BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(2017), 267-278
DOI: 10.7251/BIMVI1702267R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

PREŠIĆ TYPE FIXED POINT THEOREM FOR SIX MAPS IN D^*- METRIC SPACES

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Abstract. In this paper, we obtain a Prešic type fixed point theorem for three pairs of jointly $3k$-weakly compatible maps in D^*-metric spaces.We also present an example to illustrate our main theorem. We also obtain four corollaries for four maps, three maps,two maps and a single map. We also give some probable modifications of Theorems of [**5, 12, 13**] in G-metric spaces.

1. Introduction and Preliminaries

In 1922, Banach [**6**] proved a theorem which is known as Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Later many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In 1992, Dhage [**1**] introduced generalized metric space or D-metric space and proved several results.

Naidu et al [**9**], [**10**], [**11**] observed that almost all fixed point theorems in D-metric spaces are not valid or of doubtful validity and modified some fixed point theorems in D- metric spaces. As a probable modification of D- metric spaces, Sedghi et al. [**8**] introduced D^*- metric spaces and Mustafa et al. [**14**] introduced G-metric spaces.

On the other hand, amongst the various generalizations of Banach contraction principle, Prešić [**7**] gave a contractive condition and proved a Banach type fixed point theorem which was useful to solve certain difference equations.Throughout this paper $\mathcal{N}$ denotes the set of all positive integers.

Actually Prešić [**7**] proved the following.

2010 *Mathematics Subject Classification.* 47H10; 54E50; 54H25.

Key words and phrases. D^*-Complete metric spaces, Prešić type theorem, Jointly $3k$-weakly compatible mappings.

THEOREM 1.1. ([**7**]) *Let* (X, d) *be a complete metric space,* k *a positive integer and* $f : X^k \to X$*. Suppose that*

$$d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \leqslant \sum_{i=1}^{k} q_i d(x_i, x_{i+1})$$

holds for all $x_1, x_2, \cdots, x_k, x_{k+1} \in X$*, where* $q_i \geqslant 0$ *and* $\sum_{i=1}^{k} q_i \in [0, 1)$*. Then* f *has a unique fixed point* x^**. Moreover, for any arbitrary points* $x_1, x_2, \cdots, x_{k+1}$ *in* X*, the sequence* $\{x_n\}$ *defined by* $x_{n+k} = f(x_n, x_{n+1}, \cdots, x_{n+k-1})$*, for all* $n \in \mathcal{N}$ *converges to* x^**.*

Later Ćirić and Prešić [**4**] generalized the above theorem as follows.

THEOREM 1.2 ([**4**]). *Let* (X, d) *be a complete metric space,* k *a positive integer and* $f : X^k \to X$*. Suppose that*

$$d(f(x_1, x_2, \cdots, x_k), f(x_2, x_3, \cdots, x_{k+1})) \leqslant \lambda \max\{d(x_i, x_{i+1}) : 1 \leqslant i \leqslant k\}$$

holds for all $x_1, x_2, \cdots, x_k, x_{k+1}$ *in* X*, where* $\lambda \in [0, 1)$*. Then* f *has a fixed point* $x^* \in X$*. Moreover, for any arbitrary points* $x_1, x_2, \cdots, x_{k+1}$ *in* X*, the sequence* $\{x_n\}$ *defined by* $x_{n+k} = f(x_n, x_{n+1}, \cdots, x_{n+k-1})$*, for all* $n \in \mathcal{N}$ *converges to* x^**. Moreover, if* $d(f(u, u, \cdots, u), f(v, v, \cdots, v)) < d(u, v)$ *holds for all* $u, v \in X$ *with* $u \neq v$*, then* x^* *is the unique fixed point of* f*.*

Recently Rao et al. [**2**], [**3**] obtained some Prešić type theorems for two and three maps in metric spaces. Now we give the following definition of [**2**], [**3**].

DEFINITION 1.1. *Let* X *be a non empty set and* $T : X^{2k} \to X$ *and* $f : X \to X$*. The pair* (f, T) *is said to be* $2k$*-weakly compatible if* $f(T(x, x, ..., x, x)) = T(fx, fx, ..., fx, fx)$ *whenever* $x \in X$ *such that* $fx = T(x, x, ..., x, x)$*.*

Using this definition, Rao et al. [**2**] proved the following

THEOREM 1.3 ([**2**]). *Let* (X, d) *be a metric space,* k *a positive integer and* $S, T : X^{2k} \to X, f : X \to X$ *be mappings satisfying*

$$\text{(1.3.1)} \quad d\begin{pmatrix} S(x_1, x_2, \cdots, x_{2k}), \\ T(x_2, x_3, \cdots, x_{2k+1}) \end{pmatrix} \leqslant \lambda \max\{d(fx_i, fx_{i+1}) : 1 \leqslant i \leqslant 2k\}$$
for all $x_1, x_2, \cdots, x_{2k}, x_{2k+1}$ *in* X*,*

$$\text{(1.3.2)} \quad d\begin{pmatrix} T(y_1, y_2, \cdots, y_{2k}), \\ S(y_2, y_3, \cdots, y_{2k+1}) \end{pmatrix} \leqslant \lambda \max\{d(fy_i, fy_{i+1}) : 1 \leqslant i \leqslant 2k\}$$
for all $y_1, y_2, \cdots, y_{2k}, y_{2k+1}$ *in* X*, where* $0 < \lambda < 1$

(1.3.3) $d(S(u, \cdots, u), T(v, \cdots, v)) < d(fu, fv)$*, for all* $u, v \in X$ *with* $u \neq v$

(1.3.4) *Suppose that* $f(X)$ *is complete and either* (f, S) *or* (f, T) *is a* $2k-$ *weakly compatible pair.*

Then there exists a unique point $p \in X$ *such that* $fp = p = S(p, \cdots, p) = T(p, \cdots, p)$*.*

In this paper we obtain a Prešić type common fixed point theorem for six mappings in D^*-metric spaces and present an example to illustrate our main theorem.

Now we give some known definitions and lemmas which are useful for further discussion.

Definition 1.2 ([**14**]). *Let X be a non-empty set and let $G : X^3 \to \mathcal{R}^+$ be a function satisfying the following properties :*

(G_1): *$G(x, y, z) = 0$ if and only if $x = y = z$,*

(G_2): *$0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,*

(G_3): *$G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,*

(G_4): *$G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables,*

(G_5): *$G(x, y, z) \leqslant G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.*

Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Recently Dhasmana [**5**] and Gairola et al. [**12**], [**13**] proved Prešić type fixed and common fixed point theorems in G-metric spaces. They are the following theorems

Theorem 1.4 (Theorem 2.1, [**5**]). *Let (X, G) be a complete G-metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$(1.4.1) \quad G\begin{pmatrix} T(x_1, x_2, \cdots, x_k), \\ T(x_2, x_3, \cdots, x_{k+1}), \\ T(x_3, x_4, \cdots, x_{k+2}) \end{pmatrix} \leqslant \lambda \max\{G(x_i, x_{i+1}, x_{i+2}) : 1 \leqslant i \leqslant k\}$$

where $\lambda \in (0, 1)$ is constant and $x_1, x_2, \cdots, x_{k+2}$ are arbitrary elements in X. Then there exists a point x in X such that $x = T(x, x, \cdots, x)$.

Moreover, if $x_1, x_2, \cdots, x_k$ are arbitrary points in X and for $n \in \mathcal{N}$, $x_{n+k} = T(x_n, x_{n+1}, \cdots, x_{n+k-1})$ then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \cdots, \lim x_n)$.

If, in addition we suppose that on diagonal $\Delta \subset x^k$,

$$G\begin{pmatrix} T(u, u, \cdots, u), \\ T(v, v, \cdots, v), \\ T(w, w, \cdots, w) \end{pmatrix} < G(u, v, w)$$

holds for all $u, v, w \in X$ with $u \neq v \neq w$, then x is unique point in X with $T(x, x, \cdots, x) = x$.

Theorem 1.5 (Theorem 3.1, [**12**]). *Let (X, G) be a G-metric space, k a positive integer and $T : X^k \to X$, $f : X \to X$ be mappings satisfying the following conditions*

$(1.5.1) \quad T(X^k) \subseteq f(X)$,

$$(1.5.2)\ G\begin{pmatrix} T(x_1, x_2, \cdots, x_k), \\ T(x_2, x_3, \cdots, x_{k+1}), \\ T(x_3, x_4, \cdots, x_{k+2}) \end{pmatrix} \leqslant \lambda \max\{G(fx_i, fx_{i+1}, fx_{i+2})/1 \leqslant i \leqslant k\}$$

for all $x_1, x_2, \cdots, x_{k+2} \in X$, where $0 \leqslant \lambda < 1$;

$(1.5.3) \quad d(T(u, \cdots, u), T(v, \cdots, v), T(w, \cdots, w)) < G(fu, fv, fw)$, *for all* $u, v, w \in X$ *with* $u \neq v \neq w$,

(1.5.4) $f(X)$ *is* G*-complete and if the pair* (f, T) *is coincidently commuting.*

Then there exist a unique point $p \in X$ *such that* $fp = p = T(p, p, \cdots, p)$.

THEOREM 1.6 (Theorem 3.1, [**13**]). *Let* (X, G) *be a* G*-metric space,* k *a positive integer and* $S, T, R : X^k \to X$, $f : X \to X$ *be mappings satisfying the following conditions*

(1.6.1) $S(X^k) \cup T(X^k) \cup R(X^k) \subseteq f(X)$,

(1.6.2) $G\begin{pmatrix} S(x_1, x_2, \cdots, x_k), \\ T(x_2, x_3, \cdots, x_{k+1}), \\ R(x_3, x_4, \cdots, x_{k+2}) \end{pmatrix} \leqslant \lambda \max\{G(fx_i, fx_{i+1}, fx_{i+2}), 1 \leqslant i \leqslant k\}$

for all $x_1, x_2, \cdots, x_{k+2} \in X$,

(1.6.3) $G\begin{pmatrix} T(y_1, y_2, \cdots, y_k), \\ R(y_2, y_3, \cdots, y_{k+1}), \\ S(y_3, y_4, \cdots, y_{k+2}) \end{pmatrix} \leqslant \lambda \max\{G(fy_i, fy_{i+1}, fy_{i+2}), 1 \leqslant i \leqslant k\}$

for all $y_1, y_2, \cdots, y_{k+2} \in X$,

(1.6.4) $G\begin{pmatrix} R(z_1, z_2, \cdots, z_k), \\ S(z_2, z_3, \cdots, z_{k+1}), \\ T(z_3, z_4, \cdots, z_{k+2}) \end{pmatrix} \leqslant \lambda \max\{G(fz_i, fz_{i+1}, fz_{i+2}), 1 \leqslant i \leqslant k\}$

for all $z_1, z_2, \cdots, z_{k+2} \in X$,

(1.6.5) $d(S(u, \cdots, u), T(v, \cdots, v), R(w, \cdots, w)) < G(fu, fv, fw)$,

for all $u, v, w \in X$ *with* $u \neq v \neq w$.

Suppose that $f(X)$ *is complete and one of* (f, S), (f, T) *or* (f, R) *is coincidentally commuting pair. Then there exists a unique point* $p \in X$ *such that* $fp = p = S(p, p, \cdots, p) = T(p, p, \cdots, p) = R(p, p, \cdots, p)$.

We observed that in these theorems the authors [**5, 12, 13**] wrongly used the condition (G_3) in proving Cauchy sequences.

We also observed the following:

(*i*) In Theorem 2.1 of [**5**], the condition (2.1.2) is also wrongly used. In Page 13 line 21 from below $y \neq x \neq z, \cdots$, which is a contradiction. From this we can not conclude $y = x = z$ only. There are some more possibilities namely $x = y$ or $y = z$ or $x = z$.

(*ii*) In Theorem 3.1 of [**12**], the condition (3.3) is wrongly used two times. In Page 199, line 5 from above and line 10 from above.

(*iii*) In Theorem 3.1 of [**13**], the condition (5) is wrongly used two times. In line 3 from below of Page 403 and line 4 from above of Page 404.

DEFINITION 1.3 ([**8**]). *Let* X *be a non-empty set and* $D^* : X^3 \to \mathcal{R}^+$ *be a function satisfying :*

(1.3.1)*:* $D^*(x, y, z) = 0$ *if and only if* $x = y = z$,

(1.3.2)*:* $D^*(x, y, z) = D^*(p\{x, y, z\})$, *where* p *is a permutation function,* (1.3.3): $D^*(x,y,z) \leqslant D^*(x,y,a) + D^*(a,z,z)$

Then the function D^* is called a D^*- metric and the pair (X, D^*) is called a D^*-metric space.

REMARK 1.1 ([**8**]). *In a D^*-metric space, we have $D^*(x,x,y) = D^*(x,y,y)$ for all $x, y \in X$.*

DEFINITION 1.4 ([**8**]). *Let (X, D^*) be a D^*-metric space. For $r > 0$, define $B_{D^*}(x,r) = \{y \in X : D^*(x,y,y) < r\}$*

DEFINITION 1.5 ([**8**]). *Let (X, D^*) be a D^*-metric space.*

(i) If for every $x \in A \subset X$, there exists $r > 0$ such that $B_{D^*}(x,r) \subset A$, then A is called an open subset of X.

(ii) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if and only if $\lim_{n\to\infty} D^*(x_n, x_n, x) = 0$.

(iii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if

$$\lim_{n,m\to\infty} D^*(x_n, x_n, x_m) = 0.$$

(iv) (X, D^*) is said to be complete if every cauchy sequence is convergent in X.

LEMMA 1.1 ([**8**]). *Let (X, D^*) be a D^*-metric space. Then D^* is continuous in all its three variables.*

LEMMA 1.2 ([**8**]). *If a sequence $\{x_n\}$ in (X, D^*) converges to $x \in X$ then x is unique. Also $\{x_n\}$ is a Cauchy sequence in X.*

DEFINITION 1.6 ([**8**]). *Let (X, D^*) be a D^*-metric space. Then D^* is called of first type if $D^*(x,x,y) \leqslant D^*(x,y,z)$ for all $x, y, z \in X$.*

Now we give the following definition.

DEFINITION 1.7. *The pairs (S, f), (T, g), (R, h) are jointly $3k$-weakly compatible if*

$f(S(x,x,\cdots,x)) = S(fx, fx, \cdots, fx)$, $g(T(x,x,\cdots,x)) = T(gx, gx, \cdots, gx)$ *and* $h(R(x,x,\cdots,x)) = R(hx, hx, \cdots, hx)$ *whenever there exists $x \in X$ such that* $fx = S(x,x,\cdots,x)$, $gx = T(x,x,\cdots,x)$ *and* $hx = R(x,,x,\cdots,x)$.

Now we are ready to prove our main theorem.

2. Main Results

THEOREM 2.1. *Let (X, D^*) be a complete D^*-metric space where D^* is of first type, k a positive integer and $S, T, R : X^{3k} \to X$ and $f, g, h : X \to X$ be mappings satisfying*

(2.1.1) $S(X^{3k}) \subseteq g(X)$, $T(X^{3k}) \subseteq h(X)$, $R(X^{3k}) \subseteq f(X)$,

(2.1.2) $$D^*(S(x_1, x_2, \cdots, x_{3k}), T(y_1, y_2, \cdots, y_{3k}), R(z_1, z_2, \cdots, z_{3k}))$$
$$\leqslant \lambda \max \left\{ \begin{array}{c} D^*(gx_1, hy_1, fz_1), D^*(hx_2, fy_2, gz_2), \\ D^*(fx_3, gy_3, hz_3), \cdots, D^*(gx_{3k-2}, hy_{3k-2}, fz_{3k-2}), \\ D^*(hx_{3k-1}, fy_{3k-1}, gz_{3k-1}), D^*(fx_{3k}, gy_{3k}, hz_{3k}) \end{array} \right\}$$

for all $x_1, x_2, \cdots, x_{3k}, y_1, y_2, \cdots, y_{3k}, z_1, z_2, \cdots, z_{3k} \in X$ *and* $0 < \lambda < 1$,

(2.1.3) *The pairs* (S, f), (T, g) *and* (R, h) *are jointly* $3k$*-weakly compatible pairs.*

(2.1.4) *Suppose* $z = fu = gu = hu$ *for some* $u \in X$ *whenever there exists a sequence* $\{y_{3k+n}\}_{n=1}^{\infty}$ *in* X *such that* $y_{3k+n} \to z \in X$ *as* $n \to \infty$.

Then z *is the unique point in* X *such that*

$$fz = gz = hz = z = s(z, z, \cdots, z) = T(z, z, \cdots, z) = R(z, z, \cdots, z).$$

Proof. Suppose $x_1, x_2, \cdots, x_{3k}$ are arbitrary points in X. Define

$$\begin{aligned} y_{3k+3n-2} &= S(x_{3n-2}, x_{3n-1}, \cdots, x_{3k+3n-3}) = gx_{3k+3n-2}, \\ y_{3k+3n-1} &= T(x_{3n-1}, x_{3n}, \cdots, x_{3k+3n-2}) = hx_{3k+3n-1}, \\ y_{3k+3n} &= R(x_{3n}, x_{3n+1}, \cdots, x_{3k+3n-1}) = fx_{3k+3n} \ for\ n = 1, 2, \cdots \end{aligned}$$

Let

$$\begin{aligned} \alpha_{3n-2} &= D^*(gx_{3n-2}, hx_{3n-1}, fx_{3n}), \\ \alpha_{3n-1} &= D^*(hx_{3n-1}, fx_{3n}, gx_{3n+1}), \\ \alpha_{3n} &= D^*(fx_{3n}, gx_{3n+1}, hx_{3n+2}),\ n = 1, 2, \cdots \end{aligned}$$

and let $\theta = \lambda^{\frac{1}{3k}}$ and $\mu = \max\{\frac{\alpha_1}{\theta}, \frac{\alpha_2}{\theta^2}, \cdots, \frac{\alpha_{3k}}{\theta^{3k}}\}$. Then $\theta < 1$ and by the selection of μ, we have

(2.1) $$\alpha_n \leqslant \mu\theta^n,\ for\ n = 1, 2, \cdots, 3k.$$

Consider

(2.2)

$$\begin{aligned} \alpha_{3k+1} &= D^*(gx_{3k+1}, hx_{3k+2}, fx_{3k+3}) \\ &= D^*(S(x_1, x_2, \cdots, x_{3k}), T(x_2, x_3, \cdots, x_{3k+1}), R(x_3, x_4, \cdots, x_{3k+2})) \\ &\leqslant \lambda \max \left\{ \begin{array}{c} D^*(gx_1, hx_2, fx_3), D^*(hx_2, fx_3, gx_4), \\ D^*(fx_3, gx_4, hx_5), \cdots, D^*(gx_{3k-2}, hx_{3k-1}, fx_{3k}), \\ D^*(hx_{3k-1}, fx_{3k}, gx_{3k+1}), D^*(fx_{3k}, gx_{3k+1}, hx_{3k+2}) \end{array} \right\} \\ &= \lambda \max\{\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{3k-2}, \alpha_{3k-1}, \alpha_{3k}\} \\ &\leqslant \lambda \max\{\mu\theta, \mu\theta^2, \mu\theta^3, \cdots, \mu\theta^{3k-2}, \mu\theta^{3k-1}, \mu\theta^{3k}\},\ from\ (2.1) \\ &= \lambda\mu\theta = \theta^{3k}\mu\theta \\ &= \mu\theta^{3k+1} \end{aligned}$$

(2.3)

$$\begin{aligned} \alpha_{3k+2} &= D^*(hx_{3k+2}, fx_{3k+3}, gx_{3k+4}) = D^*(gx_{3k+4}, hx_{3k+2}, fx_{3k+3}) \\ &= D^*(S(x_4, x_5, \cdots, x_{3k+3}), T(x_2, x_3, \cdots, x_{3k+1}), R(x_3, x_4, \cdots, x_{3k+2})) \\ &\leqslant \lambda \max \left\{ \begin{array}{c} D^*(gx_4, hx_2, fx_3), D^*(hx_5, fx_3, gx_4), \\ D^*(fx_6, gx_4, hx_5), \cdots, D^*(gx_{3k+1}, hx_{3k-1}, fx_{3k}), \\ D^*(hx_{3k+2}, fx_{3k}, gx_{3k+1}), D^*(fx_{3k+3}, gx_{3k+1}, hx_{3k+2}) \end{array} \right\} \\ &= \lambda \max\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \cdots, \alpha_{3k-1}, \alpha_{3k}, \alpha_{3k+1}\} \\ &\leqslant \lambda \max\{\mu\theta^2, \mu\theta^3, \mu\theta^4, \mu\theta^5, \cdots, \mu\theta^{3k-1}, \mu\theta^{3k}, \mu\theta^{3k+1}\},\ from\ (2.1), (2.2) \\ &= \lambda\mu\theta^2 = \theta^{3k}\mu\theta^2 \\ &= \mu\theta^{3k+2} \end{aligned}$$

(2.4)

$$\begin{aligned}\alpha_{3k+3} &= D^*(fx_{3k+3}, gx_{3k+4}, hx_{3k+5}) = D^*(gx_{3k+4}, hx_{3k+5}, fx_{3k+3}) \\ &= D^*(S(x_4, x_5, \cdots, x_{3k+3}), T(x_5, x_6, \cdots, x_{3k+4}), R(x_3, x_4, \cdots, x_{3k+2})) \\ &\leqslant \lambda \max \left\{ \begin{array}{c} D^*(gx_4, hx_5, fx_3), D^*(hx_5, fx_6, gx_4), \\ D^*(fx_6, gx_7, hx_5), \cdots, D^*(gx_{3k+1}, hx_{3k+2}, fx_{3k}), \\ D^*(hx_{3k+2}, fx_{3k+3}, gx_{3k+1}), D^*(fx_{3k+3}, gx_{3k+4}, hx_{3k+2}) \end{array} \right\} \\ &= \lambda \max\{\alpha_3, \alpha_4, \alpha_5, \cdots, \alpha_{3k}, \alpha_{3k+1}, \alpha_{3k+2}\} \\ &\leqslant \lambda \max\{\mu\theta^3, \mu\theta^4, \mu\theta^5, \cdots, \mu\theta^{3k}, \mu\theta^{3k+1}, \mu\theta^{3k+2}\}, \ \mathit{from}\ (2.1), (2.2), (2.3) \\ &= \lambda\mu\theta^3 = \theta^{3k}\mu\theta^3 \\ &= \mu\theta^{3k+3}.\end{aligned}$$

Continuing in this way, we get

$$\alpha_n \leqslant \mu\theta^n \ \mathit{for\ all}\ n \in \mathcal{N}. \tag{2.5}$$

Consider

$$\begin{aligned}&D^*(y_{3k+3n+1}, y_{3k+3n+2}, y_{3k+3n+3}) \\ &\quad= D^*(gx_{3k+3n+1}, hx_{3k+3n+2}, fx_{3k+3n+3}) \\ &\quad= D^* \left(\begin{array}{c} S(x_{3n+1}, x_{3n+2}, \cdots, x_{3k+3n}), \\ T(x_{3n+2}, x_{3n+3}, \cdots, x_{3k+3n+1}), \\ R(x_{3n+3}, x_{3n+4}, \cdots, x_{3k+3n+2}) \end{array} \right) \\ &\quad\leqslant \lambda \max \left\{ \begin{array}{c} D^*(gx_{3n+1}, hx_{3n+2}, fx_{3n+3}), D^*(hx_{3n+2}, fx_{3n+3}, gx_{3n+4}), \\ D^*(fx_{3n+3}, gx_{3n+4}, hx_{3n+5}), \cdots, \\ D^*(gx_{3k+3n-2}, hx_{3k+3n-1}, fx_{3k+3n}), \\ D^*(hx_{3k+3n-1}, fx_{3k+3n}, gx_{3k+3n+1}), \\ D^*(fx_{3k+3n}, gx_{3k+3n+1}, hx_{3k+3n+2}) \end{array} \right\} \\ &\quad= \lambda \max\{\alpha_{3n+1}, \alpha_{3n+2}, \alpha_{3n+3}, \cdots, \alpha_{3k+3n-2}, \alpha_{3k+3n-1}, \alpha_{3k+3n}\} \\ &\quad\leqslant \lambda \max\{\mu\theta^{3n+1}, \mu\theta^{3n+2}, \mu\theta^{3n+3}, \cdots, \mu\theta^{3k+3n-2}, \mu\theta^{3k+3n-1}, \mu\theta^{3k+3n}\} \\ &\qquad\qquad \mathit{from}\ (2.5) \\ &\quad= \lambda\mu\theta^{3n+1} = \theta^{3k}\mu\theta^{3n+1} \\ &\quad= \mu\theta^{3k+3n+1}\end{aligned}$$

Similarly, we have

$$D^*(y_{3k+3n+2}, y_{3k+3n+3}, y_{3k+3n+4}) \leqslant \mu\theta^{3k+3n+2}$$

and

$$D^*(y_{3k+3n+3}, y_{3k+3n+4}, y_{3k+3n+5}) \leqslant \mu\theta^{3k+3n+3}$$

Thus

$$D^*(y_{3k+n}, y_{3k+n+1}, y_{3k+3n+2}) \leqslant \mu\theta^{3k+n}, \ n = 1, 2, \cdots \tag{2.6}$$

Since D^* is of first type, we have

$$\begin{aligned}D^*(y_{3k+n}, y_{3k+n}, y_{3k+3n+1}) &\leqslant D^*(y_{3k+n}, y_{3k+n+1}, y_{3k+3n+2}) \\ &\leqslant \mu\theta^{3k+n}, \ n = 1, 2, \cdots, \ \mathit{from}(2.6).\end{aligned} \tag{2.7}$$

Now for $m > n$, consider

$$
\begin{aligned}
&D^*(y_{3k+n}, y_{3k+n}, y_{3k+m}) \\
&\quad \leqslant D^*(y_{3k+n}, y_{3k+n}, y_{3k+n+1}) + D^*(y_{3k+n+1}, y_{3k+n+1}, y_{3k+n+2}) + \cdots \\
&\qquad\qquad + D^*(y_{3k+m-1}, y_{3k+m-1}, y_{3k+m}) \\
&\quad \leqslant \mu\theta^{3k+n} + \mu\theta^{3k+n+1} + \mu\theta^{3k+n+2} + \cdots + \mu\theta^{3k+m-1} \\
&\quad \leqslant \mu\theta^{3k}\tfrac{\theta^n}{1-\theta}, \ \textit{since } 0 < \theta < 1 \\
&\quad \to 0 \text{ as } n \to \infty, m \to \infty.
\end{aligned}
$$

Hence the sequence $\{y_{3k+n}\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} y_{3k+n} = z. \tag{2.8}$$

By (2.1.4), there exists $u \in X$ such that

$$z = fu = gu = hu \tag{2.9}$$

Now consider

$$
D^*\begin{pmatrix} S(u, u, \cdots, u), \\ T(x_{3n-1}, x_{3n}, \cdots, x_{3k+3n-2}), \\ R(x_{3n}, x_{3n+1}, \cdots, x_{3k+3n-1}) \end{pmatrix}
\leqslant \lambda \max \left\{ \begin{array}{c} D^*(gu, hx_{3n-1}, fx_{3n}), D^*(hu, fx_{3n}, gx_{3n+1}), \\ D^*(fu, gx_{3n+1}, hx_{3n+2}), \cdots, D^*(gu, hx_{3k+3n-4}, fx_{3k+3n-3}), \\ D^*(hu, fx_{3k+3n-3}, gx_{3k+3n-2}), D^*(fu, gx_{3k+3n-2}, hx_{3k+3n-1}) \end{array} \right\}
$$

Letting $n \to \infty$ and using (2.8), (2.9) we get

$$D^*(S(u, u, \cdots, u, u), fu, fu) \leqslant \lambda(0)$$

which in turn yields that

$$S(u, u, \cdots, u) = fu \tag{2.10}$$

Similarly we can show that

$$T(u, u, \cdots, u) = gu \tag{2.11}$$

$$R(u, u, \cdots, u) = hu \tag{2.12}$$

Since the pairs (S, f), (T, g) and (R, h) are jointly $3k$-weakly compatible and from (2.10),(2.11),(2.12), we have

$$fz = f(fu) = f(S(u, u, \cdots, u)) = S(fu, fu, \cdots, fu) = S(z, z, \cdots, z) \tag{2.13}$$

$$gz = T(z, z, \cdots, z) \tag{2.14}$$

and

$$hz = R(z, z, \cdots, z) \tag{2.15}$$

Now consider

$$\begin{aligned} D^*(fz,z,z) &= D^*\begin{pmatrix} S(z,z,\cdots,z), \\ T(u,u,\cdots,u), \\ R(u,u,\cdots,u) \end{pmatrix}, \ \textit{from } (2.11),(2.12) \textit{ and } (2.13) \\ &\leqslant \lambda\max\left\{\begin{array}{c} D^*(gz,hu,fu), D^*(hz,fu,gu), \\ D^*(fz,gu,hu),\cdots, D^*(gz,hu,fu), \\ D^*(hz,fu,gu), D^*(fz,gu,hu) \end{array}\right\} \\ &= \lambda\max\{D^*(gz,z,z), D^*(hz,z,z), D^*(fz,z,z)\}. \end{aligned}$$

Similarly, we can show that

$$D^*(z,gz,z) \leqslant \lambda\max\{D^*(gz,z,z), D^*(hz,z,z), D^*(fz,z,z)\},$$
$$D^*(z,z,hz) \leqslant \lambda\max\{D^*(gz,z,z), D^*(hz,z,z), D^*(fz,z,z)\}.$$

Thus

$$\max\left\{\begin{array}{c} D^*(fz,z,z), \\ D^*(gz,z,z), \\ D^*(hz,z,z) \end{array}\right\} \leqslant \lambda\max\left\{\begin{array}{c} D^*(fz,z,z), \\ D^*(gz,z,z), \\ D^*(hz,z,z) \end{array}\right\}$$

which implies that $fz = gz = hz = z$.

Now from (2.13),(2.14) and (2.15) we have

$$fz = gz = hz = z = S(z,z,\cdots,z) = T(z,z,\cdots,z) = R(z,z,\cdots,z). \tag{2.16}$$

Suppose there exists $z' \in X$ such that

$$fz' = gz' = hz' = z' = S(z',z',\cdots,z') = T(z',z',\cdots,z') = R(z',z',\cdots,z').$$

Then

$$\begin{aligned} D^*(z,z,z') &= D^*(S(z,z,\cdots,z), T(z,z,\cdots,z), R(z',z',\cdots,z')) \\ &\leqslant \lambda\max\left\{\begin{array}{c} D^*(gz,hz,fz'), D^*(hz,fz,gz'), \\ D^*(fz,gz,hz'),\cdots, D^*(gz,hz,fz'), \\ D^*(hz,fz,gz'), D^*(fz,gz,hz') \end{array}\right\} \\ &= \lambda D^*(z,z,z') \end{aligned}$$

which implies that $z' = z$.

Thus z is the unique point in X satisfying (2.16). □

Now we give an example to illustrate our main Theorem 2.1.

Example 2.1. Let $X = [0,1]$ and $D^*(x,y,z) = |x-y| + |y-z| + |z-x|$ and $k = 1$. Define

$$S(x,y,z) = \frac{2x+3y^2+4z^3}{72},\ T(x,y,z) = \frac{3x^2+4y^3+2z}{72},\ R(x,y,z) = \frac{4x^3+2y+3z^2}{72}$$
$$fx = \frac{x^3}{6},\ gx = \frac{x}{6} \text{ and } hx = \frac{x^2}{4}$$

for all $x, y, z, u, v, w, p, q, r \in X$. Consider

$$
\begin{aligned}
&D^*(S(x,y,z),T(u,v,w),R(p,q,r)) \\
&\quad = \left|\tfrac{2x+3y^2+4z^3}{72} - \tfrac{3u^2+4v^3+2w}{72}\right| + \left|\tfrac{3u^2+4v^3+2w}{72} - \tfrac{4p^3+2q+3r^2}{72}\right| \\
&\quad + \left|\tfrac{4p^3+2q+3r^2}{72} - \tfrac{2x+3y^2+4z^3}{72}\right| \\
&\quad \leqslant \tfrac{1}{72}\left[\begin{array}{c} \{\left|2x-3u^2\right| + \left|3u^2-4p^3\right| + \left|4p^3-2x\right|\} \\ +\{\left|3y^2-4v^3\right| + \left|4v^3-2q\right| + \left|2q-3y^2\right|\} \\ +\{\left|4z^3-2w\right| + \left|2w-3r^2\right| + \left|3r^2-4z^3\right|\} \end{array}\right] \\
&\quad = \tfrac{1}{6}\left[\begin{array}{c} \{\left|\tfrac{x}{6}-\tfrac{u^2}{4}\right| + \left|\tfrac{u^2}{4}-\tfrac{p^3}{3}\right| + \left|\tfrac{p^3}{3}-\tfrac{x}{6}\right|\} \\ +\{\left|\tfrac{y^2}{4}-\tfrac{v^3}{3}\right| + \left|\tfrac{v^3}{3}-\tfrac{q}{6}\right| + \left|\tfrac{q}{6}-\tfrac{y^2}{4}\right|\} \\ +\{\left|\tfrac{z^3}{3}-\tfrac{w}{6}\right| + \left|\tfrac{w}{6}-\tfrac{r^2}{4}\right| + \left|\tfrac{r^2}{4}-\tfrac{z^3}{3}\right|\} \end{array}\right] \\
&\quad = \tfrac{1}{6}[D^*(gx,hu,fp) + D^*(hy,fv,gq) + D^*(fz,gw,hr)] \\
&\quad \leqslant \tfrac{1}{2}\max\{D^*(gx,hu,fp), D^*(hy,fv,gq), D^*(fz,gw,hr)\}.
\end{aligned}
$$

Thus the condition (2.1.2) of Theorem 2.1 is satisfied. Clearly

$$fx = S(x,x,\cdots,x),\ gx = T(x,x,\cdots,x) \text{ and } hx = R(x,x,\cdots,x)$$

imply that $x = 0$ and

$$f(S(0,0,\cdots,0)) = S(f0,f0,\cdots,f0),\ g(T(0,0,\cdots,0)) = T(g0,g0,\cdots,g0)$$

and $h(R(0,0,\cdots,0)) = R(h0,h0,\cdots,h0)$. Hence the condition (2.1.3) is satisfied. One can easily verify (2.1.1) and (2.1.4).

Clearly, 0 is the unique point in X such that

$$f0 = g0 = h0 = 0 = S(0,0,...,0,0) = T(0,0,...,0,0) = R(0,0,...,0,0).$$

Corollary 2.1. *Let (X, D^*) be a D^*-metric space, where D^* is of first type, k a positive integer and $S, T, R : X^{3k} \to X$ and $f : X \to X$ be mappings satisfying*

(2.1.1)* $S(X^{3k}) \subseteq f(X)$, $T(X^{3k}) \subseteq f(X)$, $R(X^{3k}) \subseteq f(X)$,

(2.1.2)* $D^*\begin{pmatrix} S(x_1,x_2,\cdots,x_{3k}), \\ T(y_1,y_2,\cdots,y_{3k}), \\ R(z_1,z_2,\cdots,z_{3k}) \end{pmatrix} \leqslant \lambda \max\{D^*(fx_i,fy_i,fz_i)/1 \leqslant i \leqslant 3k\}$

for all $x_1,x_2,\cdots,x_{3k},y_1,y_2,\cdots,y_{3k},z_1,z_2,\cdots,z_{3k} \in X$ and $0 < \lambda < 1$,

(2.1.3)* *One of the pairs (S,f), (T,f) and (R,f) is $3k-$weakly compatible,*

(2.1.4)* *$f(X)$ is a complete subspace of X.*

Then there exists a unique $z \in X$ such that

$$fz = z = S(z,z,\cdots,z) = T(z,z,\cdots,z) = R(z,z,\cdots,z).$$

Corollary 2.2. *Let (X, D^*) be a D^*-metric space, where D^* is of first type, k a positive integer and $S : X^k \to X$ and $f : X \to X$ be mappings satisfying*

(2.2.1) $D^*\begin{pmatrix} S(x_1,x_2,\cdots,x_k), \\ S(y_1,y_2,\cdots,y_k), \\ S(z_1,z_2,\cdots,z_k) \end{pmatrix} \leqslant \lambda \max\{D^*(fx_i,fy_i,fz_i)/1 \leqslant i \leqslant 3k\}$

for all $x_1,x_2,\cdots,x_k,y_1,y_2,\cdots,y_k,z_1,z_2,\cdots,z_k \in X$ and $0 < \lambda < 1$, (2.2.2) *$S(X^k) \subseteq f(X)$,*

(2.2.3) *The pair (S, f) is $k-$weakly compatible,*

(2.2.4) *$f(X)$ is a complete subspace of X.*

Then there exists a unique $z \in X$ such that $fz = z = S(z, z, \cdots, z)$.

Corollary 2.3. *Let (X, D^*) be a complete D^*-metric space,where D^* is of first type, k a positive integer and $S, T, R : X^{3k} \to X$ be mappings satisfying*

$$(2.3.1) \quad D^* \begin{pmatrix} S(x_1, x_2, \cdots, x_{3k}), \\ T(y_1, y_2, \cdots, y_{3k}), \\ R(z_1, z_2, \cdots, z_{3k}) \end{pmatrix} \leqslant \lambda \max \{D^*(x_i, y_i, z_i)/1 \leqslant i \leqslant 3k\}$$

for all $x_1, x_2, \cdots, x_{3k}, y_1, y_2, \cdots, y_{3k}, z_1, z_2, \cdots, z_{3k} \in X$ and $0 < \lambda < 1$.

Then there exists a unique point $z \in X$ such that

$$z = S(z, z, \cdots, z) = T(z, z, \cdots, z) = R(z, z, \cdots, z).$$

Corollary 2.4. *Let (X, D^*) be a complete D^*-metric space,where D^* is of first type, k a positive integer and $S : X^{3k} \to X$ be mappings satisfying*

$$(2.4.1) \quad D^* \begin{pmatrix} S(x_1, x_2, \cdots, x_{3k}), \\ S(y_1, y_2, \cdots, y_{3k}), \\ S(z_1, z_2, \cdots, z_{3k}) \end{pmatrix} \leqslant \lambda \max \{D^*(x_i, y_i, z_i)/1 \leqslant i \leqslant 3k\}$$

for all $x_1, x_2, \cdots, x_{3k}, y_1, y_2, \cdots, y_{3k}, z_1, z_2, \cdots, z_{3k} \in X$ and $0 < \lambda < 1$.

Then there exists a unique point $z \in X$ such that $z = S(z, z, \cdots, z)$.

Remark 2.1. *Now we give probable modifications of Theorems of* [**5, 12, 13**]*:*

(i) In Theorem 2.1 *of* [**5**]*, Theorem* 3.1 *of* [**12**]*, Theorem* 3.1 *of* [**13**] *one has to assume $G(x, x, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$,instead of (G_3).*

(ii) In (2.1.2) *of Theorem* 2.1 *of* [**5**]*, in* (3.3) *of Theorem* 3.1 *of* [**12**] *and in* (5) *of Theorem* 3.1 *of* [**13**] *one has to assume that any two of u,v,w are different instead of $u \neq v \neq w$.*

Acknowledgement. The authors are thankful to the referees for their valuable suggestions.

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Received by editors 12.06.2016; Revised version 25.12.2016; Available online 16.01.2017.

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