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THE THEORY OF DERIVATIONS IN ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of a derivation in an Almost Distributive Lattice (ADL) and derive some important properties of derivations in ADLs. Also we introduce the concepts of a principal derivation, an isotone derivation and the fixed set of a derivation. We derive important results on derivations in Heyting ADLs.

1. Introduction

The notation of derivation, introduced from the analytic theory, is helpful for the research of structure and property in an algebraic system. Several authors ([5],[2]) have studied derivations in rings and near rings after Posner [9] has given the definition of the derivation in ring theory. The concept of a derivation in lattices was introduced by G.Szasz in 1974 [14]. X. L. Xin et al. [15] applied the notion of derivation in the ring theory to lattices and investigated some properties. Later, several authors ([1], [3], [4], [6], [7], [8] and [17]) have worked on this concept.

In 1980, the concept of an Almost Distributive Lattice(ADL) was introduced by U.M.Swamy and G.C Rao [4]. This class of ADLs include most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other.

In this paper, we introduce the concept of a derivation in an ADL and investigate some important properties. Also, we introduce the concept of an isotone derivation, a principal derivation in ADLs and investigate the relations among them. We give some equivalent conditions under which a derivation on an ADL becomes the identity map, a monomorphism, an epimorphism. Also, we establish a set of conditions which are sufficient for a derivation on an ADL with a maximal

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element to become an isotone derivation. We define $Fix_d(L)$, the fixed set of a derivation d in an ADL L and prove that it is an ideal of L if d is an isotone derivation. Also, we derive a necessary and sufficient condition for $Fix_d(L)$ to be a prime ideal of L. We prove that the set of all isotone derivations on an ADL L is itself an ADL. We derive a set of sufficient conditions in terms of principal derivations for an ADL to become a Heyting ADL. We introduce a congruence relation ϕ_a, induced by $a \in L$, on an ADL L and derive some useful properties of ϕ_a. We prove that the set $\mathcal{P}(L)$ of all principal derivations on an ADL L is a distributive lattice under pointwise operations and it is isomorphic to the lattice $P\mathcal{I}(L)$ $(P\mathcal{F}(L))$ of all principal ideals (filters) of L. Finally, we prove that the lattice $\mathcal{P}(L)$ is dually isomorphic to $\{\phi_a/a \in L\}$.

2. Preliminaries

In this section, we recollect certain basic concepts and certain important results on Almost Distributive Lattices.

Definition 2.1. [3] *An algebra* $(L, \vee, \wedge)$ *of type* $(2, 2)$ *is called an Almost Distributive Lattice, if it satisfies the following axioms:*
$L_1:$ $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ $(RD\wedge)$
$L_2:$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $(LD\wedge)$
$L_3:$ $(a \vee b) \wedge b = b$
$L_4:$ $(a \vee b) \wedge a = a$
$L_5:$ $a \vee (a \wedge b) = a$, *for all* $a, b, c \in L$.

Definition 2.2. [3] *Let* X *be any non-empty set. Define, for any* $x, y \in L$, $x \vee y = x$ *and* $x \wedge y = y$. *Then* $(X, \vee, \wedge)$ *is an ADL and such an ADL, we call discrete ADL.*

Throughout this paper L stands for an ADL $(L, \vee, \wedge)$ unless otherwise specified.

Lemma 2.1. [3] *For any* $a, b \in L$, *we have*
(i) $a \wedge a = a$
(ii) $a \vee a = a$.
(iii) $(a \wedge b) \vee b = b$
(iv) $a \wedge (a \vee b) = a$
(v) $a \vee (b \wedge a) = a$.
(vi) $a \vee b = a$ *if and only if* $a \wedge b = b$
(vii) $a \vee b = b$ *if and only if* $a \wedge b = a$.

Definition 2.3. [3] *For any* $a, b \in L$, *we say that* a *is less than or equal to* b *and write* $a \leqslant b$, *if* $a \wedge b = a$ *or , equivalently,* $a \vee b = b$.

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Theorem 2.1. [3] *For any* $a, b, c \in L$, *we have the following*
(i) The relation $\leqslant$ *is a partial ordering on* L.
(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. $(LD\vee)$

(iii) $(a \vee b) \vee a = a \vee b = a \vee (b \vee a)$.
(iv) $(a \vee b) \wedge c = (b \vee a) \wedge c$.
(v) The operation $\wedge$ *is associative in* L.
(vi) $a \wedge b \wedge c = b \wedge a \wedge c$.

Theorem 2.2. [3] *For any* $a, b \in L$, *the following are equivalent.*
(i) $(a \wedge b) \vee a = a$
(ii) $a \wedge (b \vee a) = a$
(iii) $(b \wedge a) \vee b = b$
(iv) $b \wedge (a \vee b) = b$
(v) $a \wedge b = b \wedge a$
(vi) $a \vee b = b \vee a$
(vii) *The supremum of* a *and* b *exists in* L *and equals to* $a \vee b$
$(viii)$ *there exists* $x \in L$ *such that* $a \leqslant x$ *and* $b \leqslant x$
(ix) *the infimum of* a *and* b *exists in* L *and equals to* $a \wedge b$.

Definition 2.5. [3] L *is said to be associative, if the operation* $\vee$ *in* L *is associative.*

Theorem 2.3. [3] *The following are equivalent.*
(i) L *is a distributive lattice.*
(ii) *the poset* $(L, \leqslant)$ *is directed above.*
(iii) $a \wedge (b \vee a) = a$, *for all* $a, b \in L$.
(iv) *the operation* $\vee$ *is commutative in* L.
(v) *the operation* $\wedge$ *is commutative in* L.
(vi) *the relation* $\theta := \{(a, b) \in L \times L \mid a \wedge b = b\}$ *is anti-symmetric.*
(vii) *the relation* θ *defined in* (vi) *is a partial order on* L.

Lemma 2.2. [3] *For any* $a, b, c, d \in L$,*we have the following*
(i) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(ii) $a \wedge b = b \wedge a$ *whenever* $a \leqslant b$.
(iii) $[a \vee (b \vee c)] \wedge d = [(a \vee b) \vee c] \wedge d$.
(iv) $a \leqslant b$ *implies* $a \wedge c \leqslant b \wedge c$, $c \wedge a \leqslant c \wedge b$ *and* $c \vee a \leqslant c \vee b$.

Definition 2.6. [3] *An element* $0 \in L$ *is called zero element of* L, *if* $0 \wedge a = 0$ *for all* $a \in L$.

Lemma 2.3. [3] *If* L *has* 0, *then for any* $a, b \in L$, *we have the following*
(i) $a \vee 0 = a$, (ii) $0 \vee a = a$ *and* (iii) $a \wedge 0 = 0$.
(iv) $a \wedge b = 0$ *if and only if* $b \wedge a = 0$.

An element $x \in L$ is called maximal if, for any $y \in L$, $x \leqslant y$ implies $x = y$. We immediately have the following.

Lemma 2.4. [3] *For any* $m \in L$, *the following are equivalent:*
(1) m *is maximal*
(2) $m \vee x = m$ *for all* $x \in L$
(3) $m \wedge x = x$ *for all* $x \in L$.

Definition 2.7. [17] *L is called an almost chain if, for any $x, y \in L$,*
$x \wedge y = y$ or $y \wedge x = x$.
If L has a maximal element m, then this is equivalent to $x \wedge m \leqslant y \wedge m$ or $y \wedge m \leqslant x \wedge m$ for all $x, y \in L$.

Definition 2.8. [3]

(1) *A non-empty subset I of L is said to be an ideal if, $a \vee b \in I$ for all $a, b \in L$ and $a \wedge x \in I$ for any $a \in I,\ x \in L$.*
(2) *A proper ideal P of L is called a prime ideal if for any $x, y \in L,\ x \wedge y \in P$ implies that $x \in P$ or $y \in P$.*
(3) *A non-empty subset F of L is said to be a filter if, $a \wedge b \in F$ for all $a, b \in F$ and $x \vee a \in F$ for any $a \in F,\ x \in L$.*

Theorem 2.4. [3] *For any $a, b \in L$ we have the following*

(1) *$(a] = \{a \wedge x / x \in L\}$ is the smallest ideal containing 'a' and is called the principal ideal of L generated by 'a'.*
(2) *The set $\mathcal{I}(L)$ of all ideals of L forms a distributive lattice under set inclusion in which the glb and lub of I and J are respectively $I \wedge J = I \cap J$ and $I \vee J = \{x \vee y / x \in I$ and $y \in J\}$.*
(3) *$(a] \vee (b] = (a \vee b] = (b \vee a]$ and $(a] \wedge (b] = (a \wedge b] = (b \wedge a]$.*

Though lattice theoretic duality principle does not hold good in an ADL, we have the following.

Theorem 2.5. [3] *For any $a, b \in L$ we have the following*

(1) *$[a) = \{x \vee a / x \in L\}$ is the smallest filter containing 'a' and is called the principal filter of L generated by 'a'.*
(2) *The set $\mathcal{F}(L)$ of all filters of L forms a distributive lattice under set inclusion in which the glb and lub of F and G are respectively by $F \wedge G = F \cup G$ and $F \vee G = \{x \wedge y / x \in F$ and $y \in G\}$.*
(3) *$[a) \vee [b) = [a \wedge b) = [b \wedge a)$ and $[a) \wedge [b) = [a \vee b) = [b \vee a)$*
(4) *$(a] = (b]$ if and only if $[a) = [b)$*
(5) *The class $P\mathcal{I}(L)(P\mathcal{F}(L))$ of all principal ideals (filters) of L is a sublattice of the distributive lattice $\mathcal{I}(L)(\mathcal{F}(L))$ of all ideals (filters) of L. Moreover , the lattice $P\mathcal{I}(L)$ is 'dually isomorphic' onto the lattice $P\mathcal{F}(L)$.*

Definition 2.9. [11] *Let $(L, \vee, \wedge, 0, m)$ be an ADL with 0 and a maximal element m. Suppose $\rightarrow$ is a binary operation on L satisfying the following conditions for all $x, y, z \in L$.*

(1) $x \rightarrow x = m$
(2) $(x \rightarrow y) \wedge y = y$
(3) $x \wedge (x \rightarrow y) = x \wedge y \wedge m$
(4) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$
(5) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

Then $(L, \vee, \wedge, \rightarrow, 0, m)$ is called a Heyting Almost Distributive lattice (HADL).

3. Derivations in ADLs

We begin this section with the following definition of a derivation in an ADL.

Definition 3.1. *A function $d : L \to L$ is called a derivation on L, if*
$$d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy) \text{ for all } x, y \in L.$$

Example 3.1. The identity map on L is a derivation on L. This is called the identity derivation on L.

Example 3.2. If L has 0, define a function d on L by $dx = 0$ for all $x \in L$. Then, d is a derivation on L, and it is called the zero derivation on L.

Example 3.3. In a discrete ADL $L = \{0, a, b\}$, if we define a function d on L by $d0 = 0$, $da = b$, $db = a$, then d is not a derivation on L.

Example 3.4. Let L_1 and L_2 be two ADLs and d_1 and d_2 are derivations on L_1 and L_2 respectively. Then, $d_1 \times d_2$ is a derivation on $L_1 \times L_2$ where $(d_1 \times d_2)(x, y) = (d_1x, d_2y)$, for all $x \in L_1, y \in L_2$.

Lemma 3.1. *Let d be a derivation on L, then the following hold:*
(i) $dx \leqslant x$, for any $x \in L$
(ii) $dx \wedge dy \leqslant d(x \wedge y)$ for all $x, y \in L$
(iii) If I is an ideal of L, then $dI \subseteq I$
(iv) If L has 0, then $d0 = 0$.

Proof. (i) If $x \in L$, then $dx = d(x \wedge x) = (dx \wedge x) \vee (x \wedge dx) = dx \wedge x$(by Lemma 2.1). Therefore, $dx \leqslant x$.

(ii) Let $x, y \in L$. We have $d(x \wedge y) = (dx \wedge y) \vee (x \wedge dy)$. Therefore, $dx \wedge y \leqslant d(x \wedge y)$. Now, by(i) above , we get that $dx \wedge dy \leqslant dx \wedge y \leqslant d(x \wedge y)$.

(iii) If $a \in I$, then by(i) above, $da \leqslant a$ and hence $da \in I$. Thus, $dI \subseteq I$.

(iv) If L has 0, then by(i) above, $d0 \leqslant 0$. Thus, $0 \leqslant d0 \leqslant 0$ and hence $d0 = 0$. □

Theorem 3.1. *If d is a derivation on L a discrete ADL with 0 , then d is either a zero derivation or the identity derivation on L.*

Proof. Suppose $da \neq 0$ for some $a(\neq 0) \in L$.Then, $da = d(a \wedge a) = (da \wedge a) \vee (a \wedge da) = da \wedge a = a$.Therefore d is either a zero derivation or the identity derivation. □

Definition 3.2. *A derivation d on L is called,*
(1) an isotone derivation, if $da \leqslant db$ for all $a, b \in L$ with $a \leqslant b$.
(2) a monomorphic derivation, if d is an injection.
(3) an epimorphic derivation, if d is a surjection.

Example 3.5. Every constant map on an ADL L is an isotone map , but not a derivation.

Example 3.6. Let $L_1 = \{0, x, y, z\}$ be a discrete ADL and consider d_1 as the identity derivation on L_1. Let $L_2 = \{0, a, b, 1\}$ be a chain and define d_2 on L_2 by

$d_2x = a\ if\ x = 1$ and $d_2x = x$ otherwise. Then d_2 is a derivation on L_2. Observe that $d_1 \times d_2$ is a non-isotone derivation on the ADL $L_1 \times L_2$.

DEFINITION 3.3. *Let L be an ADL and $a \in L$. Define a function d_a on L by $d_a x = a \wedge x$ for all $x \in L$. Then, d_a is a derivation on L and is called a principal derivation on L induced by a.*

THEOREM 3.2. *Every principal derivation on L is an isotone derivation .*

PROOF. Let d_a be the principal derivation on L induced by $a \in L$. Now, for $x, y \in L$ with $x \leqslant y$, we have

$$d_a x = d_a(x \wedge y) = a \wedge x \wedge y = a \wedge x \wedge a \wedge y = d_a x \wedge d_a y.$$

Thus $d_a x \leqslant d_a y$ and hence d_a is an isotone derivation . □

LEMMA 3.2. *Suppose L has a maximal element m. Then, $(dm \wedge x) \leqslant dx$ for all $x \in L$.*

PROOF. For $x \in L$, $dx = d(m \wedge x) = (dm \wedge x) \vee (m \wedge dx)$. Hence $(dm \wedge x) \leqslant dx$. □

COROLLARY 3.1. *Suppose m is a maximal element of L and d is a derivation on L. Then, we have ,*
(1) If $x \in L$, $x \geqslant dm$ then $dx \geqslant dm$.
(2) If $x \in L$, $x \leqslant dm$ then $dx = x$.

PROOF. (1) If $x \in L$ and $x \geqslant dm$ then $dm = (dm \wedge x) \leqslant dx$ by above Lemma.
(2) If $x \in L$ and $x \leqslant dm$, then by above Lemma, $dx = (dm \wedge x) \vee dx = x \vee dx = x$. □

LEMMA 3.3. *Let d be a derivation on L. If $y \leqslant x$ and $dx = x$ then $dy = y$.*

PROOF. Let $x, y \in L$ with $y \leqslant x$ and $dx = x$. Now,

$$dy = d(y \wedge x) = (dy \wedge x) \vee (y \wedge dx) = (dy \wedge x) \vee (y \wedge x) = (dy \wedge x) \vee y.$$

Since $dy \leqslant y \leqslant x$, we get $dy = dy \wedge x$. Thus, $dy = dy \vee y = y$. □

LEMMA 3.4. *Let d be an isotone derivation on L. Then, $d(x \vee y) \leqslant dx \vee dy$ for all $x, y \in L$.*

PROOF. Let d be an isotone derivation on L and $x, y \in L$. Now

$$dx = d[(x \vee y) \wedge x] = [d(x \vee y) \wedge x] \vee [(x \vee y) \wedge dx] = [d(x \vee y) \wedge x] \vee dx = [d(x \vee y) \vee dx] \wedge x.$$

Since d is isotone and $x \leqslant x \vee y$ implies $dx \leqslant d(x \vee y)$. Therefore, $dx = d(x \vee y) \wedge x$. Also,

$$dy = d[(x \vee y) \wedge y] = [d(x \vee y) \wedge y] \vee [(x \vee y) \wedge dy] = [d(x \vee y) \wedge y] \vee [(y \vee x) \wedge dy].$$

Since $dy \leqslant y \leqslant y \vee x$, we get $(y \vee x) \wedge dy = dy$. Thus,

$$dy = [d(x \vee y) \wedge y] \vee dy = [d(x \vee y) \vee dy] \wedge y.$$

Now,

$$d(x \vee y) \wedge (dx \vee dy) = d(x \vee y) \wedge [[d(x \vee y) \wedge x] \vee [[d(x \vee y) \vee dy] \wedge y]] = [d(x \vee y) \wedge x] \vee [d(x \vee y) \wedge y] = d(x \vee y) \wedge (x \vee y) = d(x \vee y).$$

Therefore, $d(x \vee y) \leqslant dx \vee dy$. □

THEOREM 3.3. *Let m be a maximal element of L and d be a derivation on L. Then $dm = m$ if and only if d is the identity derivation.*

PROOF. Suppose $dm = m$. For any $x \in L$,

$$dx = d(m \wedge x) = (dm \wedge x) \vee (m \wedge dx) = (m \wedge x) \vee dx = x \vee dx = x.$$

Therefore, d is the identity map on L. The converse is obvious. □

LEMMA 3.5. *Let d be a derivation on L. Then, $d^2x = dx$ for all $x \in L$.*

PROOF. For any $x \in L$, $d^2x = d(dx) \leqslant dx \leqslant x$. Now,

$$d^2x = d(dx) = d(dx \wedge x) = (d^2x \wedge x) \vee (dx \wedge dx) = d^2x \vee dx = dx.$$

□

THEOREM 3.4. *Let d be a derivation on L. Then, the following are equivalent.*
(1) d is the identity map
(2) $d(x \vee y) = (x \vee dy) \wedge (dx \vee y)$ for all $x, y \in L$.
(3) d is a monomorphic derivation.
(4) d is an epimorphic derivation.

PROOF. Clearly (1) implies (2), (3) and (4).
If (2) holds, then for any $x \in L$, $dx = d(x \vee x) = (x \vee dx) \wedge (dx \vee x) = x \wedge x = x$. Therefore, d is the identity map.
Suppose (3) holds and $da \neq a$ for some $a \in L$. Write $da = a_1$. Then, $da_1 \leqslant a_1 < a$. Now, $da_1 = d(a_1 \wedge a) = (da_1 \wedge a) \vee (a_1 \wedge da) = da_1 \vee a_1 = a_1 = da$, which is contradiction since d is monomorphic.
Finally suppose (4) holds and $x \in L$. Then $x = dy$ for some $y \in L$. Now, $dx = d(dy) = d^2y = dy = x$. Therefore, d is the identity map. □

THEOREM 3.5. *Let m be a maximal element of L and d be a derivation on L. Then the following are equivalent.*
(1) d is isotone
(2) $dx = dm \wedge x$ for all $x \in L$
(3) $d(x \wedge y) = dx \wedge y$ for all $x, y \in L$
(4) $d(x \wedge y) = dx \wedge dy$ for all $x, y \in L$
(5) $d(x \vee y) = dx \vee dy$ for all $x, y \in L$.

PROOF. (1) ⇒ (2): Suppose d is an isotone and $x \in L$. Then

$$dx = d(m \wedge x) = (dm \wedge x) \vee (m \wedge dx) = (dm \wedge x) \vee dx.$$

Therefore, $dm \wedge x \leqslant dx$. Also,

$$dx = dx \wedge x = (dx \wedge m) \wedge x \leqslant d(x \wedge m) \wedge x \leqslant dm \wedge x$$

since d is isotone. Therefore, $dm \wedge x = dx$.

(2) $\Rightarrow$ (4): Assume (2) and $x, y \in L$. Then $d(x \wedge y) = dm \wedge x \wedge y = dx \wedge dy$. Thus, we get (4).

(2) $\Rightarrow$ (5): Assume (2) and $x, y \in L$. Then $d(x \vee y) = dm \wedge (x \vee y) = (dm \wedge x) \vee (dm \wedge y) = dx \vee dy$. Thus, we get (5).

(4) $\Rightarrow$ (1): Trivial.

(5) $\Rightarrow$ (1): Trivial.

Thus (1), (2), (4) and (5) are equivalent.

(2) $\Rightarrow$ (3): For any $x, y \in L$, $d(x \wedge y) = dm \wedge x \wedge y = dx \wedge y$.

(3) $\Rightarrow$ (2): For any $x, y \in L$, $dx = d(m \wedge x) = dm \wedge x$. □

Definition 3.4. *Let d be a derivation on L. We define*

$$Fix_d(L) = \{x \in L / dx = x\}.$$

Theorem 3.6. *Let L be an ADL with a maximal element m and d be an isotone derivation on L. Then, $Fix_d(L)$ is an ideal of L.*

Proof. By Lemma 3.5, $dx \in Fix_d(L)$ for any $x \in L$ and thus $\phi \neq Fix_d(L) \subseteq L$. Also, by Lemma 3.3, $Fix_d(L)$ is an initial segment of L. Now, let $x, y \in Fix_d(L)$. By Theorem 3.5, we have, $d(x \vee y) = dx \vee dy = x \vee y$. Hence, $Fix_d(L)$ is an ideal of L. □

Lemma 3.6. *Let d_1 and d_2 be two isotone derivations on L. Then $d_1 = d_2$ if and only if $Fix_{d_1}(L) = Fix_{d_2}(L)$.*

Proof. If $d_1 = d_2$ then clearly $Fix_{d_1}(L) = Fix_{d_2}(L)$. Suppose $Fix_{d_1}(L) = Fix_{d_2}(L)$. For any $x \in L$, $d_1(d_1x) = d_1x$, thus $d_1x \in Fix_{d_1}(L)$. So that $d_1x \in Fix_{d_2}(L)$. Therefore, $d_2(d_1x) = d_1x$ and hence $d_2d_1 = d_1$. Similarly, we get that $d_1d_2 = d_2$. Since d_1, d_2 are isotones and $d_1x \leqslant x$, we get $d_2d_1x \leqslant d_2x$ thus, $d_2d_1 \leqslant d_2$. That is $d_1 \leqslant d_2$. By symmetry we get $d_2 = d_1$. □

Theorem 3.7. *Let m be a maximal element of L and $\mathcal{D}(L)$ be the set of all isotone derivations on L. Then $(\mathcal{D}(L), \vee, \wedge)$ is an ADL where for $d_1,\ d_2 \in \mathcal{D}(L)$, $(d_1 \wedge d_2)x = d_1x \wedge d_2x$ and $(d_1 \vee d_2)x = d_1x \vee d_2x$ for all $x, y \in L$.*

Proof. Let $d_1, d_2 \in \mathcal{D}(L)$ and $x, y \in L$. Then

$$[(d_1 \vee d_2)x] \wedge y = (d_1x \vee d_2x) \wedge y = (d_1x \wedge y) \vee (d_2x \wedge y) = d_1(x \wedge y) \vee d_2(x \wedge y) = (d_1 \vee d_2)(x \wedge y)$$

and

$$x \wedge (d_1 \vee d_2)y = x \wedge (d_1y \vee d_2y) = (x \wedge d_1y) \vee (x \wedge d_2y) = (x \wedge d_1m \wedge y) \vee (x \wedge d_2m \wedge y) = (d_1m \wedge x \wedge y) \vee (d_2m \wedge x \wedge y) = (d_1x \wedge y) \vee (d_2x \wedge y) = d_1(x \wedge y) \vee d_2(x \wedge y) = (d_1 \vee d_2)(x \wedge y).$$

Now, $(d_1 \vee d_2)(x \wedge y) = [(d_1 \vee d_2)x \wedge y] \vee [x \wedge (d_1 \vee d_2)y]$ and hence $d_1 \vee d_2$ is a derivation on L. Also,

$(d_1 \vee d_2)x = d_1x \vee d_2x = (d_1m \wedge x) \vee (d_2m \wedge x) = (d_1m \vee d_2m) \wedge x = (d_1 \vee d_2)m \wedge x.$

Therefore, by Theorem 3.5 $d_1 \vee d_2$ is an isotone derivation on L. Now,

$(d_1 \wedge d_2)x \wedge y = d_1x \wedge d_2x \wedge y = d_1x \wedge y \wedge d_2x \wedge y = d_1(x \wedge y) \wedge d_2(x \wedge y) = (d_1 \wedge d_2)(x \wedge y).$

Again,

$$x \wedge (d_1 \wedge d_2)y = x \wedge d_1y \wedge d_2y = x \wedge d_1m \wedge y \wedge d_2m \wedge y =$$
$$d_1m \wedge x \wedge y \wedge d_2m \wedge x \wedge y = d_1(x \wedge y) \wedge d_2(x \wedge y) = (d_1 \wedge d_2)(x \wedge y).$$

Therefore, $(d_1 \wedge d_2)(x \wedge y) = [(d_1 \wedge d_2)x \wedge y] \vee [x \wedge (d_1 \wedge d_2)y]$ and hence $d_1 \wedge d_2$ is a derivation on L. Also,

$$(d_1 \wedge d_2)x = d_1x \wedge d_2x = d_1m \wedge x \wedge d_2m \wedge x = d_1m \wedge d_2m \wedge x = (d_1 \wedge d_2)m \wedge x.$$

Therefore, by Theorem 3.5, $d_1 \wedge d_2$ is an isotone derivation on L.

Therefore, $\mathcal{D}(L)$ is closed under $\wedge$ *and* $\vee$ and clearly it satisfies the properties of an ADL. □

THEOREM 3.8. *Let m be a maximal element of L and $\mathcal{F} = \{Fix_d(L)/d \in \mathcal{D}(L)\}$. For $d_1, d_2 \in \mathcal{D}(L)$, if we define $Fix_{d_1}(L) \vee Fix_{d_2}(L) = Fix_{d_1 \vee d_2}(L)$ and $Fix_{d_1}(L) \wedge Fix_{d_2}(L) = Fix_{d_1 \wedge d_2}(L)$, then $(\mathcal{F}, \vee, \wedge)$ is an ADL and it is isomorphic to $\mathcal{D}(L)$.*

PROOF. Define $Fix_{d_1}(L) \vee Fix_{d_2}(L) = Fix_{d_1 \vee d_2}(L)$ and $Fix_{d_1}(L) \wedge Fix_{d_2}(L) = Fix_{d_1 \wedge d_2}(L)$, for any $d_1, d_2 \in \mathcal{D}(L)$. Then by Theorem 3.7, we get that $\mathcal{F}$ is closed under $\vee$ *and* $\wedge$. Since $(\mathcal{D}(L), \vee, \wedge)$ is an ADL, we can verify that $(\mathcal{F}, \vee, \wedge)$ is an ADL. Now, define ϕ: $\mathcal{D}(L) \to \mathcal{F}$ by $\phi(d) = Fix_d(L)$. By Lemma 3.6, ϕ is well-defined and injective. Clearly ϕ is surjective. Also, for any $d_1, d_2 \in \mathcal{D}(L)$, $\phi(d_1 \wedge d_2) = Fix_{d_1 \wedge d_2}(L) = Fix_{d_1}(L) \wedge Fix_{d_2}(L) = \phi(d_1) \wedge \phi(d_2)$ and $\phi(d_1 \vee d_2) = Fix_{d_1 \vee d_2}(L) = Fix_{d_1}(L) \vee Fix_{d_2}(L)$. Hence, ϕ is an isomorphism. □

LEMMA 3.7. *Let m be a maximal element of L and d be an isotone epimorphic derivation on L. Then, dm is a maximal element in L.*

PROOF. Let $x \in L$. Since d is epimorphic, $dy = x$ for some $y \in L$. Now, $dm \wedge x = dm \wedge dy = d(m \wedge y) = dy = x$ and hence $dm \vee x = dm$. Thus, dm is a maximal element in L. □

The following theorem gives a necessary and sufficient condition for $Fix_d(L)$ to be a prime ideal.

THEOREM 3.9. *Let m be a maximal element of L. Then the following are equivalent.*
(1) L is an almost chain.
(2) For any isotone derivation d, $Fix_d(L)$ is a prime ideal.

Proof. (1) $\Rightarrow$ (2): Suppose L is an Almost Chain and let d be an isotone derivation on L. Let $x, y \in L$ such that $x \wedge y \in Fix_d(L)$. Since L is an Almost Chain $x \wedge m \leqslant y \wedge m$ *or* $y \wedge m \leqslant x \wedge m$. Without loss of generality assume $x \wedge m \leqslant y \wedge m$. Then $dx = dx \wedge x = dx \wedge m \wedge x = d(x \wedge m) \wedge x = d(x \wedge y \wedge m) \wedge x = x \wedge m \wedge x = x$. Therefore, $x \in Fix_d(L)$.

(2) $\Rightarrow$ (1): Assume (2). Let $x, y \in L$. Consider the principal derivation $d_{x\wedge y}$ induced by $x \wedge y$. By Theorem 3.2, $d_{x\wedge y}$ is an isotone derivation on L and $d_{x\wedge y}(x \wedge y) = x \wedge y$, so that $x \wedge y \in Fix_{d_{x\wedge y}}(L)$. Hence, by our assumption, we get either $x \in Fix_{d_{x\wedge y}}(L)$ or $y \in Fix_{d_{x\wedge y}}(L)$. Without loss of generality assume $x \in Fix_{d_{x\wedge y}}(L)$. Now, $(x \wedge m) \wedge (y \wedge m) = y \wedge x \wedge m = [(x \wedge y) \wedge x] \wedge m = d_{x\wedge y}(x) \wedge m = x \wedge m$ and hence $x \wedge m \leqslant y \wedge m$. Therefore, L is an Almost Chain. □

Theorem 3.10. *Let m be a maximal element of L and $a \in L$. Then $Fix_{d_a}(L)$ is a principal ideal.*

Proof. Let $a \in L$. By Theorem 3.2 and by Theorem 3.6, $Fix_{d_a}(L)$ is an ideal of L. Now, let $x \in L$. Then

$$x \in Fix_{d_a}(L) \Longleftrightarrow d_a x = x \Longleftrightarrow a \wedge x = x \Longleftrightarrow x \in (a].$$

Hence, $Fix_{d_a}(L) = (a]$. □

Theorem 3.11. *If I is a principal ideal of L, then there exists unique isotone derivation d such that $Fix_d(L) = I$.*

Proof. Let $I = (a]$ be a principal ideal of L where $a \in L$ and d_a be the principal derivation on L induced by a. Now, we have

$$x \in Fix_{d_a}(L) \Longleftrightarrow d_a x = x \Longleftrightarrow a \wedge x = x \Longleftrightarrow x \in (a].$$

Therefore, $Fix_{d_a}(L) = I$. Uniqueness of d follows from Lemma 3.6. □

Now, we introduce the concepts of a weak ideal and a principal weak ideal in an ADL in the following.

Definition 3.5. *A nonempty subset I of L is said to be a weak ideal if it satisfies the following.*
(i) $x, y \in I \Rightarrow x \vee y \in I$
(ii)$x \in I, a \in L$ *and* $a \leqslant x$ *implies* $a \in I$.

It can be observe that, for $a \in L$, $(a) = \{x \wedge a / x \in L\}$ is the smallest weak ideal containing $'a'$ and it is called the principal weak ideal generated by $'a'$ in L.

Lemma 3.8. *For $a, b \in L$, then $S_a(b) = \{x \wedge m / x \in L, d_a(x \wedge m) \leqslant b \wedge m\}$ is a weak ideal in L where d_a is the principal derivation induced by a on L.*

Proof. Let $a, b \in L$. We have $d_a(b \wedge m) = a \wedge b \wedge m \leqslant b \wedge m$. Thus $b \wedge m \in S_a(b)$ and hence $\phi \neq S_a(b) \subseteq L$. Let $x, y \in L$ such that $x \leqslant y$ and $y \in S_a(b)$. Thus,

$$x = x \wedge y = x \wedge y \wedge m$$

$$a \wedge x \wedge y \wedge m \wedge m = a \wedge x \wedge y \wedge m \leqslant a \wedge y \wedge m \leqslant b \wedge m$$

and hence $x \in S_a(b)$. Now, let $x, y \in S_a(b)$. Thus,

$$x \vee y = (x \wedge m) \vee (y \wedge m) = (x \vee y) \wedge m$$
$$a \wedge (x \vee y) \wedge m = (x \wedge a \wedge m) \vee (y \wedge a \wedge m) \leqslant b \wedge m$$

and hence $x \vee y \in S_a(b)$. Therefore, $S_a(b)$ is a weak ideal of L. □

THEOREM 3.12. *Let m be a maximal element of L. Then the following are equivalent.*

(1) *L is a Heyting ADL with a maximal element m.*
(2) *For $a, b \in L$, $S_a(b)$ has greatest element.*
(3) *For $a \in L,\ b \in Fix_{d_a}(L)$, $S_a(b)$ has greatest element.*
(4) *For $a \in L,\ b \in Fix_{d_a}(L)$, $S_a(b)$ is a principal weak ideal of L.*

PROOF. (1) ⇒ (2): Let $a, b \in L$. We prove that $(a \to b) \wedge m$ is the greatest element of $S_a(b)$. Since $a \wedge (a \to b) \wedge m \leqslant b \wedge m$, we get that $(a \to b) \wedge m \in S_a(b)$. Let $x \wedge m \in S_a(b)$. Then $a \wedge x \wedge m \leqslant b \wedge m$. Thus, $x \wedge m \leqslant (a \to x) \wedge m = a \to (x \wedge m) = a \to (a \wedge x \wedge m) \leqslant a \to (b \wedge m) = (a \to b) \wedge m$ and hence $(a \to b) \wedge m$ is the greatest element of $S_a(b)$.

(2) ⇒ (3) is trivial and

(3) ⇒ (4) follows from Lemma 3.8.

(4) ⇒ (1): Assume (4) and $a, b \in L$. Then $a \wedge b \in Fix_{d_a}(L)$ since $d_a(a \wedge b) = a \wedge a \wedge b = a \wedge b$. Hence, by (4), $S_a(a \wedge b)$ is a principal weak ideal. Write $S_a(a \wedge b) = (p)$ for some $p \in L$. Now, define $a \to b = p$. Clearly $a \to b$ is well defined (since $(p) = (q) \iff p = q$).

Now we verify that $(L, \vee, \wedge, \to)$ is a Heyting ADL. Let $a, b \in L$.

(i) Observe that $S_a(a) = (m)$. Hence $a \to a = m$.

(ii) Since $a \wedge b \wedge m \leqslant a \wedge b \wedge m$, we get $b \wedge m \in S_a(a \wedge b)$ and hence $b \wedge m \leqslant a \to b$. Therefore, $b \wedge m = b \wedge m \wedge (a \to b)$. Thus, $(a \to b) \wedge b = b \wedge (a \to b) \wedge b = b \wedge m \wedge b = b$.

(iii) Clearly $a \wedge (a \to b) \leqslant a \wedge b \wedge m$. Also from above, $b \wedge m \leqslant (a \to b)$ and hence $a \wedge b \wedge m \leqslant a \wedge (a \to b)$. Therefore, $a \wedge (a \to b) = a \wedge b \wedge m$.

(iv) By (iii), $a \wedge [a \to (b \wedge c)] = a \wedge b \wedge c \wedge m \leqslant a \wedge b \wedge m$. So that $a \to (b \wedge c) \in S_a(a \wedge b)$ and hence $a \to (b \wedge c) \leqslant a \to b$. Similarly we get $a \to (b \wedge c) \leqslant a \to c$. Now, $a \wedge (a \to b) \wedge (a \to c) = a \wedge b \wedge m \wedge a \wedge c \wedge m = a \wedge b \wedge c \wedge m$ and hence $(a \to b) \wedge (a \to c) \in S_a(a \wedge b \wedge c)$. Therefore, $(a \to b) \wedge (a \to c) \leqslant a \to (b \wedge c)$. Thus, $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

(v) Let $a \wedge m \leqslant b \wedge m$. Then $a \wedge (b \to c) \leqslant b \wedge (b \to c) \leqslant b \wedge c \wedge m$. So that $a \wedge (b \to c) = a \wedge a \wedge (b \to c) \leqslant a \wedge b \wedge c \wedge m = a \wedge c \wedge m$. Thus, $b \to c \in S_a(a \wedge c)$. Therefore, $b \to c \leqslant a \to c$. Therefore, we get $(a \vee b) \to c \leqslant (a \to c) \wedge (b \to c)$. On the other hand

$$(a \vee b) \wedge (a \to c) \wedge (b \to c) = [(a \wedge (a \to c) \wedge (b \to c))] \vee [(b \wedge (a \to c) \wedge (b \to c))] \leqslant$$
$$[a \wedge c \wedge (b \to c)] \vee [b \wedge c \wedge (a \to c)] = (a \wedge c \wedge m) \vee (b \wedge c \wedge m) = (a \vee b) \wedge c \wedge m.$$

Thus, $(a \to c) \wedge (b \to c) \in S_{a \vee b}((a \vee b) \wedge c)$ and hence $(a \to c) \wedge (b \to c) \leqslant (a \vee b) \to c$. Therefore, $(L, \vee, \wedge, \to)$ is a Heyting ADL. □

THEOREM 3.13. *Let P be a prime ideal of L. Then there exists a derivation d on L such that $Fix_d(L) = P$.*

PROOF. Let P be a prime ideal of L. Choose $a \in P$. Define , for any $x \in L$, $dx = x$ *if* $x \in P$ *and* $dx = a \wedge x$ otherwise. If $x \notin P$ and $y \notin P$ then $x \wedge y \notin P$. Thus, $d(x \wedge y) = a \wedge x \wedge y = [(a \wedge x) \wedge y] \vee [x \wedge (a \wedge y)] = (dx \wedge y) \vee (x \wedge dy)$. Now assume that $x \in P$. Then $x \wedge y \in P$ and $(dx \wedge y) \vee (x \wedge dy) = (x \wedge y) \vee (x \wedge dy) = x \wedge (y \vee dy) = x \wedge y = d(x \wedge y)$. Therefore, d is a derivation on L. Also, if $x \in P$ then by the definition of d, $x \in Fix_d(L)$. Suppose $x \in Fix_d(L)$. Then $dx = x$. If $x \notin P$, then $x = a \wedge x \in P$ and hence we get $x \in P$. Thus $Fix_d(L) = P$. □

DEFINITION 3.6. *Let $(L, \vee, \wedge, 0)$ be an ADL. For any $a \in L$, define $\phi_a = \{(x, y) \in L \times L / d_a(x) = d_a(y)\}$ where d_a is the principal derivation induced by a on L.*

LEMMA 3.9. *Let L be an ADL. Then for any $a \in L$, ϕ_a is a congruence relation on L.*

PROOF. Clearly ϕ_a is an equivalence relation on L. Now, let (x, y),$(p, q) \in \phi_a$. Then $a\wedge x = a\wedge y$ and $a\wedge p = a\wedge q$. Now, $a\wedge x\wedge p = a\wedge x\wedge a\wedge p = a\wedge y\wedge a\wedge q = a\wedge y\wedge q$ and $a \wedge (x \vee p) = (a \wedge x) \vee (a \wedge p) = (a \wedge y) \vee (a \wedge q) = a \wedge (y \vee q)$. Therefore, $(x \wedge p, y \wedge q), (x \vee p, y \vee q) \in \phi_a$. Hence, ϕ_a is a congruence relation on L. □

LEMMA 3.10. *For any $a, b \in L$, the following hold.*
(1) $\phi_{a\wedge b} = \phi_{b\wedge a}$
(2) $\phi_{a\vee b} = \phi_{b\vee a}$
(3) $\phi_a \cap \phi_b = \phi_{a\vee b}$
(4) $\phi_a o \phi_b = \phi_{a\wedge b} = \phi_a \vee \phi_b$.

PROOF. Since $a \wedge b \wedge x = b \wedge a \wedge x$ and $(a \vee b) \wedge x = (b \vee a) \wedge x$, we get that $\phi_{a\wedge b} = \phi_{b\wedge a}$ and $\phi_{a\vee b} = \phi_{b\vee a}$. Again,

$$(x, y) \in \phi_a \wedge \phi_b \Longleftrightarrow a \wedge x = a \wedge y \textit{ and } b \wedge x = b \wedge y$$
$$\Longleftrightarrow (a \vee b) \wedge x = (a \vee b) \wedge y \Longleftrightarrow (x, y) \in \phi_{a\vee b}.$$

Thus $\phi_{a\vee b} = \phi_a \cap \phi_b$.

Now, if $(x, y) \in \phi_a o \phi_b$, then there exists $z \in L$ such that $(x, z) \in \phi_b$ and $(z, y) \in \phi_a$. So that $b \wedge x = b \wedge z$ and $a \wedge z = a \wedge y$. Now,

$$(a \wedge b) \wedge x = a \wedge b \wedge x = a \wedge b \wedge z = b \wedge a \wedge z = b \wedge a \wedge y = a \wedge b \wedge y.$$

Thus $(x, y) \in \phi_{a\wedge b}$. Therefore, $\phi_a o \phi_b \subseteq \phi_{a\wedge b}$.

Also, if $(x, y) \in \phi_{a\wedge b}$, then $a \wedge b \wedge x = a \wedge b \wedge y$. Now take $z = (b \wedge x) \vee (a \wedge y)$. Then,

$$b \wedge z = b \wedge [(b \wedge x) \vee (a \wedge y)] = (b \wedge x) \vee (b \wedge a \wedge y) = (b \wedge x) \vee (a \wedge b \wedge y) =$$
$$(b \wedge x) \vee (a \wedge b \wedge x) = b \wedge x \textit{ and } a \wedge z = a \wedge [(b \wedge x) \vee (a \wedge y)] =$$
$$(a \wedge b \wedge x) \vee (a \wedge y) = (a \wedge b \wedge y) \vee (a \wedge y) = [b \wedge (a \wedge y)] \vee (a \wedge y) = a \wedge y.$$

Hence, $(x, y) \in \phi_a o \phi_b$. Therefore $\phi_{a\wedge b} \subseteq \phi_a o \phi_b$ and hence $\phi_a o \phi_b = \phi_{a\wedge b}$. By symmetry and by (1) we get that $\phi_b o \phi_a = \phi_{b\wedge a} = \phi_{a\wedge b}$. Hence, $\phi_{a\wedge b} = \phi_a \vee \phi_b$. □

THEOREM 3.14. *Let L be an ADL. Then, the set of all principal derivations $\mathcal{P}(L) = \{d_a/a \in L\}$ is a distributive lattice with the following operations,*

$$d_a \vee d_b = d_{a\vee b} \text{ and } d_a \wedge d_b = d_{a\wedge b} \text{ for all } a, b \in L.$$

Also, $\mathcal{P}(L)$ is isomorphic to $P\mathcal{I}(L)$ as well as $P\mathcal{F}(L)$.

PROOF. Let $a, b \in L$. For any $x \in L$,

$$(d_a \vee d_b)x = d_a x \vee d_b x = (a \wedge x) \vee (b \wedge x) = (a \vee b) \wedge x = d_{a\vee b}x.$$

Therefore, $d_a \vee d_b = d_{a\vee b} \in \mathcal{P}(L)$. Also,

$$(d_a \wedge d_b)x = d_a x \wedge d_b x = a \wedge x \wedge b \wedge x = a \wedge b \wedge x = d_{a\wedge b}x.$$

Therefore, $d_a \wedge d_b = d_{a\wedge b} \in \mathcal{P}(L)$. Hence $\mathcal{P}(L)$ is closed under $\vee$ and $\wedge$ and hence $\mathcal{P}(L)$ is a sub-ADL of $\mathcal{D}(L)$. Also, for any $x \in L$, $d_{a\wedge b}x = a \wedge b \wedge x = b \wedge a \wedge x = d_{b\wedge a}x$. Thus $d_{a\wedge b} = d_{b\wedge a}$. Therefore $d_a \wedge d_b = d_b \wedge d_a$. Hence, $\mathcal{P}(L)$ is a distributive lattice. Now, define $\psi : \mathcal{P}(L) \to P\mathcal{I}(L)$ by $\psi(d_a) = (a]$ for all $a \in L$. By Lemma 3.6 , Theorem 3.10 and Theorem 3.11 we get that ψ is bijection. Now, for $a, b \in L$, $\psi(d_a \vee d_b) = \psi(d_{a\vee b}) = (a \vee b] = (a] \vee (b]$ *and* $\psi(d_a \wedge d_b) = \psi(d_{a\wedge b}) = (a \wedge b] = (a] \wedge (b]$. Therefore, ψ is an isomorphism. Since $P\mathcal{I}(L)$ is isomorphic to $P\mathcal{F}(L)$, we get that $\mathcal{P}(L)$ is isomorphic to $P\mathcal{F}(L)$. □

Finally we conclude this paper with the following theorem.

THEOREM 3.15. *$\mathcal{C} = \{\phi_a/a \in L\}$ is dually isomorphic to $\mathcal{P}(L)$, the set of all principal derivations on L.*

PROOF. Define $\psi : \mathcal{C} \to \mathcal{P}(L)$ by $\psi(d_a) = \phi_a$ for all $a \in L$.
Let $a, b \in L$ such that $d_a = d_b$. Now, for any $x, y \in L$,

$$(x, y) \in \phi_a \Longleftrightarrow a \wedge x = a \wedge y \Longleftrightarrow d_a x = d_a y \Longleftrightarrow d_b x = d_b y \Longleftrightarrow b \wedge x = b \wedge y \Longleftrightarrow (x, y) \in \phi_b.$$

Thus $\phi_a = \phi_b$ and hence ψ is well defined.
On the other hand , let $\phi_a = \phi_b$. For any $x \in L$,

$$(x, a \wedge x) \in \phi_a \Rightarrow (x, a \wedge x) \in \phi_b \Rightarrow b \wedge x = b \wedge a \wedge x \leqslant a \wedge x,$$

by symmetry, we get that $a \wedge x = b \wedge x$ and hence $d_a = d_b$. Now, for $a, b \in L$, by Lemma 3.10,

$$\psi(a\wedge b) = \phi_{a\wedge b} = \phi_a \vee \phi_b = \psi(a) \vee \psi(b) \text{ and } \psi(a\vee b) = \phi_{a\vee b} = \phi_a \wedge \phi_b = \psi(a) \wedge \psi(b).$$

Thus, ψ is a dual isomorphism. □

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