\frac{d(x_1,x_2)d(x_0,x_1)}{1+d(x_0,x_1)+d(x_1,x_2)}\right| \\
&\quad + \lambda_7(x_0,x_1)\left|\frac{d(x_0,x_2)d(x_0,x_1)}{1+d(x_0,x_1)+d(x_1,x_2)}\right| \\
&\quad + \lambda_8(x_0,x_1)\left|\frac{d(x_0,x_1)[d(x_0,x_2)+d(x_1,x_2)]}{1+d(x_0,x_1)+d(x_1,x_2)}\right| \\
&\leqslant \lambda_1(x_0,x_1)|d(x_0,x_1)| + \lambda_2(x_0,x_1)|d(x_1,x_2)| \\
&\quad + \lambda_4(x_0,x_1)|d(x_0,x_2)| + \lambda_6(x_0,x_1)|d(x_1,x_2)| \\
&\quad + \lambda_7(x_0,x_1)|d(x_0,x_2)| + \lambda_8(x_0,x_1)|d(x_0,x_2)+d(x_1,x_2)| \\
&\leqslant \lambda_1(x_0,x_1)|d(x_0,x_1)| + \lambda_2(x_0,x_1)|d(x_1,x_2)| + \lambda_4(x_0,x_1)|d(x_0,x_1)| \\
&\quad + \lambda_4(x_0,x_1)|d(x_1,x_2)| + \lambda_6(x_0,x_1)|d(x_1,x_2)| \\
&\quad + \lambda_7(x_0,x_1)|d(x_0,x_1)| + \lambda_7(x_0,x_1)|d(x_1,x_2)| \\
&\quad + \lambda_8(x_0,x_1)|d(x_0,x_1)| + 2\lambda_8(x_0,x_1)|d(x_1,x_2)|.
\end{aligned}$$

Consequently

$$|d(x_1,x_2)| \leqslant k|d(x_0,x_1)|,$$

where

$$k = \frac{\lambda_1(x_0,x_1)+\lambda_4(x_0,x_1)+\lambda_7(x_0,x_1)+\lambda_8(x_0,x_1)}{1-(\lambda_2(x_0,x_1)+\lambda_4(x_0,x_1)+\lambda_6(x_0,x_1)+\lambda_7(x_0,x_1)+2\lambda_8(x_0,x_1))} < 1.$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that for $n = 0, 1, 2, \dots$ we have

$$|d(x_n,x_{n+1})| \leqslant k^n|d(x_0,x_1)|, \qquad x_{2n+1} \in Sx_{2n} \text{ and } x_{2n+2} \in Tx_{2n}.$$

Now for $m > n$, we have

$$\begin{aligned}
|d(x_n,x_m)| &\leqslant |d(x_n,x_{n+1})| + |d(x_{n+1},x_{n+2})| + \cdots + |d(x_{m-1},x_m)| \\
&\leqslant [k^n + k^{n+1} + \cdots + k^{m-1}]|d(x_0,x_1)| \\
&\leqslant \left[\frac{k^n}{1-k}\right]|d(x_0,x_1)|.
\end{aligned}$$

Thus,

$$|d(x_n,x_m)| \leqslant \left[\frac{k^n}{1-k}\right]|d(x_0,x_1)| \longrightarrow 0 \text{ as } n,m \longrightarrow \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now, we show that $u \in Su$ and $u \in Tu$. From (3.2), we have

$$\begin{aligned}
&\lambda_1(x_{2n},u)d(x_{2n},u) + \lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad + \lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)} + \lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&\quad + \lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad + \lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad + \lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad + \lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad + \lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)} \in s(Sx_{2n},Tu).
\end{aligned}$$

Which implies

$$\begin{aligned}
&\lambda_1(x_{2n},u)d(x_{2n},u) + \lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad + \lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)} + \lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&\quad + \lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad + \lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad + \lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad + \lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad + \lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)} \in \left(\bigcap_{v\in Sx_{2n}} s(v,Tu)\right).
\end{aligned}$$

$$\begin{aligned}
&\lambda_1(x_{2n},u)d(x_{2n},u)+\lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}+\lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\in s(v,Tu),\forall v\in Sx_{2n}.
\end{aligned}$$

Since $x_{2n+1}\in Sx_{2n}$, we get

$$\begin{aligned}
&\lambda_1(x_{2n},u)d(x_{2n},u)+\lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}+\lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\in s(x_{2n+1},Tu).
\end{aligned}$$

$$\begin{aligned}
&\lambda_1(x_{2n},u)d(x_{2n},u)+\lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}+\lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad\in\Bigg(\bigcup_{a^*\in Tu}s(d(x_{2n+1},a^*))\Bigg).
\end{aligned}$$

So it implies that, there exists some $u_n \in Tu$ such that

$$\begin{aligned}
&\lambda_1(x_{2n},u)d(x_{2n},u)+\lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}+\lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&\quad+\lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad+\lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&\quad\in s(d(x_{2n+1},u_n)).
\end{aligned}$$

Applying the definition of $s(.)$, Proposition 3.1 and the g.l.b property of S and T, we get

$$\begin{aligned}
d(x_{2n+1},u_n) \preceq{}& \lambda_1(x_{2n},u)d(x_{2n},u)+\lambda_2(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&+\lambda_3(x_{2n},u)\frac{d(u,Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&+\lambda_4(x_{2n},u)\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tu)}{1+d(x_{2n},u)}\\
&+\lambda_5(x_{2n},u)\frac{d(u,Sx_{2n})d(u,Tu)}{1+d(x_{2n},u)}\\
&+\lambda_6(x_{2n},u)\frac{d(u,Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&+\lambda_7(x_{2n},u)\frac{d(x_{2n},Tu)[d(x_{2n},Sx_{2n})+d(u,Sx_{2n})]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&+\lambda_8(x_{2n},u)\frac{d(x_{2n},Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}\\
&+\lambda_9(x_{2n},u)\frac{d(u,Sx_{2n})[d(x_{2n},Tu)+d(u,Tu)]}{1+d(x_{2n},u)+d(Sx_{2n},Tu)}.
\end{aligned}$$

$$\begin{aligned}
d(x_{2n+1},u_n) \preceq{}& \lambda_1(x_0,u)d(x_{2n},u)+\lambda_2(x_0,u)\frac{d(x_{2n},x_{2n+1})d(u,u_n)}{1+d(x_{2n},u)}\\
&+\lambda_3(x_0,u)\frac{d(u,x_{2n+1})d(x_{2n},u_n)}{1+d(x_{2n},u)}\\
&+\lambda_4(x_0,u)\frac{d(x_{2n},x_{2n+1})d(x_{2n},u_n)}{1+d(x_{2n},u)}+\lambda_5(x_0,u)\frac{d(u,x_{2n+1})d(u,u_n)}{1+d(x_{2n},u)}\\
&+\lambda_6(x_0,u)\frac{d(u,u_n)[d(x_{2n},x_{2n+1})+d(u,x_{2n+1})]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\\
&+\lambda_7(x_0,u)\frac{d(x_{2n},u_n)[d(x_{2n},x_{2n+1})+d(u,x_{2n+1})]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\\
&+\lambda_8(x_0,u)\frac{d(x_{2n},x_{2n+1})[d(x_{2n},u_n)+d(u,u_n)]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\\
&+\lambda_9(x_0,u)\frac{d(u,x_{2n+1})[d(x_{2n},u_n)+d(u,u_n)]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}.
\end{aligned}$$

From (D_4) of Definition 2.1, we have

$$d(u,u_n) \preceq d(u,x_{2n+1})+d(x_{2n+1},u_n).$$

Thus

$$\begin{aligned} d(u,u_n) \preceq\ & d(u,x_{2n+1}) + \lambda_1(x_0,u)d(x_{2n},u) + \lambda_2(x_0,u)\frac{d(x_{2n},x_{2n+1})d(u,u_n)}{1+d(x_{2n},u)} \\ & + \lambda_3(x_0,u)\frac{d(u,x_{2n+1})d(x_{2n},u_n)}{1+d(x_{2n},u)} + \lambda_4(x_0,u)\frac{d(x_{2n},x_{2n+1})d(x_{2n},u_n)}{1+d(x_{2n},u)} \\ & + \lambda_5(x_0,u)\frac{d(u,x_{2n+1})d(u,u_n)}{1+d(x_{2n},u)} \\ & + \lambda_6(x_0,u)\frac{d(u,u_n)[d(x_{2n},x_{2n+1})+d(u,x_{2n+1})]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)} \\ & + \lambda_7(x_0,u)\frac{d(x_{2n},u_n)[d(x_{2n},x_{2n+1})+d(u,x_{2n+1})]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)} \\ & + \lambda_8(x_0,u)\frac{d(x_{2n},x_{2n+1})[d(x_{2n},u_n)+d(u,u_n)]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)} \\ & + \lambda_9(x_0,u)\frac{d(u,x_{2n+1})[d(x_{2n},u_n)+d(u,u_n)]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}. \end{aligned}$$

This implies that

$$\begin{aligned} |d(u,u_n)| \leqslant & |d(u,x_{2n+1})| + \lambda_1(x_0,u)|d(x_{2n},u)| \\ & + \lambda_2(x_0,u)\left|\frac{d(x_{2n},x_{2n+1})d(u,u_n)}{1+d(x_{2n},u)}\right| + \lambda_3(x_0,u)\left|\frac{d(u,x_{2n+1})d(x_{2n},u_n)}{1+d(x_{2n},u)}\right| \\ & + \lambda_4(x_0,u)\left|\frac{d(x_{2n},x_{2n+1})d(x_{2n},u_n)}{1+d(x_{2n},u)}\right| + \lambda_5(x_0,u)\left|\frac{d(u,x_{2n+1})d(u,u_n)}{1+d(x_{2n},u)}\right| \\ & + \lambda_6(x_0,u)\left|\frac{d(u,u_n)[d(x_{2n},x_{2n+1})+d(u,x_{2n+1})]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\right| \\ & + \lambda_7(x_0,u)\left|\frac{d(x_{2n},u_n)[d(x_{2n},x_{2n+1})+d(u,x_{2n+1})]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\right| \\ & + \lambda_8(x_0,u)\left|\frac{d(x_{2n},x_{2n+1})[d(x_{2n},u_n)+d(u,u_n)]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\right| \\ & + \lambda_9(x_0,u)\left|\frac{d(u,x_{2n+1})[d(x_{2n},u_n)+d(u,u_n)]}{1+d(x_{2n},u)+d(x_{2n+1},u_n)}\right|. \end{aligned}$$

By letting $n \longrightarrow \infty$ in the above inequality, we have

$$|d(u,u_n)| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Thus, $u_n \to u$ as $n \to \infty$ by Lemma 2.1, also since Tu is closed then $u \in Tu$. Similarly, taking $x = x_{2n+1}$ and $y = u$ in (3.2) it follows that $u \in Su$. Therefore, S and T have a common fixed point.
(Uniqueness:) Let $t \neq u$ be another common fixed point of S and T. Then by (3.2),

we get

$$\begin{aligned}\lambda_1(u,t)d(u,t)&+\lambda_2(u,t)\frac{d(u,Su)d(t,Tt)}{1+d(u,t)}+\lambda_3(u,t)\frac{d(t,Su)d(u,Tt)}{1+d(u,t)}\\&+\lambda_4(u,t)\frac{d(u,Su)d(u,Tt)}{1+d(u,t)}+\lambda_5(u,t)\frac{d(t,Su)d(t,Tt)}{1+d(u,t)}\\&+\lambda_6(u,t)\frac{d(t,Tt)[d(u,Su)+d(t,Su)]}{1+d(u,t)+d(Su,Tt)}\\&+\lambda_7(u,t)\frac{d(u,Tt)[d(u,Su)+d(t,Su)]}{1+d(u,t)+d(Su,Tt)}\\&+\lambda_8(u,t)\frac{d(u,Su)[d(u,Tt)+d(t,Tt)]}{1+d(u,t)+d(Su,Tt)}\\&+\lambda_9(u,t)\frac{d(t,Su)[d(u,Tt)+d(t,Tt)]}{1+d(u,t)+d(Su,Tt)}\in s(Su,Tt).\end{aligned}$$

Which implies

$$\begin{aligned}d(u,t)\preceq\lambda_1(u,t)d(u,t)&+\lambda_2(u,t)\frac{d(u,Su)d(t,Tt)}{1+d(u,t)}+\lambda_3(u,t)\frac{d(t,Su)d(u,Tt)}{1+d(u,t)}\\&+\lambda_4(u,t)\frac{d(u,Su)d(u,Tt)}{1+d(u,t)}+\lambda_5(u,t)\frac{d(t,Su)d(t,Tt)}{1+d(u,t)}\\&+\lambda_6(u,t)\frac{d(t,Tt)[d(u,Su)+d(t,Su)]}{1+d(u,t)+d(Su,Tt)}\\&+\lambda_7(u,t)\frac{d(u,Tt)[d(u,Su)+d(t,Su)]}{1+d(u,t)+d(Su,Tt)}\\&+\lambda_8(u,t)\frac{d(u,Su)[d(u,Tt)+d(t,Tt)]}{1+d(u,t)+d(Su,Tt)}\\&+\lambda_9(u,t)\frac{d(t,Su)[d(u,Tt)+d(t,Tt)]}{1+d(u,t)+d(Su,Tt)}.\end{aligned}$$

Thus

$$\begin{aligned}|d(u,t)|\leqslant\lambda_1(u,t)|d(u,t)|&+\lambda_2(u,t)\left|\frac{d(u,u)d(t,t)}{1+d(u,t)}\right|+\lambda_3(u,t)\left|\frac{d(t,u)d(u,t)}{1+d(u,t)}\right|\\&+\lambda_4(u,t)\left|\frac{d(u,u)d(u,t)}{1+d(u,t)}\right|+\lambda_5(u,t)\left|\frac{d(t,u)d(t,t)}{1+d(u,t)}\right|\\&+\lambda_6(u,t)\left|\frac{d(t,t)[d(u,u)+d(t,u)]}{1+d(u,t)+d(u,t)}\right|\\&+\lambda_7(u,t)\left|\frac{d(u,t)[d(u,u)+d(t,u)]}{1+d(u,t)+d(u,t)}\right|\\&+\lambda_8(u,t)\left|\frac{d(u,u)[d(u,t)+d(t,t)]}{1+d(u,t)+d(u,t)}\right|\\&+\lambda_9(u,t)\left|\frac{d(t,u)[d(u,t)+d(t,t)]}{1+d(u,t)+d(u,t)}\right|.\end{aligned}$$

$$\begin{aligned}|d(u,t)| &\leqslant \lambda_1(u,t)|d(u,t)| + \lambda_3(u,t)\left|\frac{d(t,u)d(u,t)}{1+d(u,t)}\right| \\ &\qquad + \lambda_7(u,t)\left|\frac{d(u,t)d(t,u)}{1+2d(u,t)}\right| + \lambda_9(u,t)\left|\frac{d(t,u)d(u,t)}{1+2d(u,t)}\right| \\ &\leqslant [\lambda_1(u,t)+\lambda_3(u,t)+\lambda_7(u,t)+\lambda_9(u,t)]|d(u,t)| \\ &\leqslant \gamma|d(u,t)|.\end{aligned}$$

A contradiction, since $\gamma < 1$. Thus, $u = t$ is a unique common fixed point of S and T in X. □

In similar way with Theorem 3.1, we can deduce the following results from Theorem 3.2 above.

COROLLARY 3.5. *If $S = T$ and all the other conditions of Theorem 3.2 hold, then T has a unique fixed point.*

COROLLARY 3.6. *Suppose that $\lambda_9 = 0$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.7. *Suppose that $\lambda_8 = 0$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.8. *Suppose that $\lambda_i = 0$ for $i = 8, 9$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.9. *Suppose that $\lambda_7 = 0$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.10. *Suppose that $\lambda_i = 0$ for $i = 7, 8, 9$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.11. *Suppose that $\lambda_6 = 0$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.12. *Suppose that $\lambda_i = 0$ for $i = 6, 8, 9$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.13. *Suppose that $\lambda_i = 0$ for $i = 2, 6, 7$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

COROLLARY 3.14. *Suppose that $\lambda_i = 0$ for $i = 2, 6, 7, 8, 9$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

Corollary 3.15. *Suppose that $\lambda_i = 0$ for $i = 3, 6, 7$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

Corollary 3.16. *Suppose that $\lambda_i = 0$ for $i = 3, 6, 7, 8, 9$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.2 hold, then S and T have a unique common fixed point.*

Remark 3.2. *In Theorem 3.2 above, if $\lambda_i = 0$ for some $i \in \{1, 2, 3, \dots, 9\}$, then Theorem 3.2 will reduce to many and different results in the literature (see* [**9, 10, 11, 22**]*).*

Theorem 3.3. *Let (X, d) be a complete complex valued metric space, $\lambda_i : X \longrightarrow [0, 1)$ for $i = 1, 2, 3, ..., 9$ and $S, T : X \longrightarrow CB(X)$ be a multi-valued mappings with g.l.b property such that for all $x, y \in X$ the following conditions holds:*

i. $\lambda_i(TSx) \leqslant \lambda_i(x)$ *and* $\lambda_i(STx) \leqslant \lambda_i(x)$;

ii.

$$\begin{aligned}
\lambda_1(x)d(x,y) &+ \lambda_2(x)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + \lambda_3(x)\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} \\
&+ \lambda_4(x)\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + \lambda_5(x)\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)} \\
&+ \lambda_6(x)\frac{d(y,Ty)[d(x,Sx)+d(y,Sx)]}{1+d(x,y)+d(Sx,Ty)} \\
&+ \lambda_7(x)\frac{d(x,Ty)[d(x,Sx)+d(y,Sx)]}{1+d(x,y)+d(Sx,Ty)} \\
&+ \lambda_8(x)\frac{d(x,Sx)[d(x,Ty)+d(y,Ty)]}{1+d(x,y)+d(Sx,Ty)} \\
&+ \lambda_9(x)\frac{d(y,Sx)[d(x,Ty)+d(y,Ty)]}{1+d(x,y)+d(Sx,Ty)} \in s(Sx,Ty),
\end{aligned}$$

where

$$\lambda_1(x) + \lambda_2(x) + \lambda_3(x) + 2[\lambda_4(x) + \lambda_5(x) + \lambda_7(x) + \lambda_9(x)] + 3[\lambda_6(x) + \lambda_8(x)] < 1.$$

Then, S and T have a unique common fixed point.

Proof. The proof follows from the prove of Theorem 3.2 and using Lemma 3.2. □

Consequently, one can deduce the following results from Theorem 3.3 above.

Corollary 3.17. *If $S = T$ and all the other conditions of Theorem 3.3 hold, then T has a unique fixed point.*

Corollary 3.18. *Suppose that $\lambda_i = 0$ for some $i \in \{1, 2, 3, \dots, 9\}$ and $S, T : X \longrightarrow CB(X)$ be multi-valued mappings such that all the other conditions of Theorem 3.3 hold, then S and T have a unique common fixed point.*

REMARK 3.3. *In Theorem 3.3 above, if $\lambda_i = 0$ for some $i \in \{1, 2, 3, \dots, 9\}$, then Theorem 3.3 will reduce to many and different results in the literature (see* [**3, 10, 17, 21, 22**]*).*

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