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A New Design Approach to Discrete-Time Sliding Mode Controllers With Disturbance Compensation

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***Abstract*— A new design approach to discrete time sliding-mode-based controllers with disturbance compensation is proposed in this paper. The approach is applicable for linear time invariant plants with matched disturbances. Starting from the known methods for disturbance estimation (*i*) using discrete time nominal plant model and (*ii*) using original sliding mode control design method, full integration of those two algorithms into single one is proposed. Besides, the proposed algorithm can be additionally simplified for a small sampling time. The simplified algorithm does not directly depend on the equivalent control but only on the present and previous values of the sliding variable and previous value of the control. The obtained results are compared with the corresponding system with disturbance estimator based on sliding variable measurement. It is established that both methods give identical results in the nominal case. The method is illustrated in a positional servo system design. Comparative analysis of different methods is done by computer simulation.**

Keywords- electrical drives; sliding mode control; speed control; position control; discrete-time variable structure control

I. INTRODUCTION

There are three basic requirements in the design of high-quality automatic control systems: (*i*) achieving the required accuracy of reference tracking; (*ii*) reaching the defined state at the desired speed and (*iii*) robustness to external disturbances (from the load) or internal disturbances (parameter variations and uncertainties). Disturbances affecting the object can be classified into disturbances that act exclusively through the control channel (always the case with the first order plants), disturbances that act outside the control channel, and disturbances of the combined type. This paper deals with plants having the first type of disturbances, which are said to meet the matching conditions. Control of such plants using continuous time sliding modes leads to complete system invariance to matched disturbances [1]. However, the time discretization of continuous sliding mode control algorithms prevents occurrence of ideal sliding regime. Consequently, a quasi-sliding mode [2] arises due to time delay in data processing, which represents a chaotic zig-zag motion of the state point around the given sliding manifold. This creates unwanted chattering, i.e. high-frequency vibrations in electromechanical systems that causes wear of transmission elements (gears, beds, carriers). Algorithms that considerably reduce chattering are based on the realization of the so-called ideal discrete time sliding mode, using equivalent control [3]-[6]. These algorithms, in the case of nominal systems, keep the system state exactly on the given sliding manifold in the sampling instants. If the plant is continuous, this means that the system state can leave the sliding manifold between two consecutive sampling instants, once again resulting in the quasi-sliding mode. However, the quality of such sliding mode is better than the previous approach. The third approach in the discrete time sliding mode design is to deliberately provide a zigzag motion around the given sliding surface [7], with requirement for the system state to cross the sliding surface during each sampling period. This also implies deterministic chattering. Some methods do not require the mandatory sliding surface intersection at each sampling period [8], which mitigates chattering. Also, there are methods that convert continuous time control algorithms into discrete ones without pronounced chattering, using the implicit implementation of the signum function [9], [10].

Loss of invariance to matched disturbances characterizes all discrete time methods of realizing the quasi-sliding regime since disturbances are not subjected to time discretization. Thus, discrete time sliding modes are not invariant, but they have a certain robustness. To improve robustness, several possibilities were proposed: increasing the sampling frequency (the shorter the sampling period, the closer the system is to its continuous counterpart), introducing an additional switching component to the equivalent control that defines the ideal discrete time sliding mode (which induces additional

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chattering), more adequate selection of the sliding variable [11], introduction of the disturbance estimator for its compensation.

The focus of this paper is on the application of a disturbance estimator, which is based on the nominal control plant model. Such an estimator estimates the disturbance with a delay of one sampling period and it is usually implemented in a system with the discrete time sliding mode based on the application of equivalent control. In addition, a brief review of the estimator based on the integral of the sliding variable is given.

The main contribution of the paper is threefold: (1) the integration of the disturbance estimator of the first type into the discrete sliding mode control algorithm design; (2) the simplification of the control algorithm for systems with a higher sampling frequency. The given approach generates a new way of designing discrete time servo systems, in which equivalent control does not participate explicitly; (3) the establishing equivalence between the two methods of disturbance compensation. The proposed approach will be illustrated on the second-order positional servo system.

The paper is organized as follows. The second section lists the known results that form the basis of the proposed algorithm. Two disturbance estimation methods are given. The first one is based on the nominal system model and can be applied for all types of discrete time control. The second method uses the integral of the sliding variable and is applicable only to systems with a discrete time sliding mode. An original design method of discrete time sliding control systems is proposed. The third section describes the method of incorporating the disturbance estimator into the discrete time sliding mode control algorithm, and its modification for systems with higher sampling frequencies. The fourth section is devoted to a servo system design, where simulation is used to compare the proposed method with other procedures. The paper ends with conclusions and a list of references on which the paper relies.

II. Preliminary Analysis

Consider a linear time invariant continuous time dynamic system,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{D}\boldsymbol{f}(t), \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the measurable state vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ and $\boldsymbol{f}(t) \in \mathbb{R}^l$ are the control and disturbance vectors, respectively. Matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ are constant of appropriate dimensions. It is assumed that the disturbance enters the system through control channel, i.e. fulfills the matching conditions [1]

$$\mathrm{rank}([\boldsymbol{B}\ \boldsymbol{D}]) = \mathrm{rank}\,(\boldsymbol{B}), \tag{2}$$

and also, it holds $|\boldsymbol{f}(t)| \leq \boldsymbol{f}_0 < \infty$; $|\dot{\boldsymbol{f}}(t)| \leq \boldsymbol{f}_1 < \infty$.

Time discretization of the system (1) with the sampling period T, leads to the discrete time model ($\boldsymbol{x}(kT) \equiv \boldsymbol{x}_k$, $k \in N_0$)

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d\boldsymbol{u}_k + \boldsymbol{v}_k, \tag{3}$$

where $\boldsymbol{A}_d = e^{\boldsymbol{A}T}$; $\boldsymbol{B}_d = \int_0^T e^{\boldsymbol{A}\tau}d\tau\,\boldsymbol{B}$,

$\boldsymbol{v}_k = \int_0^T e^{\boldsymbol{A}\tau}\boldsymbol{D}\boldsymbol{f}(k+1)T-\tau)d\tau.$

A. Disturbance estimation methods

This section presents two simple disturbance estimation methods in digital control systems. The first one is universal and can be applied to any digital control system. The second one is applicable only to discrete time sliding mode control systems. Both methods give disturbance estimate delayed by one sampling period.

A1. The first estimation method

This method is based on discrete time model of nominal plant (3), and first time applied in discrete-time sliding mode systems in [5].

The model (3) can be rewritten into equivalent form

$$\boldsymbol{x}_k = \boldsymbol{A}_d\boldsymbol{x}_{k-1} + \boldsymbol{B}_d\boldsymbol{u}_{k-1} + \boldsymbol{v}_{k-1}, \tag{4}$$

from which it can be found

$$\boldsymbol{v}_{k-1} = \boldsymbol{x}_k - \boldsymbol{A}_d\boldsymbol{x}_{k-1} - \boldsymbol{B}_d\boldsymbol{u}_{k-1}. \tag{5}$$

In general case, $\boldsymbol{v}_k$ does not meet the matching conditions (2). However, if the sampling period T is sufficiently small, the unmatched part of the disturbances can be neglected compering to the matched part, which will be applied below. In that case, $\boldsymbol{v}_k = \boldsymbol{B}_d\boldsymbol{d}_k$, and (3) can be rewritten as

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d(\boldsymbol{u}_k + \boldsymbol{d}_k). \tag{6}$$

Using the estimated value $\boldsymbol{v}_{k-1} = \boldsymbol{B}_d\boldsymbol{d}_{k-1}$ for the compensation with the opposite sign in (6) with respect to $\boldsymbol{d}_k$, it is obtained,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}_d\boldsymbol{x}_k + \boldsymbol{B}_d(\boldsymbol{u}_k + \boldsymbol{d}_k - \boldsymbol{d}_{k-1}). \tag{7}$$

If the disturbance is continuous and bounded (as assumed), the uncompensated disturbance part $\boldsymbol{d}_k - \boldsymbol{d}_{k-1}$ is of order $O(T^2)$, [5].

A2. The second estimation method

This method is based on the fact that the disturbance, which fulfills (2), is directly reflected with a time delay of one sampling period in the sliding variable, which is available for measurement. However, unlike the first estimation method, the measured signal cannot be directly added to the control for disturbance compensation but must be additionally integrated. This is like the classical introduction of proportional-integral (PI) control into the channel of system error signal. In this case, a sliding variable is used instead of the error signal, where the sliding variable is a linear combination of the state variables and should be zero as well. More on this approach will be given in the next section, after the introduction of the basic concepts of sliding modes in discrete time systems.

B. Discrete-time sliding mode with equivalent control

Application of these disturbance compensation methods into conventional (first order) discrete time sliding mode control design will be presented. An equivalent control approach will be used [3]-[5], [13].

System model (3) is extended by sliding surface equation,

$$\boldsymbol{s}_k = \boldsymbol{C}_d\boldsymbol{x}_k = 0,\ \boldsymbol{C}_d \in R^{m\times n}, \tag{8}$$

along which a discrete time sliding mode should be achieved. Hence, the first step is to define sliding variable (8), which defines the system motion in the sliding mode, based on the spectrum of the desired eigenvalues (poles) of the system. In a continuous-time realization, a control achieving the sliding mode consists of two components: the component providing sliding surface reaching and the component ensuring sliding motion. The reaching component is a discontinuous function of the $\boldsymbol{\alpha}\mathrm{sign}(\boldsymbol{s})$ type, where $\boldsymbol{\alpha}$ is a diagonal matrix with suitably chosen elements, and the sliding mode component is the equivalent control determined from the condition of satisfying the system dynamics in the sliding mode. In the non-nominal system, $\boldsymbol{\alpha}$ depends on the absolute value of the disturbance. In a discrete-time system under nominal conditions, equivalent control provides both reaching and sliding [3], i.e. the control is unique, which is not the case with continuous time systems. However, in case of non-nominal conditions, it is necessary to introduce a discontinuous control component into the discrete time system as well, in order to compensate for disturbances. But, if disturbance compensation is previously introduced, the discontinuous control component should have a much smaller amplitude (only covers $\boldsymbol{d}_k - \boldsymbol{d}_{k-1}$). This reduces chattering. In practice, discontinuous control is often not necessary. Equivalent control, with disturbance compensation, provides required system properties: fast response, high steady-state and dynamic accuracy and high robustness to disturbances that meet condition (2). These are more satisfied if T is smaller.

Equivalent control in discrete-time sliding mode control system is found from the condition that from any initial state $\boldsymbol{x_k}(0)$, the system reaches in one step $\boldsymbol{s}_{k+1}$=0, i.e. from the condition [3]-[5]

$$\boldsymbol{s}_{k+1} = \boldsymbol{C}_d \boldsymbol{x}_{k+1} = 0. \tag{9}$$

If $\boldsymbol{C}_d$ is known, replacement of (3) into (9) and solving the obtained equation with respect to $\boldsymbol{u}_k$ gives,

$$\boldsymbol{u}_k = -(\boldsymbol{C}_d\boldsymbol{B}_d)^{-1}(\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k - \boldsymbol{C}_d\boldsymbol{v}_k). \tag{10}$$

Since generally $\boldsymbol{v}_k$ cannot be measured, the equivalent control for the nominal system can be defined as

$$\boldsymbol{u}_{k,eq} = -(\boldsymbol{C}_d\boldsymbol{B}_d)^{-1}\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k. \tag{11}$$

To compensate disturbance, a discontinuous component is usually added to the equivalent control, so the overall control becomes,

$$\boldsymbol{u}_k = \boldsymbol{u}_{k,eq} - \boldsymbol{\alpha}\mathrm{sign}(\boldsymbol{s}_k), \tag{12}$$

where $\boldsymbol{\alpha}$ is the diagonal matrix with adequately selected α_i.

If disturbance compensation is previously introduced, based on estimator (5), the control of the system becomes,

$$\boldsymbol{u}_k = -(\boldsymbol{C}_d\boldsymbol{B}_d)^{-1}(\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k - \boldsymbol{C}_d\boldsymbol{v}_{k-1}), \tag{13}$$

or

$$\boldsymbol{u}_k = -(\boldsymbol{C}_d\boldsymbol{B}_d)^{-1}\left(\boldsymbol{C}_d\left((\boldsymbol{A}_d - \boldsymbol{I})x_k + \boldsymbol{A}_d\boldsymbol{x}_{k-1} + \boldsymbol{B}_d\boldsymbol{u}_{k-1}\right)\right) \tag{14}$$

The control (14) will be simplified using model (6) and the original way of selection of $\boldsymbol{C}_d$ [12].

C. One way of designing sliding modes

This section gives a brief presentation of a way of designing sliding modes, proposed in [12]. It includes sliding surface determination that is based on the equivalent control of the nominal system (1). The method is applicable for both continuous and discrete time sliding regimes. The method is based on the fact that the selection of the matrix $\boldsymbol{C}_d$ is not unique and that $\boldsymbol{C}_d$ can be specified in advance. The method starts from the condition that the matrix $\boldsymbol{C}_d\boldsymbol{B}_d$ must be non-singular but can have different values at the designer's choice, including the value of the unit matrix, which is the basic assumption of this approach. Then, based on the spectrum of desired eigenvalues of the system in the continuous time domain, which gives the designer a clear insight into the system dynamics, the corresponding desired eigenvalues can easily be found in the discrete shift (z) domain or in the discrete δ-domain [13]. In addition, the procedure relies on the system state feedback control design method based on pole placement.

It is known that the spectrum of desired system eigenvalues in the sliding mode of the first order (conventional sliding modes) contains m zero eigenvalues (equal to the number of the control inputs) and $n - m$ desired stable eigenvalues. These eigenvalues determine the state feedback, defined by the gain matrix $\boldsymbol{K}$ such that the matrix (***A***-***BK***) [1] has the desired eigenvalues.

The procedure starts from the system of the equations[2]

$$\mathbf{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}) = \mathbf{0}\,, \tag{15a}$$

$$\boldsymbol{C}\boldsymbol{B} = \boldsymbol{I}_m. \tag{15b}$$

Since matrix $\boldsymbol{C}\boldsymbol{B} = \boldsymbol{I}_m$ has full rank, relation (15a) can be written in the form,

$$\mathbf{C}\boldsymbol{A} = \boldsymbol{K}, \tag{16}$$

which is equivalent to $(\boldsymbol{C}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}$ if $\boldsymbol{C}\boldsymbol{B} \neq I_m$. Now, (15) can be expressed as

$$\boldsymbol{C}[\boldsymbol{A} \quad \boldsymbol{B}] = [\boldsymbol{K} \quad \boldsymbol{I}_m]. \tag{17}$$

The solution can be obtained as

$$\boldsymbol{C} = [\boldsymbol{K} \quad \boldsymbol{I}_m][\boldsymbol{A} \quad \boldsymbol{B}]^{\dagger}, \tag{18}$$

where $[\boldsymbol{A} \quad \boldsymbol{B}]^{\dagger}$ is the pseudoinverse matrix.

Remark 1: Formula (18) for calculation of matrix $\boldsymbol{C}$, which defines the sliding surface, has been derived from the conditions $\boldsymbol{C}\boldsymbol{B} = \boldsymbol{I}_m$ and $\mathbf{C}\boldsymbol{A} = \boldsymbol{K}$. In large number of publications on this topic, matrix $\boldsymbol{C}$ is determined using different approaches or it is predetermined, when the condition $\boldsymbol{C}\boldsymbol{B} = \boldsymbol{I}_m$ is not fulfilled. Since the matrix $\boldsymbol{C}$ can be scaled by any positive factor, under condition that the equivalent gain remains unchanged, i.e. $\left(\overline{\boldsymbol{C}}\boldsymbol{B}\right)^{-1} * \overline{\boldsymbol{C}}\boldsymbol{A} = \boldsymbol{K}$, where the matrix $\overline{\boldsymbol{C}}$ satisfies $\overline{\boldsymbol{C}} * \boldsymbol{B} \neq \boldsymbol{I}_m$. Then the matrix $\boldsymbol{C}$ that fulfills $\boldsymbol{C}\boldsymbol{B} = \boldsymbol{I}_m$ can be easily determined from

[1] The compensated nominal system is described by $\boldsymbol{x}_{k+1}$ =(***A***-***BK***) $\boldsymbol{x}_k$.

[2] Here the matrix notations do not subscript d, since the procedure is identical both for continuous and discrete time sliding modes.

$$\boldsymbol{C} = (\overline{\boldsymbol{C}}\boldsymbol{B})^{-1} * \overline{\boldsymbol{C}}. \quad (19)$$

Finaly, it will be proved that a matched disturbance is directly reflected on the sliding variable (8). According to (6), (11), $\boldsymbol{C}_d\boldsymbol{B}_d = \boldsymbol{I}_m$ and $\boldsymbol{u}_k = \boldsymbol{u}_{k,eq} = -\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k$, the sliding variable dynamics can be expressed as

$$\boldsymbol{s}_{k+1} = \boldsymbol{C}_d\boldsymbol{x}_{k+1} = \boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k + (\boldsymbol{u}_k + \boldsymbol{d}_k), \quad (20)$$

from which it follows $\boldsymbol{d}_k = \boldsymbol{s}_{k+1}$, or $\boldsymbol{d}_{k-1} = \boldsymbol{s}_k$.

It was shown in [14] how the compensation can be achieved by integrating the sliding variable. This approach was taken in [15] and [16] on positional and speed servosystems, respectively, having the experimental conformation.

III. Controller with the Incorporated Disturbance Estimator

Starting from (6) rewritten as

$$\boldsymbol{x}_k = \boldsymbol{A}_d\boldsymbol{x}_{k-1} + \boldsymbol{B}_d(\boldsymbol{u}_{k-1} + \boldsymbol{d}_{k-1}), \quad (21)$$

multiplying both sides by $\boldsymbol{C}_d$ and assuming $\boldsymbol{C}_d\boldsymbol{B}_d = \boldsymbol{I}_m$, it is obtained,

$$\boldsymbol{d}_{k-1} = \boldsymbol{C}_d\boldsymbol{x}_k - \boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_{k-1} - \boldsymbol{u}_{k-1} = \boldsymbol{s}_k - \boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_{k-1} - \boldsymbol{u}_{k-1} \quad (22)$$

The control that provides $\boldsymbol{s}_{k+1} = \boldsymbol{C}_d\boldsymbol{x}_{k+1} = 0$ according to (6) is

$$\boldsymbol{u}_k = -\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k - \boldsymbol{d}_k. \quad (23)$$

Replacement of $\boldsymbol{d}_{k-1}$ from (20) instead of $\boldsymbol{d}_k$ into (23), yields

$$\boldsymbol{u}_k = -\boldsymbol{C}_d\boldsymbol{A}_d(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}) - \boldsymbol{s}_k + \boldsymbol{u}_{k-1}. \quad (24)$$

Relation (24) represents the complete (overall) sliding control, including disturbance estimator (5), that is (22), in which the first right hand term is the difference between the equivalent control from k and $k-1$ sampling instant. It will be shown that this term consists of two parts, which one of them is not dominant and can be neglected in practice, under certain conditions.

A. The reduced controller

The relation (24) can be simplified for sufficiently small T. Consider the term $\boldsymbol{C}_d\boldsymbol{A}_d(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})$, using possible representation of matrix $\boldsymbol{A}_d$

$$\boldsymbol{A}_d = e^{\boldsymbol{A}T} = \boldsymbol{I}_n + \boldsymbol{A}T + \frac{\boldsymbol{A}^2T^2}{2} + \cdots. \quad (25)$$

Then, (24) can be expanded as

$$\boldsymbol{u}_k = -\boldsymbol{C}_d(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}) - \boldsymbol{C}_d\boldsymbol{A}T(\boldsymbol{x}_k - \boldsymbol{x}_{k-1}) - \cdots$$
$$-\boldsymbol{s}_k + \boldsymbol{u}_{k-1}. \quad (26)$$

Value of $\boldsymbol{x}_k$ in the sliding mode is of order $O(T)$ while $(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})$ is of order $O(T^2)$ [17])]. Therefore, the second right hand side term $\boldsymbol{C}_d\boldsymbol{A}T(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})$ is of order $O(T^3)$ and can be neglected if $T \ll 1$. Finally, it is obtained,

$$\boldsymbol{u}_k = -2\boldsymbol{s}_k + \boldsymbol{s}_{k-1} + \boldsymbol{u}_{k-1}. \quad (27)$$

Error that arises by such approximation is of order $O(T^2)$ with respect to the sliding variable and of order $O(T)$ [17] with respect to $\boldsymbol{x}_k$, since $\boldsymbol{d}_k - \boldsymbol{d}_{k-1}$ is of order $O(T^2)$. It can be easily shown that using (27), the considered errors will remain of the same order with additional term $\boldsymbol{C}_d\boldsymbol{A}T(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})$, which can be neglected since it is of order lower by one.

In this way, it was showed that by measuring only the sliding variable at the k-th moment and its value and control from the previous sampling, high-quality control of the system can be realized, assuming that the sampling period is small enough and the system is without unmodeled dynamics. In other words, the equivalent control is explicitly absent in the control.

The same result can be obtained if the disturbance estimation by integrating the sliding variable is applied. Then the control becomes

$$\boldsymbol{u}_k = -\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}_k - u_{int,k}, \quad (28a)$$

$$\boldsymbol{u}_{int,k} = \boldsymbol{u}_{int,k-1} + \boldsymbol{K}_{int}\boldsymbol{s}_k, \quad (28b)$$

where $\boldsymbol{K}_{int}$ is the diagonal matrix of gains of the integral compensator. Its values may be chosen from the interval $0 \leq k_{int,i} \leq 1$, $i = 1, \ldots, m$. For $\boldsymbol{K}_{int} = \boldsymbol{I}$, constant disturbances are eliminated in one sampling interval, and for $k_{int,i} < 1$ compensation is asymptotic that is faster for $k_{int,i} \to 1$. Disturbances that are not constant are attenuated even more if their rate of change is lower in relation to the system sampling frequency.

In the same manner can be shown that if (25) is used, description (28) results in

$$\boldsymbol{u}_k = -\boldsymbol{s}_k - \boldsymbol{u}_{int,k}, \quad (29a)$$

$$\boldsymbol{u}_{i,k} = \boldsymbol{u}_{int,k-1} + \boldsymbol{K}_{int}\boldsymbol{s}_k. \quad (29b)$$

As already mentioned, both algorithms estimate the disturbance with a delay of one sampling period. However, are the controls (24) and (28) identical in everything since they have different expressions? The following proposition indicates the relation of both the complete controls (24) and (28) and the reduced versions (27) and (29), respectively.

Proposition 1: Control laws defined by (24) and (28) are identical, as well as (27) and (29) under nominal conditions and for $\boldsymbol{K}_{int} = \boldsymbol{I}_m$.

Proof:

Transform (24) into complex z-domain

$$\boldsymbol{u}(z) = -\boldsymbol{C}_d\boldsymbol{A}_d\frac{z-1}{z}\boldsymbol{x}(z) - \boldsymbol{s}(z) + \frac{1}{z}\boldsymbol{u}(z). \quad (30)$$

The control can be found from (30) as

$$\boldsymbol{u}(z) = -\boldsymbol{C}_d\boldsymbol{A}_d\boldsymbol{x}(z) - \boldsymbol{s}(z)\frac{z}{z-1}. \quad (31)$$

Transform (28b) into complex z-domain, yielding,

$$\boldsymbol{u}_i(z) = \frac{z}{z-1}\boldsymbol{K}_{int}\boldsymbol{s}(z). \quad (32)$$

If (32) is replaced into the transformed (28a), for $\boldsymbol{K}_{int} = \boldsymbol{I}_m$ the result is identical to (31).

Further, by replacement of (32), which is the complex representative of (29b), into the complex representative of (29a), and by multiplying the resulting relation by $z-1$ and going back to the time domain, it is obtained,

$$\boldsymbol{u}_{i,k} = -(\boldsymbol{I}_m + \boldsymbol{K}_i)\boldsymbol{s}_k + \boldsymbol{s}_k + \boldsymbol{u}_{i,k-1}. \quad (33)$$

For $\boldsymbol{K}_{int} = \boldsymbol{I}_m$, (33) becomes equal to (27), i.e. (27) and (29) are the equivalent controls. ■

Remark 2: It can be concluded form (29) that the estimator based on the integral of the sliding variable has the possibility of tunning integral gain $\boldsymbol{K}_{int}$, which can have particular importance in case of systems having unmodeled dynamics, when lowering the integral gain can eliminate or alleviate the chattering [20].

Remark 3: Both estimators have integral action. This introduces overshoot in reaching the sliding surface even in systems without input signal saturation. The simplest way to eliminate the overshoot in systems with control (13) is to postpone activation of the estimator by one sampling period from the initial time. For the systems with control (24) or (25), integrators initial states should be set to $-\boldsymbol{s}(0)$, i.e. $\boldsymbol{u}_{k-1}(0) = -\boldsymbol{s}(0)$. For the systems with control (28) and (29) the easiest way is to delay the integrator input by one sampling period.

Summarizing, system stability is ensured by selection of a stable sliding surface, by choosing a control that reaches the sliding surface in one step, as well as by additional compensation of disturbances.

In practice, discrete time sliding mode control can have very large magnitudes, which requires the introduction of restrictions in the controller or restrictions are present in the plant itself. In that case, stability problems and problems of integrator windup arise, because controls (13), (24), (27), (28) and (29) contain integral action. In addition, the dynamics and system performance with reduced controls will be analyzed. These controls are performed under the assumption that the sampling period T is small enough. However, by neglecting the equivalent control component, the conditions for reaching the sliding surface and the system stability can be endangered. This means that in that case the stability conditions of the system should be examined in relation to the sampling period. This especially applies to the system with the disturbance estimator of the first type, which does not have the possibility of adjusting the integral gain. The system with reduced control with an estimator of the second type has a structure similar to the discrete realization of the super twisting algorithm (STA) [21]. STA is obtained from the given structure by introducing the non-linear discontinuous term $k\sqrt{|\boldsymbol{s}|}\ \mathrm{sign}(\boldsymbol{s})$ into the control circuit (29a), instead of the linear term $\boldsymbol{s}$. That possibility requires additional analysis, which is beyond the scope of this paper. Stability analyzes of similar systems with complete algorithms with or without control constraints can be found in [5], [18].

Here, an original way of eliminating the problem of integrator windup will be pointed out, which resulted from the realization of the system with the first method of disturbance compensation. The integrator should contain a limiter that is in the direct branch and has a positive feedback loop via the element of one-time delay (see Fig. 1).

IV. An Illustrative Example

This section gives an example of a typical positional servo system, which is described by a second order model (1) with scalar control,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & a\end{bmatrix} \cdot \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ b\end{bmatrix}(u(t) + f(t)). \quad (34)$$

State variables x_1, x_2 are position and velocity that are available for measurement. Such systems are treated in many papers. In [15] is considered a system with disturbance compensation using integration of the sliding variable (26), while in [19] the disturbance compensation of type (5) is used. This section uses data from [19][3] to compare with the proposed controller (22) and its simplified variant (27), as well as with the disturbance compensation using sliding variable integration (28) and (29).

Plant parameters [19] are: $a = -144$, $b = 6$, initial state $x(0) = [-1 \quad 0]^{\mathrm{T}}$, sampling period $T = 1$ ms and $\bar{\boldsymbol{c}} = [0.5 \quad 0.5]$. An external disturbance that affects the system is given by,

$$f(t) = -\begin{cases} 1 \text{ for } 0 \le t \le 30; \\ 1 + 2.2\sin(0.5\pi t) \text{ for } 30 \le t \le 60; \\ 1 + 8.8\sin(0.5\pi t) \text{ for } 60 \le t \le 90. \end{cases} \quad (35)$$

Here, the control constraint $|u_k| \le 50$ will be imposed.

Based on the system parameters, to compare the systems with identical dynamical properties, system eigenvalues in [19] is determined (notice that $\boldsymbol{c}\boldsymbol{b}_d \ne 1$) using

$$p_d = \mathrm{eig}(\boldsymbol{A}_d - \boldsymbol{b}_d(\boldsymbol{c}\boldsymbol{b}_d)^{-1}\boldsymbol{c}\boldsymbol{A}_d) = [0 \quad 0.999]. \quad (36)$$

Now, the system is redesigned $\boldsymbol{c}_d\boldsymbol{b}_d = 1$.

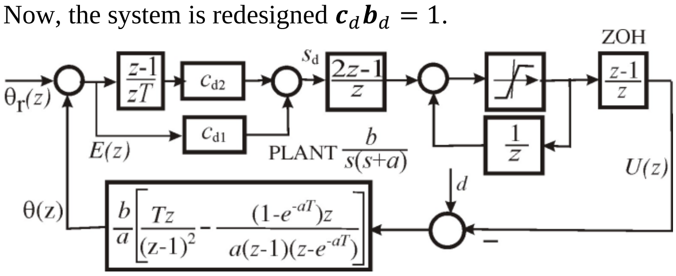

Figure 1. Block diagram of the servo system according to algorithm (27) with control signal limitation and anti-windup structure.

The following Matlab commands determine vectors $\boldsymbol{k}_d$ and $\boldsymbol{c}_d$ using our design procedure:

```
A= [0 1;0 -144];b=[0;6];T=0.001;
%Shift domain design
[Ad, bd]=c2d(A, b, T);pd=[0 0.999];
kd=acker(Ad, bd, pd);
cd= [kd 1]*pinv([Ad bd]);
```

The obtained values are:

$$\boldsymbol{k}_d = \boldsymbol{c}_d\boldsymbol{A}_d = [178.954567 \quad 155.041898];$$
$$\boldsymbol{c}_d = [c_{d1} \ \ c_{d2}] = [178.954567 \quad 178.862943];$$

[3] A more complex system is proposed in [19], based on equivalent control (11) and disturbance estimator (5), i.e. with overall control (13), which is here taken as a comparison reference.

Fig. 1 shows structural block diagram of the system. Simulation results are given in Fig. 2 – Fig. 5.

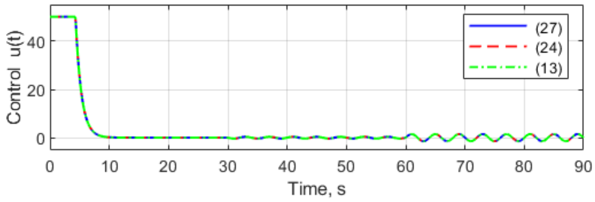

Figure 2. Control signals of the systems with accustomed disturbance estimator/compensator (13) from [19], with the proposed control (24) and with the simplified control (27).

Fig. 2 shows control signals of the considered control algorithms. It can be observed that, except in the starting instant, the signals are identical in the whole time-range. Consequently, system responses are identical as well, which can be confirmed in Fig. 3a. The enlarged steady-state detail in Fig. 3b indicates high positioning accuracy, even under the action of varying disturbance.

To see the efficiency of the applied compensators, Fig. 3c shows the system response without disturbance compensation. It can be observed that the system has a steady state error even when constant disturbance is acting.

The same results are obtained in the case of disturbance compensation using integration of the sliding variable.

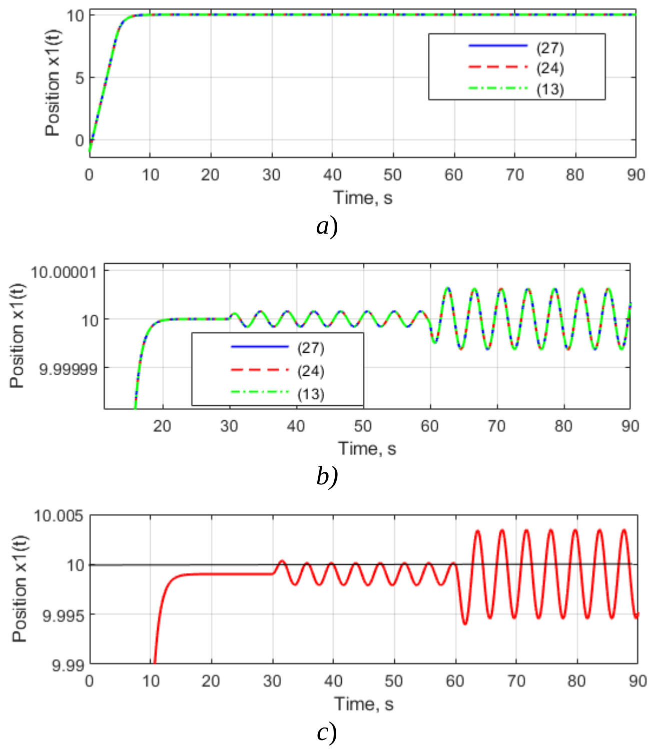

Figure 3. Comparative results of the systems with control (13) [19] and the proposed solution: a) system response to reference $10h(t)$ and disturbance (35); b) a detail of steady state behavior of the system with disturbance compensation and controls (13), (24) and (27). c) a detail of steady state behavior of the system without disturbance compensation.

Figure 4.

Comparing the behavior of the sliding variables of the system [19] and the proposed system does not make much sense because the vectors $\bar{\boldsymbol{c}}$ and $\boldsymbol{c}_d$ that define the sliding surface are very different. If, however, one of them is scaled to bring it to the same level, diagrams like those in Fig. 4 are obtained. Fig. 4a shows that the sliding variables are identical, although Fig. 4b, which represents a detail of Fig. 4a, indicates certain differences in the reaching phase. This difference is a consequence of two factors: (*i*) mismatch of the ratio of the coefficients of the vector $\bar{\boldsymbol{c}}$, which is 1 in the system [19] and 1.0005 in the proposed system; (*ii*) neglecting of $\boldsymbol{C}_d \boldsymbol{A} T(\boldsymbol{x}_k - \boldsymbol{x}_{k-1})$ in the control algorithm.

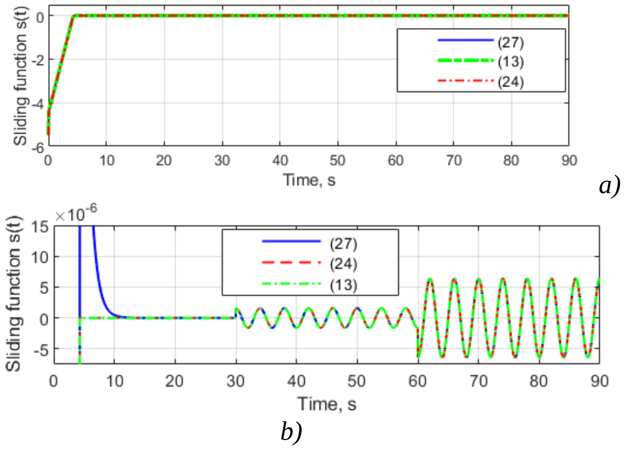

Figure 5. Sliding variables (functions) of the systems with control (13) [19] and the proposed controls (24) and (27). It should be emphasized that the sliding surface vectors $\boldsymbol{C}_d$ are different. In the proposed system coefficients are 358 times larger, so they are scaled by that factor to show that the sliding variables have the same shape.

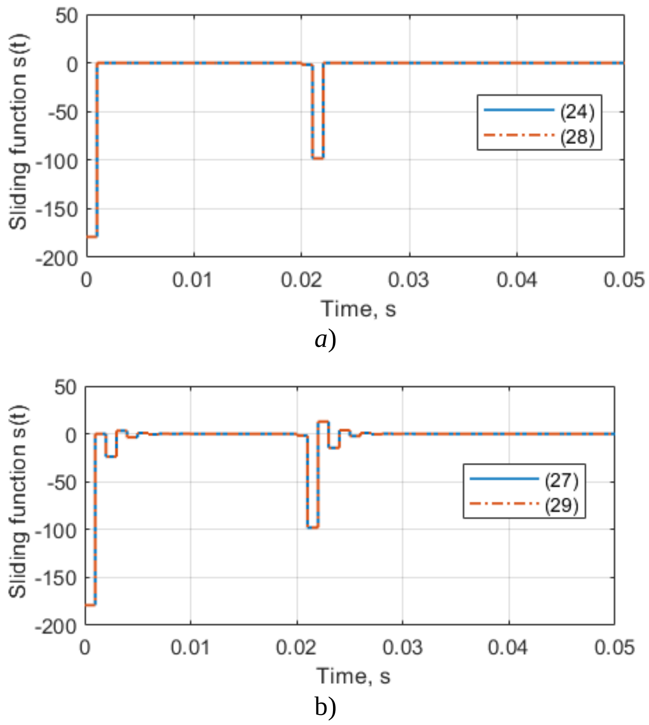

Figure 6. A detail of the reaching phase and elimination of the step disturbance $f(t) = 100h(t - 0.02)$ in the system without control limitation: *a*) system with the controls (24) and (28); b) system with the controls (27) and (29).

Fig. 5 shows details of sliding surface reaching and elimination of step disturbance $f(t) = 100h(t - 0.02)$ for the system without input limitation. It can be clearly seen from the figure that the disturbance estimation process is delayed by one sampling period, and the compensation process is realized for another discretization period. Therefore, elimination and compensation of constant disturbances takes two sampling periods at unit gain of the integrator. It is also observed that the systems with simplified controls have a damped oscillatory character in the reaching mode, which indicates stability problems if the sampling period is not small enough.

V. Conclusions

The paper analyzes the control system of a linear time invariant continual plant using discrete time sliding mode control with a disturbance estimator/compensator. Two types of estimators were considered. Both estimate a disturbance with a delay of one sampling period. One is based on the nominal discrete time model of the plant, and the other on the fact that the disturbance, which enters through the plant control channel, is directly "seen" in the sliding variable, from which it is extracted by a known estimation procedure. It is shown in the paper that these estimators can be simply incorporated into the control algorithm. Particular contribution of the paper is the simplification of the control algorithm for systems with a small sampling period, which shows that the control in the sliding mode does not explicitly depend on the equivalent control, which is widely used in the theory and practice of discrete time sliding modes, but only on the current and previous value of the sliding variable and on the previous control signal value. On the example of a positional servo system, it was demonstrated by simulation that the proposed simplification is fully justified for discrete time sliding modes with a small sampling period. The further research direction is to study the impact of the used approximations and neglections on the system stability and the properties of the system with the considered estimators in the case of the presence of unmodeled dynamics.

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