BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(2017), 217-229

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

EDGE a-ZAGREB INDICES AND ITS COINDICES OF GRAPH OPERATIONS

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ABSTRACT. In this paper, the edge a-Zagreb indices and its coindices of some graph operations, such as generalized hierarchical product, Cartesian Product, join, composition of two graphs are obtained. Using the results obtained here, we deduce the F-indices and its coindices for the above graph operation. Finally, we have computed the edge a-Zagreb Index, F-index and their coindices of some important classes of graphs.

1. Introduction

A *chemical graph* is a graph whose vertices denote atoms and edges denote bonds between those atoms of any underlying chemical structure. A *topological index* for a(chemical) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. In the current chemical literature, a large number of graph-based structure descriptors (topological indices) have been put forward, that all depend only on the degrees of the vertices of the molecular graph. More details on vertex-degree-based topological indices and on their comparative study can be found in [**4**, **5**, **7**, **8**]. The topological indices are graph invariants which has been used for examing quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, see [**3**].

For a (molecular) graph G, The *first Zagreb index* $M_1(G)$ is the equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is the equal to the sum of the products of the degrees of pairs of adjacent vertices, that is,

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2010 *Mathematics Subject Classification.* 05C12; 05C76.

Key words and phrases. edge a-Zagreb indices and coindices, generalized hierarchical product, join, composition of two graphs.

$$M_1(G) = \sum_{u\in V(G)} d_G^2(u) = \sum_{uv\in E(G)} (d_G(u) + d_G(v)),\ M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The *first* and *second Zagreb coindices* were first introduced by Ashrafi et al. [**1**]. They are defined as follows:

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v)),\ \overline{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v).$$

The *forgotten topological index*or *F-index* was introduced by Furtula and Gutman [**6**], and it is defined as

$$F = F(G) = \sum_{u\in V(G)} d_G^3(u) = \sum_{uv\in E(G)} (d_G^2(u) + d_G^2(v)).$$

In this sequence, the *forgotten topological coindex* or *F-coindex* is defined as

$$\overline{F}(G) = \sum_{uv\notin E(G)} (d_G^2(u) + d_G^2(v)).$$

Mansour and Song [**19**] are introduced the *vertex a-Zagreb index,edge a-Zagreb index*, and *edge a-Zagreb Coindex*. They are defined as follows

$$N_a(G) = \sum_{v\in V(G)} d^a(v),\ Z_a(G) = \sum_{uv\in E(G)} (d^a(u) + d^a(v)) \text{ and}$$
$$\overline{Z}_a(G) = \sum_{uv\notin E(G)} (d^a(u) + d^a(v)).$$

One can observe that

$$N_0(G) = |V(G)|, N_1(G) = Z_0(G) = 2|E(G)|,\ N_2(G) = Z_1(G) = M_1(G) \text{ and}$$
$$N_3(G) = Z_2(G) = F(G).$$

Similarly,

$$\overline{Z_0}(G) = 2\binom{|V(G)|}{2} - 2|E(G)|,\ \overline{Z}_1(G) = \overline{M}_1(G), \text{ and } \overline{Z}_2(G) = \overline{F}(G).$$

Li et al. [**15, 16**] studied the vertex a-Zagreb index in the name general first Zagreb index. The same graph invariant is also being studied by various authors in the name of general zeroth-order Randić index, see [**18, 21, 23, 24**]. For more details, see [**2, 10–13, 17, 20, 25, 27**] and [**28**].

Khalifeh et al. [**14**] obtained the first and second Zagreb indices of the Cartesian, join, composition, disjunction and symmetric difference of two graphs. Ashrafi et al. [**1**] obtained the first and second Zagreb coindices of the Cartesian, join, composition, disjunction and symmetric difference of two graphs. Graovac and Pisanki [**9**] were the first to consider the problem of computing topological indices of product graphs. Furtula and Gutman [**6**], established a few basic properties of the forgotten topological index and show that it can significantly enhance the physico-chemical applicability of the first Zagreb index. The hyper Zagreb indices and coindices of some graph operations are presented in [**22, 26**]. In this paper, the edge a-Zagreb Indices and its coindices of some graph operations, such as generalized hierarchical product, Cartesian Product, join, composition of two graphs are obtained. Using the results obtained here, we deduce the F-indices and its coindices for the above graph operation. Finally, we have computed the edge a-Zagreb Index, F-index and their coindices of some important classes of graphs.

2. Main results

Let G be a graph on n vertices and m edges. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices of G in which two vertices u and v are adjacent in $\overline{G}$ if and only if they are nonadjacent in G. Obviously, $E(G) \cup E(\overline{G}) = E(K_n)$ and $\left|E(\overline{G})\right| = \binom{n}{2} - m$. The degree of a vertex v in G is denoted by $d_G(v)$; the degree of the same vertex in $\overline{G}$ is given by $d_{\overline{G}}(v) = n - 1 - d_G(v)$.

THEOREM 2.1. ([**19**]) *Let G be a simple graph on n vertices and m edges. For all $a \geqslant 1$, we have*

$$Z_a(\overline{G}) = n(n-1)^{a+1} - \sum_{j=0}^{a} \binom{a+1}{j+1}(-1)^j (n-1)^{a-j} Z_j(G).$$

THEOREM 2.2. ([**19**]) *Let G be a simple graph on n vertices and m edges. For all $a \geqslant 1$, we have*

$$\overline{Z}_a(G) = (n-1)Z_{a-1}(G) - Z_a(G).$$

THEOREM 2.3. ([**19**]) *Let $a \geqslant 0$, and let G be a simple graph. Then holds*

$$\overline{Z}_a(\overline{G}) = \sum_{j=0}^{a} \binom{a}{j}(-1)^j (n-1)^{a-j} Z_j(G).$$

3. Generalized hierarchical product and Cartesian product

A graph G with a specified vertex subset $U \subseteq V(G)$ is denoted by $G(U)$. Let G and H be two graphs with a nonempty vertex subset $U \subseteq V(G)$. Then the *generalized hierarchical product,* denoted by $G(U) \sqcap H$, is the graph with the vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent if and only if $g = g' \in U$ and $hh' \in E(H)$ or, $gg' \in E(G)$ and $h = h'$. The *Cartesian product,* $G \,\Box\, H$, of the graphs G and H has the vertex set $V(G \,\Box\, H) = V(G) \times V(H)$ and $(u, x)(v, y)$ is an edge of $G \,\Box\, H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \,\Box\, H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \,\Box\, H$. But in the generalized hierarchical product, to each vertex $u \in U$, there is an isomorphic copy of H and to each vertex $v \in V(G)$, there is an isomorphic copy of G. In particular, if $U = V(G)$, then $G \,\Box\, H = G(U) \sqcap H$.

The following lemma is follows from the structure of the generalized hierarchical product of two graphs.

LEMMA 3.1. *Let G and H be graphs with $S \subseteq V(G)$. Then we have*

(*i*) *The number of vertices of $G(S) \sqcap H$ is $|V(G)|\,|V(H)|$.*

(*ii*) *The number of edges of $G(S) \sqcap H$ is $|E(G)|\,|V(H)| + |E(H)|\,|S|$.*

(*iii*) *If $s \in S$ and $v \in V(H)$, then the degree of a vertex (s, v) in $G(S) \sqcap H$ is $d_{G(S)}(s) + d_H(v)$.*

(*iv*) *If $s \notin S$ and $v \in V(H)$, then the degree of a vertex (s, v) in $G(S) \sqcap H$ is $d_{G(S)}(s)$.*

THEOREM 3.1. *Let G and H be two connected graphs with $S \subseteq V(G)$. Then*

$$Z_a(G(S) \sqcap H) = \sum_{t=0}^{a+1} \binom{a+1}{t} N_t(H) \Big(\sum_{u_i \in S} d_{G(S)}^{a-t+1}(u_i) \Big) + |V(H)| \sum_{u_i \notin S} d_{G(S)}^{a+1}(u_i).$$

Proof. Let S be a nonempty subset of $V(G)$. Hence

$$Z_a(G(S) \sqcap H) = \sum_{(u_i,v_j) \in V(G(S) \sqcap H)} (d_{G(S) \sqcap H}((u_i, v_j)))^{a+1}$$

$$= \sum_{u_i \in S} \sum_{v_j \in V(H)} (d_{G(S)}(u_i) + d_H(v_j))^{a+1} + \sum_{u_i \notin S} \sum_{v_j \in V(H)} (d_{G(S)}(u_i))^{a+1},$$

by Lemma 3.1,

$$= \sum_{u_i \in S} \sum_{v_j \in V(H)} \Big(\sum_{t=0}^{a+1} \binom{a+1}{t} d_{G(S)}^{a-t+1}(u_i) d_H^t(v_j) \Big) + \sum_{u_i \notin S} \sum_{v_j \in V(H)} d_{G(S)}^{a+1}(u_i)$$

$$= \sum_{t=0}^{a+1} \binom{a+1}{t} \Big(\sum_{u_i \in S} d_{G(S)}^{a-t+1}(u_i) \Big) \Big(\sum_{v_j \in V(H)} d_H^t(v_j) \Big) + |V(H)| \sum_{u_i \notin S} d_{G(S)}^{a+1}(u_i)$$

$$= \sum_{t=0}^{a+1} \binom{a+1}{t} N_t(H) \Big(\sum_{u_i \in S} d_{G(S)}^{a-t+1}(u_i) \Big) + |V(H)| \sum_{u_i \notin S} d_{G(S)}^{a+1}(u_i).$$

□

Using Theorem 3.1 in Theorems 2.1 to 2.3, we obtain the following theorem.

Theorem 3.2. *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges respectively. Then*

(i) $Z_a(\overline{G(S) \sqcap H}) =$

$$n_1 n_2 (n_1 n_2 - 1)^{a+1} - \sum_{j=0}^{a} \binom{a+1}{j+1} (-1)^j (n_1 n_2 - 1)^{a-j} \sum_{t=0}^{j+1} \binom{j+1}{t} N_t(H) \Big(\sum_{u_i \in S} d_{G(S)}^{j-t+1}(u_i) \Big) + |V(H)| \sum_{u_i \notin U} d_{G(S)}^{j+1}(u_i).$$

(ii) $\overline{Z}_a(G(S) \sqcap H) = \sum_{t=0}^{a} \binom{a}{t} N_t(H) \Big(\sum_{u_i \in S} d_{G(S)}^{a-t}(u_i) \Big) [(n_1 n_2 - 1) - N_{a+1}(H)] +$

$$|V(H)| \sum_{u_i \notin S} d_{G(S)}^{a}(u_i)[1 + d_G(S)(u_i)].$$

(iii) $\overline{Z}_a(\overline{G(S) \sqcap H}) =$

$$\sum_{j=0}^{a} \binom{a}{j} (-1)^j (n_1 n_2 - 1)^{a-j} \sum_{t=0}^{j+1} \binom{j+1}{t} N_t(H) \Big(\sum_{u_i \in S} d_{G(S)}^{j-t+1}(u_i) \Big) + |V(H)| \sum_{u_i \notin S} d_{G(S)}^{j+1}(u_i).$$

By setting $S = V(G)$ in Theorems 3.1 and 3.2, we obtain the edge a-Zagreb indices of Cartesion product of G and H.

Corollary 3.1. *Let G and H be two graphs. Then*

(i) $Z_a(G \square H) = \sum_{t=0}^{a+1} \binom{a+1}{t} N_{a-t+1}(G) N_t(H).$

(ii) $Z_a(\overline{G\Box H}) = n_1n_2(n_1n_2-1)^{a+1} -$
$\sum_{j=0}^{a}\binom{a+1}{j+1}(-1)^j(n_1n_2-1)^{a-j}\sum_{t=0}^{j+1}\binom{j+1}{t}N_{j-t+1}(G)N_t(H)$.
(iii) $\overline{Z}_a(G\Box H) = \sum_{t=0}^{a}\binom{a}{t}N_{a-t}(G)N_t(H)[(n_1n_2-1) - N_0(G)N_{a+1}(H)]$.
(iv) $\overline{Z}_a(\overline{G\Box H}) = \sum_{j=0}^{a}\binom{a}{j}(-1)^j(n_1n_2-1)^{a-j}\sum_{t=0}^{j+1}\binom{j+1}{t}N_{j-t+1}(G)N_t(H)$.

Using Theorems 3.1 and 3.2, we obtain the first Zagreb index and coindex of $G(S)\sqcap H$.

Corollary 3.2. *Let G and H be two graphs. Then*
(i) $M_1(G(S)\sqcap H) = |V(H)|\,M_1(G(S)) + |S|\,M_1(H) + 4\,|E(H)|\sum_{u_i\in S} d_{G(S)}(u_i)$.
(ii) $\overline{M}_1(G(S)\sqcap H) = n_2\sum_{u_i\in S} d_{G(S)}(u_i)[(n_1n_2-1) - M_1(H)] + 2m_2[(n_1n_2-1) - M_1(H)] + |V(H)|\sum_{u_i\notin S} d_{G(S)}(u_i)(1+d_{G(S)}(u_i))$.

Using Theorems 3.1 and 3.2, we obtain the F-index and F-coindex of $G(S)\sqcap H$.

Corollary 3.3. (i) $F(G(S)\sqcap H) =$
$|V(H)|\,F(G(S)) + |S|\,F(H) + 6\,|E(H)|\sum_{u_i\in S} d^2_{G(S)}(u_i) + 3M_1(H)\sum_{u_i\in S} d_{G(S)}(u_i)$.
(ii) $\overline{F}(G(S)\sqcap H) =$
$n_2\sum_{u_i\in S} d^2_{G(S)}(u_i)[(n_1n_2-1) - F(H)] + 4m_2\sum_{u_i\in S} d_{G(S)}(u_i)[(n_1n_2-1) - F(H)] +$
$M_1(H)[(n_1n_2-1) - F(H)] + |V(H)|\sum_{u_i\notin S} d^2_{G(S)}(u_i)(1+d_{G(S)}(u_i))$.

By setting $S = V(G)$ in Corollaries 3.2 and 3.3, we obtain the following corollary.

Corollary 3.4. *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges respectively. Then*
(i) $M_1(G\Box H) = n_2M_1(G) + n_1M_1(H) + 8m_1m_2$.
(ii) $\overline{M}_1(G\Box H) = 2(n_1m_2 + n_2m_1)(n_1n_2-1) - 8m_1m_2 - n_2M_1(G) - n_1M_1(H)$.
(iii) $F(G\Box H) = n_2F(G) + n_1F(H) + 6m_2M_1(G) + 6m_1M_1(H)$.
(iv) $\overline{F}(G\Box H) = [(n_1n_2-1)n_1 - 6m_1]M_1(G) + [(n_1n_2-1)n_1 - 6m_2]M_1(H) - n_1F(H) - n_2F(G) + 8(n_1n_2-1)m_1m_2$.

As an application of the above results, we give the formulae of the $F-$index of $P_n\Box P_m$ (rectangular grid), $P_n\Box C_m$(C_4nanotube) and $C_n\Box C_m$(C_4 nanotorus). The formulae follows from Corollary 3.4.

Example 3.1. (i) $F(P_n\Box P_m) = 64mn - 74m - 74n + 72$.
(ii) $F(P_n\Box C_m) = 2m(32n-37)$.
(iii) $F(C_n\Box C_m) = 64mn$.
(iv) $\overline{F}(P_n\Box P_m) = (4n^2 - 20n - 8m + 12nm + 8)(n_1n_2 - 8nm + 14n - 1)$.

(v) $\overline{F}(P_n\Box C_m) = 2m(8n-7)(n_1n_2 - 8nm - 1)$.
(vi) $\overline{F}(C_n\Box C_m) = 16nm[(n_1n_2 - 1) - 8nm]$.

EXAMPLE 3.2. Let $T = T[p, q]$ be the molecular graph of a nanotorus. Then $|V(T)| = pq$ and $|E(T)| = \frac{3}{2}pq$. Clearly, $M_1(T) = 9pq$ and $F(T) = 27pq$. For a $q-$multi-walled nanotorus $G = P_n\Box T$, by Corollary 3.4, $F(G) = pq(125n - 122)$ and $\overline{F}(G) = [(n_1n_2 - 1) - nN_3(T)][(4n-6)N_0(T) + (4n-4)N_1(T) + nN_2(T)]$.

Using Corollary 3.1, we obtain the following examples.

EXAMPLE 3.3. (i) $Z_a(K_n\Box K_m) = nm(n+m-2)\sum\limits_{r=0}^{a}\binom{a}{r}(n-1)^{a-r}(m-1)^r$.

(ii) $Z_a(K_n\Box P_m) = \sum\limits_{r=0}^{a}\binom{a}{r}[2n^2(n-1)^{a-r} + 2^r(n+n^2)(n-1)^{a-r}(m-2)]$.

(iii) $Z_a(K_n\Box C_m) = nm\sum\limits_{r=0}^{a}\binom{a}{r}[2^r(n+1)(n-1)^{a-r}]$.

EXAMPLE 3.4. (i) $Z_a(P_n\Box C_m) = m\sum\limits_{r=0}^{a}\binom{a}{r}[2^{2r+3} + 2^{a+2}(n-2)]$.

(ii) $Z_a(C_n\Box C_m) = nm\sum\limits_{r=0}^{a}\binom{a}{r}2^{a+2}$.

(iii) $Z_a(P_n\Box P_m) = \sum\limits_{r=0}^{a}\binom{a}{r}[3(2)^{a-r+1}(n-2)+3(2)^{r+1}(m-2)+2^{a+2}(n-2)(m-2)+8]$.

4. Join

The *join* $G_1 + G_2$ of two graphs G_1 and G_2 is the union $G_1 \cup G_2$ together with all the edges joining $V(G_1)$ and $V(G_2)$. Hence the degree of a vertex v of $G_1 + G_2$ is

$$d_{G_1+G_2}(v) = \begin{cases} d_{G_1}(v) + |V(G_2)|, if\ v \in V(G_1) \\ d_{G_2}(v) + |V(G_1)|, if\ v \in V(G_2). \end{cases}$$

In general, for k graphs $G_1, G_2, \ldots, G_k$, the degree of a vertex v in $G_1 + G_2 + \ldots + G_k$ is given by $d_{G_1+G_2+\ldots+G_k}(v) = d_{G_i}(v) + |V(G)| - |V(G_i)|$, where v is originally a vertex of the graph G_i.

THEOREM 4.1. *Let G_i be k graphs, then for $G = G_1 + G_2 + \ldots + G_k$, $|V(G)| = |V(G_1)| + \ldots + |V(G_k)|$ then*
$Z_a(G) = \sum\limits_{i=1}^{n}\sum\limits_{r=0}^{a}\binom{a}{r}(|V|-|V_i|)^r Z_{a-r}(G_i) + \frac{1}{2}\sum\limits_{\substack{i\neq j\\ i,j=1}}^{n}|V_j|\sum\limits_{r=0}^{a}\binom{a}{r}(|V|-|V_i|)^r N_{a-r}(G_i) +$
$\frac{1}{2}\sum\limits_{\substack{i\neq j\\ i,j=1}}^{n}|V_i|\sum\limits_{r=0}^{a}\binom{a}{r}(|V|-|V_i|)^r N_{a-r}(G_j)$.

PROOF. By the definition of Z_a

$$Z_a(G) = \sum_{uv\in E(G)}(d_G^a(u) + d_G^a(v))$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{uv\in E(G_i)} \Big[(d_{G_i}(u)+|V|-|V_i|)^a + (d_{G_i}(v)+|V|-|V_i|)^a\Big] \\
&\quad + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j=1}}^{n} \sum_{u\in V_i}\sum_{v\in V_j} \Big[(d_{G_i}(u)+|V|-|V_i|)^a + (d_{G_j}(v)+|V|-|V_i|)^a\Big] \\
&= \sum_{i=1}^{n} \sum_{uv\in E(G_i)} \Big[\sum_{r=0}^{a}\tbinom{a}{r} d_{G_i}^{a-r}(u)(|V|-|V_i|)^r + \sum_{r=0}^{a}\tbinom{a}{r} d_{G_i}^{a-r}(v)(|V|-|V_i|)^r\Big] \\
&\quad + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j=1}}^{n} \sum_{u\in V_i}\sum_{v\in V_j} \Big[\sum_{r=0}^{a}\tbinom{a}{r} d_{G_i}^{a-r}(u)(|V|-|V_i|)^r + \sum_{r=0}^{a}\tbinom{a}{r} d_{G_j}^{a-r}(v)(|V|-|V_i|)^r\Big] \\
&= \sum_{i=1}^{n} \sum_{uv\in E(G_i)} \sum_{r=0}^{a}\tbinom{a}{r}(|V|-|V_i|)^r (d_{G_i}^{a-r}(u)+d_{G_i}^{a-r}(v)) \\
&\quad + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j=1}}^{n} |V_j| \sum_{r=0}^{a}\tbinom{a}{r}(|V|-|V_i|)^r \sum_{u\in V_i} d_{G_i}^{a-r}(u) \\
&\quad + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j=1}}^{n} |V_i| \sum_{r=0}^{a}\tbinom{a}{r}(|V|-|V_i|)^r \sum_{v\in V_j} d_{G_j}^{a-r}(v) \\
&= \sum_{i=1}^{n}\sum_{r=0}^{a}\tbinom{a}{r}(|V|-|V_i|)^r Z_{a-r}(G_i) + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j=1}}^{n} |V_j| \sum_{r=0}^{a}\tbinom{a}{r}(|V|-|V_i|)^r N_{a-r}(G_i) \\
&\quad + \frac{1}{2}\sum_{\substack{i\neq j\\ i,j=1}}^{n} |V_i| \sum_{r=0}^{a}\tbinom{a}{r}(|V|-|V_i|)^r N_{a-r}(G_j).
\end{aligned}
$$

□

Using Theorem 4.1, we obtain the following theorem.

THEOREM 4.2. *Let G_1 and G_2 be two graphs with n_1 and n_2 vertices respectively.* $Z_a(G_1+G_2) = \sum_{r=0}^{a}\binom{a}{r}\Big[n_2^r Z_{a-r}(G_1) + n_1^r Z_{a-r}(G_2) + \frac{n_1^r+n_2^r}{2}(n_2 N_{a-r}(G_1) + n_1 N_{a-r}(G_2))\Big]$

Using Theorem 4.2 in Theorems 2.1 to 2.3, we obtain the following theorem.

THEOREM 4.3. *Let G be a simple graph on n vertices and m edges. Then*

(i) $Z_a(\overline{G_1+G_2}) = (n_1+n_2)(n_1+n_2-1)^{a+1} - \sum_{j=0}^{a}\binom{a+1}{j+1}(-1)^j(n_1+n_2-1)^{a-j}\sum_{r=0}^{j}\binom{j}{r}\Big[n_2^r Z_{j-r}(G_1) + n_1^r Z_{j-r}(G_2) + \frac{n_1^r+n_2^r}{2}(n_2 N_{j-r}(G_1) + n_1 N_{j-r}(G_2))\Big]$.

(ii) $\overline{Z}_a(G_1+G_2) = (n_1+n_2-1)\sum_{r=0}^{a-1}\binom{a-1}{r}\Big[n_2^r Z_{a-r-1}(G_1) + n_1^r Z_{a-r-1}(G_2) + \frac{n_1^r+n_2^r}{2}$

$(n_2 N_{a-r-1}(G_1) + n_1 N_{a-r-1}(G_2))\Big] - \sum_{r=0}^{a} \binom{a}{r} \Big[n_2^r Z_{a-r}(G_1) + n_1^r Z_{a-r}(G_2) + \frac{n_1^r + n_2^r}{2}$
$(n_2 N_{a-r}(G_1) + n_1 N_{a-r}(G_2))\Big]$.

(*iii*) $\overline{Z}_a(\overline{G_1 + G_2}) = \sum_{j=0}^{a} \binom{a}{j} (-1)^j (n_1 + n_2 - 1)^{a-j} \sum_{r=0}^{j} \binom{j}{r} \Big[n_2^r Z_{j-r}(G_1) + n_1^r Z_{j-r}(G_2) +$
$\frac{n_1^r + n_2^r}{2} (n_2 N_{j-r}(G_1) + n_1 N_{j-r}(G_2))\Big]$.

Define $ID(G) = \sum_{v \in V(G)} \frac{1}{d_G(v)}$ and $ID^{'}(G) = \sum_{e=uv \in E(G)} [\frac{1}{d_G(u)} + \frac{1}{d_G(v)}]$. Note that $N_{-1}(G) = ID(G)$ and $Z_{-1}(G) = ID^{'}(G)$.

Using Theorems 4.2 and 4.3, we have following corollaries.

Corollary 4.1. (*i*) $M_1(G_1 + G_2) = M_1(G_1) + M_1(G_2) + 4(n_2 m_1 + n_1 m_2) + n_1 n_2 (n_1 + n_2)$.
(*ii*) $\overline{M}_1(G_1 + G_2) = (n_1 + n_2 - 1)[ID^{'}(G_1) + ID^{'}(G_2) + n_2 ID(G_1) + n_1 ID(G_2)] - M_1(G_1) - M_1(G_2) - 4(n_2 m_1 + n_1 m_2) - n_1 n_2 (n_1 + n_2)$.

Corollary 4.2. (*i*) $F(G_1 + G_2) = 3n_2 M_1(G_1) + 3n_1 M_1(G_2) + F(G_1) + F(G_2) + 2(n_1 + n_2)(n_2 m_1 + n_1 m_2) + 2n_2^2 m_1 + 2n_1^2 m_2 + n_1 n_2 (n_1^2 + n_2^2)$.
(*ii*) $\overline{F}(G_1 + G_2) = (n_1 - 2n_2 - 1) M_1(G_1) + (n_2 - 2n_1 - 1) M_1(G_2) - F(G_1) - F(G_2) - 2(n_1 + n_2)(n_2 m_1 + n_1 m_2) + 4(n_1 + n_2 - 1)(n_2 m_1 + n_1 m_2) + n_1 n_2 (n_1 + n_2)(n_1 + n_2 - 1) - 2n_2^2 m_1 - 2n_1^2 m_2 - n_1 n_2 (n_1^2 + n_2^2)$.

Corollary 4.3. *Let n and m be the number of vertices and edges of G, respectively. Then the edge $a-$Zagreb indices of suspension of G is given by*
$Z_a(G + K_1) = \sum_{r=0}^{a} \binom{a}{r} \Big[Z_{a-r}(G) + \frac{n^r + 1}{2} N_{a-r}(G) \Big]$.

Example 4.1. The wheel graph W_n on $(n+1)$ vertices is the suspension of C_n and the fan graph F_n on $(n+1)$ vertices is the suspension of P_n. So the edge $a-$Zagreb indices are given by

(*i*) $Z_a(P_n + K_1) = \sum_{r=0}^{a} \binom{a}{r} \Big[2 + (n-2)2^{a-r+1} + \frac{n^r + 1}{2}(2 + (n-2)2^{a-r}) \Big]$.

(*ii*) $Z_a(C_n + K_1) = \sum_{r=0}^{a} \binom{a}{r} \Big[n2^{a-r+1} + \frac{n^r + 1}{2}(n2^{a-r}) \Big]$.

Example 4.2. The Cone graph $C_{n,m}$ is defined as $C_n + \overline{K}_m$. So the edge $a-$Zagreb indices are given by

(*i*) $Z_a(C_n + \overline{K}_m) = \sum_{r=0}^{a} \binom{a}{r} \Big[nm^r 2^{a-r+1} + nm2^{a-r-1}(n^r + m^r) \Big]$.

(*ii*) $M_1(C_n + \overline{K}_m) = 4n + 4nm + 2^{-1} nm(n + m)$.

(*iii*) $F(C_n + \overline{K}_m) = 8n + 2nm(n + 2m + 6) + 2^{-1} nm(n^2 + m^2)$.

(iv) $\overline{Z}_a(C_n + \overline{K}_m) = (n + m - 1)\sum_{r=0}^{a-1}\binom{a-1}{r}\Big[nm^r 2^{a-r} + nm2^{a-r-2}(n^r + m^r)\Big] - \sum_{r=0}^{a}\binom{a}{r}\Big[nm^r 2^{a-r+1} + nm2^{a-r-1}(n^r + m^r)\Big]$.

(v) $\overline{M}_1(C_n + \overline{K}_m) = (n + m - 1)(n + 2^{-1}nm) - 4n - 4nm - 2^{-1}nm(n + m)$.

(vi) $\overline{F}(C_n + \overline{K}_m) = (n + m - 1)(4n + 4nm + 2^{-1}nm(n + m)) - 8n - 2nm(n + 2m + 6) - 2^{-1}nm(n^2 + m^2)$.

EXAMPLE 4.3. (i) $Z_a(G + K_m) = \sum_{r=0}^{a}\binom{a}{r}\Big[m^r Z_{a-r}(G) + n^r m(m-1)^{a-r+1} + \frac{n^r+m^r}{2}(mN_{a-r}(G) + nm(m-1)^{a-r})\Big]$.

(ii) $\overline{Z}_a(G + K_m) = (n + m - 1)\sum_{r=0}^{a-1}\binom{a-1}{r}\Big[m^r Z_{a-r-1}(G) + n^r m(m-1)^{a-r} + \frac{n^r+m^r}{2} (mN_{a-r-1}(G) + nm(m-1)^{a-r-1})\Big] - \sum_{r=0}^{a}\binom{a}{r}\Big[m^r Z_{a-r}(G) + n^r m(m-1)^{a-r+1} + \frac{n^r+m^r}{2}(mN_{a-r}(G) + nm(m-1)^{a-r})\Big]$.

EXAMPLE 4.4. (i) $\overline{Z}_a(P_n + K_1) = n\sum_{r=0}^{a-1}\binom{a-1}{r}\Big[2 + (n-2)2^{a-r} + \frac{n^r+1}{2}(2 + (n-2)2^{a-r-1})\Big] - \sum_{r=0}^{a}\binom{a}{r}\Big[2 + (n-2)2^{a-r+1} + \frac{n^r+1}{2}(2 + (n-2)2^{a-r})\Big]$.

(ii) $\overline{Z}_a(C_n + K_1) =$
$n\sum_{r=0}^{a-1}\binom{a-1}{r}\Big[n2^{a-r} + \frac{n^r+1}{2}(n2^{a-r-1})\Big] - \sum_{r=0}^{a}\binom{a}{r}\Big[n2^{a-r+1} + \frac{n^r+1}{2}n2^{a-r}\Big]$.

5. Composition

The *composition* of two graphs G_1 and G_2 is denoted by $G_1[G_2]$. The vertex set of $G_1[G_2]$ is $V(G_1) \times V(G_2)$ and the degree of a vertex (u, v) of $G_1[G_2]$ is given by $d_{G_1[G_2]}((u, v)) = n_2 d_{G_1}(u) + d_{G_2}(v)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1v_1 \in E(G_1)$ or $[u_1 = v_1$ *and* $u_2v_2 \in E(G_2)]$.

THEOREM 5.1. *Let G_1 and G_2 be two graphs. Then*

$$Z_a(G_1[G_2]) = \sum_{r=0}^{a}\binom{a}{r}n_2^{a-r}\Big[N_{a-r}(G_1)Z_r(G_2) + n_2 N_r(G_2)Z_{a-r}(G_1)\Big].$$

PROOF. From the structure of $G_1[G_2]$, the degree of a vertex (u, v) of $G_1[G_2]$ is given by $d_{G_1[G_2]}((u, v)) = n_2 d_{G_1}(u) + d_{G_2}(v)$ and by definition of Za, we have

$$Z_a(G_1[G_2]) = \sum_{w\in V(G_1)}\sum_{uv\in E(G_2)}\Big[(n_2 d_{G_1}(w) + d_{G_2}(u))^a + (n_2 d_{G_1}(w) + d_{G_2}(v))^a\Big]$$
$$+ \sum_{xy\in E(G_1)}\sum_{v\in V(G_2)}\sum_{u\in V(G_2)}\Big[(n_2 d_{G_1}(x) + d_{G_2}(u))^a + (n_2 d_{G_1}(y) + d_{G_2}(v))^a\Big]$$

$$= \sum_{w\in V(G_1)} \sum_{uv\in E(G_2)} \Big[\sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} d_{G_1}^{a-r}(w) d_{G_2}^{r}(u) + \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} d_{G_1}^{a-r}(w) d_{G_2}^{r}(v)\Big]$$

$$+ \sum_{xy\in E(G_1)} \sum_{v\in V(G_2)} \sum_{u\in V(G_2)} \Big[\sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} d_{G_1}^{a-r}(x) d_{G_2}^{r}(u)$$

$$+ \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} d_{G_1}^{a-r}(y) d_{G_2}^{r}(v)\Big]$$

$$= \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r}\Big(\sum_{w\in V(G_1)} d_{G_1}^{a-r}(w)\Big)\Big(\sum_{uv\in E(G_2)} d_{G_2}^{r}(u) + d_{G_2}^{r}(v)\Big)$$

$$+ \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} n_2 \sum_{xy\in E(G_1)} d_{G_1}^{a-r}(x) \sum_{u\in V(G_2)} d_{G_2}^{r}(u)$$

$$+ \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} n_2 \sum_{xy\in E(G_1)} d_{G_1}^{a-r}(y) \sum_{v\in V(G_2)} d_{G_2}^{r}(v)$$

$$= \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r} N_{a-r}(G_1) Z_r(G_2) + \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r+1} N_r(G_2) Z_{a-r}(G_1)$$

$$= \sum_{r=0}^{a}\tbinom{a}{r} n_2^{a-r}\Big[N_{a-r}(G_1) Z_r(G_2) + n_2 N_r(G_2) Z_{a-r}(G_1)\Big].$$

□

Using Theorem 5.1 in Theorems 2.1 to 2.3, we obtain the following theorem.

THEOREM 5.2. *Let G be a simple graph on n vertices and m edges. Then*

(i) $Z_a(\overline{G_1[G_2]}) = n_1 n_2 (n_1 n_2 - 1)^{a+1} - \sum_{j=0}^{a}\binom{a+1}{j+1}(-1)^j (n_1 n_2 - 1)^{a-j} \sum_{r=0}^{j}\binom{j}{r} n_2^{j-r}$ $\Big[N_{j-r}(G_1) Z_r(G_2) + n_2 N_r(G_2) Z_{j-r}(G_1)\Big]$.

(ii) $\overline{Z}_a(G_1[G_2]) = (n_1 n_2 - 1)\sum_{r=0}^{a-1}\binom{a-1}{r} n_2^{a-r-1}\Big[N_{a-r-1}(G_1) Z_r(G_2)$ $+ n_2 N_r(G_2) Z_{a-r-1}(G_1)\Big] - \sum_{r=0}^{a}\binom{a}{r} n_2^{a-r}\Big[N_{a-r}(G_1) Z_r(G_2) + n_2 N_r(G_2) Z_{a-r}(G_1)\Big]$.

(iii) $\overline{Z}_a(\overline{G_1[G_2]}) = \sum_{j=0}^{a}\binom{a}{j}(-1)^j (n_1 n_2 - 1)^{a-j} \sum_{r=0}^{j}\binom{j}{r} n_2^{j-r}\Big[N_{j-r}(G_1) Z_r(G_2)$ $+ n_2 N_r(G_2) Z_{j-r}(G_1)\Big]$.

Using Theorems 5.1 and 5.2, we obtain the following examples.

EXAMPLE 5.1. *(i)* $Z_a(K_n[K_m]) = nm(n+m-2)\sum_{r=0}^{a}\binom{a}{r} n_2^{a-r}(n-1)^{a-r}(m-1)^r$.

(ii) $Z_a(P_n[C_m]) = m\sum_{r=0}^{a}\binom{a}{r} n_2^{a-r}[(n_2+2)2^{r+1} + 2^{a+1}(n-2)(n_2+1)]$.

(iii) $Z_a(P_n[P_m]) = \sum_{r=0}^{a} \binom{a}{r} n_2^{a-r}[4(n_2+1) + (2)^{a-r+1}(n-2)(2n_2+1) + (2)^{r+1}(n_2 + 2)(m-2) + 2^{a+1}(n_2+1)(n-2)(m-2)]$.

(iv) $Z_a(C_n[C_m]) = nm \sum_{r=0}^{a} \binom{a}{r} n_2^{a-r}[2^{a+1}(n_2+1)]$.

(v) $Z_a(K_n[P_m]) = \sum_{r=0}^{a} \binom{a}{r} n(n-1)^{a-r} n_2^{a-r}[2n_2(n-1) + 2^r(m-2)(2+n_2(n-1)) + 2]$.

(vi) $Z_a(K_n[C_m]) = nm(2 + n_2(n-1)) \sum_{r=0}^{a} \binom{a}{r} n_2^{a-r} 2^r (n-1)^{a-r}$.

EXAMPLE 5.2. (i) $M_1(G_1[G_2]) = n_1 M_1(G_2) + n_2^3 M_1(G_1) + 8n_2 m_1 m_2$.

(ii) $M_1(\overline{G_1[G_2]}) = n_1 n_2 (n_1 n_2 - 1)^2 - 4(n_1 n_2 - 1)(n_1 m_2 + n_2^2 m_1) + n_2^3 M_1(G_1) + n_1 M_1(G_2) + 8n_2 m_1 m_2$.

(iii) $\overline{M}_1(\overline{G_1[G_2]}) = 2(n_1 n_2 - 1)(n_1 m_2 + n_2^2 m_1) - n_2^2 M_1(G_1) - n_1 M_1(G_2) - 8n_2 m_1 m_2$.

EXAMPLE 5.3. (i) $F(G_1[G_2]) =$
$6n_2^2 m_2 M_1(G_1) + 6n_2 m_1 M_1(G_2) + n_2^4 F(G_1) + n_1 F(G_2)$.

(ii) $F(\overline{G_1[G_2]}) =$
$n_1 n_2 (n_1 n_2 - 1)^3 - 6(n_1 n_2 - 1)^2 (n_1 m_2 + n_2^2 m_1) +$
$24(n_1 n_2 - 1) n_2 m_1 m_2 + 3n_2^2 M_1(G_1)[(n_1 n_2 - 1) n_2 - 2m_2] - n_2^4 F(G_1) - n_1 F(G_2) +$
$3M_1(G_2)[(n_1 n_2 - 1) n_1 - 2n_2 m_1]$.

(iii) $\overline{F}(G_1[G_2]) = n_2^2((n_1 n_2 - 1) n_2 - 6m_2) M_1(G_1) + ((n_1 n_2 - 1) n_1 - 6n_2 m_1) M_1(G_2) - n_2^4 F(G_1) - n_1 F(G_2) + 8(n_1 n_2 - 1) n_2 m_1 m_2$.

(iv) $\overline{F}(\overline{G_1[G_2]}) = 2(n_1 n_2 - 1)^2 (n_1 m_2 + n_2^2 m_1) - 16(n_1 n_2 - 1) n_2 m_1 m_2 - 2((n_1 n_2 - 1) n_2 - 3m_2) n_2^2 M_1(G_1) - 2((n_1 n_2 - 1) n_1 - 3n_2 m_1) M_1(G_2) + n_2^4 F(G_1) + n_1 F(G_2)$.

EXAMPLE 5.4. (i) $M_1(K_n[K_m]) = nm(n+m-2)[n_2(n-1) + m - 1]$.

(ii) $M_1(P_n[C_m]) = m[2(n_2+2)^2 + 4(n-2)(n_2+1)^2]$.

(iii) $M_1(P_n[P_m]) =$
$4(n_2+1)^2 + 2(n-2)(2n_2+1)^2 + 2(m-2)(3(n_2+2) + 4(n_2+1)(n-2))$.

(iv) $M_1(C_n[C_m]) = 4nm(n_2+1)^2$.

(v) $M_1(K_n[P_m]) = 2n((n-1)n_2+1)^2 + n(m-2)((n-1)n_2+2)^2$.

(vi) $M_1(K_n[C_m]) = nm(n_2(n-1)+2)^2$.

EXAMPLE 5.5. (i) $F(K_n[K_m]) =$
$nm(n+m-2)[n_2(n-1)(n_2(n-1) + 2(m-1)) + (m-1)^2]$.

(ii) $F(P_n[C_m]) = m[2(n_2+2)^3 + 8(n-2)(n_2+1)^3]$.

(iii) $F(P_n[P_m]) =$
$4(n_2+1)^3 + 2(n-2)(2n_2+1)^3 + 2(n_2+2)^3(m-2) + 8(n_2+1)^3(n-2)(m-2)$.

(iv) $F(C_n[C_m]) = 8nm(n_2+1)^3$.

(v) $F(K_n[P_m]) =$
$2n(n-1)^2 n_2^2((n-1)n_2+3) + 6n(n-1)n_2((n-1)n_2+1) + n(m-2)(n_2(n-1)+2)^3 + 2n$.

(vi) $F(K_n[C_m]) = nm(n_2(n-1)+2)^3$.

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Received by editors 08.09.2016; Revised version 10.10.2016: Second revised version 16.11.2016: Available online 21.11.2016.

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