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ON SOME MAPPINGS BETWEEN CO-QUASIORDERED RELATIONAL SYSTEMS

Daniel Abraham Romano

Abstract. The setting of this article is Bishop's constructive mathematics. The connections between the strong mappings, U-mappings and L-mappings of co-quasiorder relational systems we analyzed. The relations between the strong mappings of co-quasiorder relational systems defined by Novotny and other mappings are also analysed. Finally, a new mapping between two relational systems is introduced. Some properties of these mappings and connection of these mappings to the other known mappings are investigated.

1. Introduction

The constructivemathematics of Erret Bishop, i.e. BISH, forms the framework for the work in this paper. BISH originates in [**1**], in which a large part of modern analysis with constructive background was developed ([**3**]). Since then, a steady stream of publications have contributed to Bishops work (see [**2, 3, 4**]). [**4**] is the result of ten years of systematic research on topological space based on sets with apartness within the domain of BISH. Modern algebra, contrary to Bishops expectations, also proved amenable to natural, thoroughgoing, constructive treatment ([**13**]). Constructive algebra is more complicated than its classical counterpart in various ways. For example, algebraic structures as a rule do not carry decidable equality relations, and there is sometimes an abundance of substructures of many types that can prove awkward to work with.

In this paper, the basic structure will be relational systems with apartness. The study of algebraic structures in an intuitionistic setting was undertaken by Arend

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Heyting. Heyting considered structures equipped with an apartness relation in full generality. (See, for example [**9**] or [**20**]).

The main objective of this paper is to outline some of the basic concepts of mapping between relational systems with apartness.

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More background on the Constructive mathematics can be found in [**1, 4, 13**] and [**20**]. The standard reference for constructive algebra is [**18**] and [**13**]. Some basic information about relational systems with apartness reader can be found in [**10**] and [**16**].

Bishop's constructive mathematics was developed on intuitionistic logic ([**9**], [**18**] and [**20**]), that is, logic without the law of excluded middle $P \vee \neg P$. Note that in the intuitionistic logic the double negation law $P \Longleftrightarrow \neg\neg P$ also does not hold, but the following implication $P \Longrightarrow \neg\neg P$ holds even in minimal logic. Since intuitionistic logic is a part of classical logic, the results from constructive mathematics are compatible with suitable results in classical mathematics. Recall that the following deduction principle $A \vee B, \neg B \vdash A$ is acceptable in the intuitionistic logic.

What is special about the work presented in this paper? In the first place, it is the application of intuitionistic logic instead of classical logic. In mathematics based on intuitionistic logic, there is a need to determine so-called negative concepts through a positive approach. This aspect fragments classical concepts into a number of different concepts that are mutually inequivalent within mathematics founded on intuitionistic logic. For example, the classical concept of equality relation $e \subseteq A \times A$ on a set A in constructive mathematics is divided into the associated pair of concepts: equality e and coequality $q \subseteq A \times A$ relations. The concepts e and q are said be associated if $e \circ q \subseteq q$ holds. (In this way, the equality relation '$=$' and a diversity relation '$\neq$' often occur in a set.)

Of course, there is a need to understand their mutual connections, i.e. for a given equivalence relation e is there (the maximal) coequality relation q associated with e? Other interesting questions that naturally occur include what kind of structure is built by new relations, and what kind of connections occur between the structures that are generated by the associated pair of an equality e and a coequality q.

This article aims to present answers to some of these questions in connection with the concept of a special class of relations: the co-quasiorder relation (formore on this concept, see [**6, 7, 15, 17**].) Thus, this article does not intend to reiterate constructive mathematics tools and techniques, but rather to focus on some specific functions between such relational systems. This matter is deemed interesting and important because mathematicians need to investigate all the logical possibilities that appear in any structures, since mathematics can be viewed as a science of structures.

Several new results are presented in the theorems in this paper. All the results obtained are related to a certain notion of the isotones and reverse isotones of mappings between co-quasiordered relational systems.

2. Preliminaries

Let $(A, =, \neq)$ be a set in the sense of [**1, 4, 13**] and [**20**], where $\neq \subseteq A \times A$ is a binary relation on A which satisfies the following properties:

$$\neg(x \neq x),\ \ x \neq y \Longrightarrow y \neq x,\ \ x \neq y \wedge y = z \Longrightarrow x \neq z$$

called *diversity* relation. In addition, if for diversity relation $\neq$ the following implication

$$x \neq z \Longrightarrow x \neq y \vee y \neq z$$

holds, it is called *apartness* ([**9**]). Let us note that an apartness is not tight in general case. (For an apartness $\neq$ we say that it is tight if $\neg(x \neq y) \Longrightarrow x = y$ holds.) Hereafter, the relational structure $(A, =, \neq)$ is denoted simply with A. Let Y be a subset of A. The subset Y of A is a *strongly extensional subset* in A if and only if $(\forall x \in A)(y \in Y \Longrightarrow (y \neq x \vee x \in Y))$ ([**1**], [**13**]).

Let A, B and C be sets and $\alpha \subseteq A \times B$ and $\beta \subseteq B \times C$ be relations. The *filled product* ([**15**]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in A \times C : (\forall b \in B)((a, b) \in \alpha \vee (b, c) \in \beta)\}.$$

As in [**6**] and [**7**] a relation $\tau \subseteq A \times A$ is a *co-quasiorder* on A if and only if

$$\tau \subseteq \neq,\ \ \tau \subseteq \tau * \tau.$$

Note that in the earlier papers on this work, the term quasi-antiorder relation is used ([**15, 10, 16**] and [**17**]). In the present paper, (A, σ) is termed a 'co-quasiordered system'. It is easy to see that classes $a\tau$ and τb $(a, b \in A)$ of a co-quasiorder τ is strongly extensional subsets of A.

A relation α on A is an *co-order* relation ([**6**] and [**7**], [**17**]) on A if and only if

$$\alpha \subseteq \neq,\ \ \alpha \subseteq \alpha * \alpha,\ \ \neq \subseteq \alpha \cup \alpha^{-1}.$$

Note that in the earlier papers on this work, the term anti-order relation was used ([**15, 10, 16**] and [**17**]. In the present paper, (A, α) is termed a 'co-ordered system'.

EXAMPLE 2.1. **(1)** Let $X = \{a, b, c, d, e\}$ be a set. The relation $\alpha \subseteq X \times X$, defined by

$$\alpha = \{(a, b),(a, c),(a, e),(b, a),(b, c),(b, d),(b, e),(c, a),(c, b),(c, d),(d, a),(d, b),\\ (d, c),(d, e),(e, a),(e, b),(e, c),\ (e, d)\}$$

is a co-order relation on set X.

(2) Let $\sigma = \{(c, a),(c, b),(d, a),(d, b),(d, c),(e, a),(e, b),(e, c)\}$ and

$$\tau = \{(a, b),(a, c),(a, d),(a, e),(b, a),(b, c),(b, d),(b, e),(c, d),(c, e),(d, e),(e, d)\}$$

be relations on the set $X = \{a, b, c, d, e\}$. Then σ and τ are co-quasiorders on X such that $\sigma \cap \sigma^{-1} = \emptyset$, $\sigma \cup \sigma^{-1} \subset \neq$, $\tau \cap \tau^{-1} = \{(b, a), (a, b), (e, d), (d, e)\}$, $\tau \cup \tau^{-1} = \neq$, $\sigma \cup \tau = \neq$ and $\sigma \cap \tau = \emptyset$. So, the relations σ and τ are co-quasirders on X.

(3) Let $\rhd$ be a relation between an element $x \in A$ and a subset Y defined by

$$x \rhd Y \iff (\forall y \in Y)(x \neq y).$$

A subset Y of a set A is detachable in A if $(\forall x \in A)(x \in Y \vee x \rhd Y)$. ([**7**], Lemma 1) Let T be a detachable subset of a set with apartness A. Then the relation τ on A, defined by $(a, b) \in \tau \iff a \rhd T \wedge b \in T$, is a co-quasiorder on A. $\diamond$

Let $\varphi : (A, =, \neq) \longrightarrow (B, =, \neq)$ be a mapping. We say that:

(a) φ is *strongly extensional* if and only if $(\forall a, b \in A)(\varphi(a) \neq \varphi(b) \Longrightarrow a \neq b)$;

(b) φ is an *embedding* if and only if $(\forall a, b \in A)(a \neq b \Longrightarrow \varphi(a) \neq \varphi(b))$.

Let $\varphi : (A, \alpha) \longrightarrow (B, \beta)$ be a mapping of relational systems. As in the classical case, the mapping φ is called *isotone* if and only if

$$(\forall x, y \in A)((x, y) \in \alpha \Longrightarrow (\varphi(x), \varphi(y)) \in \beta);$$

the mapping φ is called *reverse isotone* if and only if

$$(\forall x, y \in A)((\varphi(x), \varphi(y)) \in \beta \Longrightarrow (x, y) \in \alpha).$$

Therefore, if a mapping $\varphi : (A, =, \neq) \longrightarrow (B, =, \neq)$ is an embedding, then φ is an isotone, and if φ is strongly extensional, then φ is a reverse isotone mapping with respect to apartness relations.

Example 2.2. **(1)** Let $X = \{a, b, c\}$, $Y = \{u, v, w\}$ be sets and let $\sigma = \{(b, a),\ (c, a),\ (c, b)\}$ and $\tau = \{(u, v), (v, u), (w, u), (w, v)\}$ be relations on X and Y respectively. Then (X, σ) and (Y, τ) are co-quasiorder relational systems. The mapping φ, defined by $\varphi(a) = u$ $\varphi(b) = v$ and $\varphi(c) = w$ is an isotone mapping between relational sistems, but it is not strong because for $(a, b) \in \varphi^{-1}(\tau)$ there do no exist elements x, y of X such that $\varphi(a) = \varphi(x)$, $\varphi(y) = \varphi(b)$ and $(x, y) \in \sigma$.

(2) If $\sigma = \{(a,b),(a,c),(a,d),(b,a),(b,c),(b,d),(c,a),(c,b),(c,d)\}$ on set $X = \{a, b, c, d\}$ and $\tau = \{(u, v), (u, w), (u, z), (v, u), (v, w), (v, z), (w, z)\}$ on set $Y = \{u, v, w, z\}$, then (X, σ) and (Y, τ) are co-quasiorder relational systems. The mapping φ, defined by $\varphi(a) = u$, $\varphi(b) = v$, $\varphi(c) = w$ and $\varphi(d) = z$, is a reverse isotone mapping, but it is not strong. $\diamond$

Let us note that if $\varphi : ((A, =, \neq), \sigma) \longrightarrow ((B, =, \neq), \tau)$ is a strongly extensional mapping of co-quasiordered systems, then the relation $\varphi^{-1}(\tau) = \{(x, y) \in A \times A : (\varphi(x), \varphi(y)) \in \tau\}$ is a co-quasiorder too. It is easy to verify the following:

$$\varphi \textit{ is isotone if } \sigma \subseteq \varphi^{-1}(\tau), \tag{2.1}$$

and

$$\varphi \textit{ is reverse isotone if } \varphi^{-1}(\tau) \subseteq \sigma. \tag{2.2}$$

For these mappings we have the following implications:

Lemma 2.1. *Let (A, α) and (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a mapping. Then:*

$$\alpha \subseteq \varphi^{-1}(\beta) \iff \varphi \circ \alpha \subseteq \beta \circ \varphi. \tag{2.3}$$

PROOF. (i) Suppose $\alpha \subseteq \varphi^{-1}(\beta)$. Let (a, b') be an arbitrary element of $\varphi \circ \alpha$. Then, there exists an element $a' \in A$ with $(a, a') \in \alpha$ and $(a', b') \in \varphi$. Thus, $(\varphi(a), \varphi(a')) \in \beta$ because φ is an isotone mapping. Now, out of $(a, \varphi(a)) \in \varphi$ and $(\varphi(a), b') \in \beta$ we have $(a, b') \in \beta \circ \varphi$.

(ii) In fact, for $(a, a') \in \alpha$ $(a, a' \in A)$ we have

$$\begin{aligned} (a, a') \in \alpha &\Longrightarrow (a, a') \in \alpha \wedge (a', \varphi(a')) \in \varphi \\ &\Longrightarrow (a, \varphi(a')) \in \varphi \circ \alpha \subseteq \beta \circ \varphi \\ &\Longrightarrow (a, \varphi(a)) \in \varphi \wedge (\varphi(a), \varphi(a')) \in \beta \\ &\Longrightarrow (a, a') \in \varphi^{-1}(\beta). \end{aligned}$$ □

LEMMA 2.2. *Let (A, α) and (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a surjective mapping. Then:*

$$\varphi^{-1}(\beta) \subseteq \alpha \Longrightarrow \beta \circ \varphi \subseteq \varphi \circ \alpha. \tag{2.4}$$

PROOF. Suppose $\varphi^{-1}(\beta) \subseteq \alpha$. If $(a, b') \in \beta \circ \varphi$, then there exists an element $b \in B$ such that $(a, b) \in \varphi$ and $(b, b') \in \beta$. Since φ is a surjective mapping, there exists an element a' of A with $(a', b') \in \varphi$. Hence, from $(\varphi(a), \varphi(a')) \in \beta$ we conclude that $(a, a') \in \alpha$ because φ is a reverse isotone mapping. Thus, finally, from $(a, a') \in \alpha \wedge (a', b') \in \varphi$ we have $(a, b') \in \varphi \circ \alpha$. □

The notion of a mapping between ordered sets is one of the fundamental notions in the study of the structure of ordered sets. In literature, for example, in [**5**], [**8**], [**11**], [**14**] and [**19**] there are several definitions of (isotone) mappings on ordered sets. Following [**5**] and [**8**] there are:
(i) isotone mappings;
(ii) strong isotone mappings;
(iii) U -mappings and L-mappings;
and following [**11**], [**14**] and [**19**]
(iv) strong mappings in Novotny sense.

This paper presents a new approach to mappings between co-quasiordered relational systems in the constructive setting.

3. The Main results

3.1. Strong mappings. Halaš and Hort ([**8**]) observed that there are several different definitions of mappings on partially ordered sets. Following ideas exposed in articles [**10, 16**], we describe some mappings between co-quasiordered relational systems. First, we modify the notion of a strong mapping in Chajda and Hoškova sense ([**5**]), as in unpublished article [**16**].

Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional mapping. We will use the kernel of the mapping φ defined by $Ker\varphi = \{(a, b) \in A \times A : \varphi(a) = \varphi(b)\}$. We define:

– Isotone mapping φ is called an *isotone strong mapping* ([**16**]) from A to B if the following holds:

$$\sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma \circ Ker\varphi. \tag{3.1}$$

– A reverse isotone mapping φ is called a *reverse isotone strong mapping* from A to B if the following holds

$$\varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi. \tag{3.2}$$

Results in the following lemma are very important in our understanding of these notions:

LEMMA 3.1. *Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional mapping. Then:*

$$(i)\ \ \sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma \circ Ker\varphi \Longleftrightarrow \varphi^{-1}(\tau) = Ker\varphi \circ \sigma \circ Ker\varphi.$$

$$(ii)\ \ \varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi \Longleftrightarrow \sigma = Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi.$$

PROOF. (i) Let $\sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma \circ Ker\varphi$ and $(a, b) \in Ker\varphi \circ \sigma \circ Ker\varphi$. Then, there exist elements x, y of A such that $\varphi(a) = \varphi(x)$, $(x, y) \in \sigma \subseteq \varphi^{-1}(\tau)$ and $\varphi(y) = \varphi(b)$. Since $\varphi^{-1}(\tau)$ is a co-quasiorder relation on A we have $(x, a) \in \varphi^{-1}(\tau) \subseteq\neq$ or $(a, b) \in \varphi^{-1}(\tau)$, or $(b, y) \in \varphi^{-1}(\tau) \subseteq\neq$. Thus, $(a, b) \in \varphi^{-1}(\tau)$, and, finally we have that $\varphi^{-1}(\tau) = Ker\varphi \circ \sigma \circ Ker\varphi$. Let us note that the opposite implication is clear.

(ii) Since the implication

$$\sigma = Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi \Longrightarrow \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi \subseteq \sigma$$

is obvious, let us prove the opposite implication. If $(a, b) \in Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi$, then there exist elements x, y of A such that $\varphi(a) = \varphi(x)$ and $(x, y) \in \varphi^{-1}(\tau)$ and $\varphi(y) = \varphi(b)$. Out of $(x, y) \in \varphi^{-1}(\tau)$ we conclude $(x, a) \in \varphi^{-1}(\tau) \subseteq\neq$ or $(a, b) \in \varphi^{-1}(\tau) \subseteq \sigma$ or $(b, y) \in \varphi^{-1}(\tau) \subseteq\neq$. Thus, $(a, b) \in \sigma$. □

3.2. U **- mappings.** The notion of a relational system was introduced by Maltsev in 1954 ([**12**]). This system was investigated by many (as for example, by Vagner ([**21**])) and also by Chajda in several of his articles. Chajda and Hoškova investigated U-mappings in the article [**5**]. The concepts of relational systems with co-order and co-quasiorder relations were analyzed in article [**15**] and [**17**].

Following ideas in the articles [**10**] and [**16**] we describe U-mappings between co-quasiorder relational systems in this section.

Let (A, α) and (B, β) be relational systems. For a binary relation α on A and $a \in A$ denote $U_\alpha(a) = \{t \in A : (a, t) \in \alpha\}$. The set $U_\alpha(a)$ (the left class of α generated by a) is called *the upper class* generated by a. A mapping $\varphi : A \longrightarrow B$ is called U*-mapping* if

$$\varphi(U_\alpha(a)) = U_\beta(\varphi(a)) \text{ for each } a \in A.$$

REMARK 3.1. **(1)**. If φ is a strongly extensional and reverse isotone surjective mapping, then $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$. Indeed, let $z \in U_\beta(\varphi(x))$, i.e. let $(\varphi(x), z) \in \beta$. Since φ is a surjective mapping, then there exists an element t of A such that $z = \varphi(t)$ and $(\varphi(x), \varphi(t)) \in \beta$. Since φ is a reverse isotone mapping, we have $(x, t) \in \alpha$. Thus, $t \in U_\alpha(x)$ and $z = \varphi(t) \in \varphi(U_\alpha(x))$.

(2). If φ is a strongly extensional and isotone surjective mapping, then $U_\beta(\varphi(x)) \supseteq \varphi(U_\alpha(x))$. Indeed, let $z \in \varphi(U_\alpha(x))$. Then there exists an element t of A such that

$z = \varphi(t)$ and $(x, t) \in \alpha$. Since φ is an isotone mapping, we have $(\varphi(x), \varphi(t)) \in \beta$. Thus, $z = \varphi(t) \in U_\beta(\varphi(x))$.

In the following theorem we prove that every strongly extensional reverse isotone strong mapping between two co-quasiorder relational systems is a U-mapping.

THEOREM 3.1. *Let $((A, =, \neq), \sigma)$ and $((B, =, \neq), \tau)$ be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional reverse isotone surjective strong mapping. Then, the mapping φ is an isotone and reverse isotone U-mapping.*

PROOF. Let z be an arbitrary element of $\varphi(U_\sigma(x))$. Then there exists an element t of $U_\sigma(x)$ such that $z = \varphi(t)$ and $(x, t) \in \sigma$. Since φ is a strong mapping, then there exist elements $a, b \in A$ such that

$$\varphi(a) = \varphi(x) \wedge (\varphi(a), \varphi(b)) \in \tau \wedge z = \varphi(t) = \varphi(b).$$

Thus, out of $(\varphi(a), \varphi(b)) \in \tau$ we have: $(\varphi(a), \varphi(x)) \in \tau \subseteq \neq$ or $(\varphi(x), \varphi(b)) \in \tau$. So, from that we conclude $\varphi(b) = \varphi(t) = z \in \{u \in B : (\varphi(x), u) \in \tau\} = U_\tau(\varphi(x))$ because the first part of the previous disjunction is impossible. Furthermore, let us assume that elements x, y of A are such that $(x, y) \in \sigma$. Then, $y \in U_\sigma(x)$ and $\varphi(y) \in \varphi(U_\sigma(x)) = U_\tau(\varphi(x))$. Hence, $(\varphi(x), \varphi(y)) \in \tau$ and φ is an isotone mapping. $\square$

It is easy to verify that the converse assertion is not valid in general. The following theorem is the main result of [**16**]:

THEOREM 3.2. *Let (A, σ) and (B, τ) be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional surjective mapping. Then:*

$$\varphi \; is \; a \; U - mapping \; if \; and \; only \; if \; \sigma \subseteq \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma \; holds; \tag{3.3}$$

$$\varphi \; is \; a \; U - mapping \; if \; and \; only \; if \; \varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \; holds. \tag{3.4}$$

PROOF. (1) ($\Longrightarrow$) Let φ be a U-mappings. Then, if (x, y) is an element of σ, i.e. if $y \in U_\sigma(x)$, then $\varphi(x) \in \varphi(U_\sigma(x)) = U_\tau(\varphi(x))$. Thus, $(\varphi(x), \varphi(y)) \in \tau$ and φ is an isotone mapping. Suppose that $(x, y) \in \varphi^{-1}(\tau)$, i.e. suppose that $(\varphi(x), \varphi(y)) \in \tau$. Then $\varphi(y) \in U_\tau((x)) = \varphi(U_\alpha(x))$. So, there exists an element t of A such that $\varphi(y) = \varphi(t)$ and $(x, t) \in \sigma$. Hence, (x, y) is an element of $Ker\varphi \circ \sigma$.

($\Longleftarrow$) Assume that φ is isotone and suppose that $\varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma$ holds. Let a be an arbitrary element of $U_\tau(\varphi(x))$. Then, there exists an element t of A such that $a = \varphi(t)$ and $(\varphi(x), \varphi(t)) \in \tau$. Thus, for the element t of A we have $a = \varphi(t)$ and $(x, t) \in \varphi^{-1}(\tau) \subseteq Ker\varphi \circ \sigma$. So, there exists an element s of A such that $(x, s) \in \sigma$ and $\varphi(s) = \varphi(t) = a$. Therefore, we have $a = \varphi(s) = \varphi(t) \in \varphi(U_\sigma(x))$. Finally, we have $U_\tau(\varphi(x)) \subseteq \varphi(U_\sigma(x))$.

(2) ($\Longrightarrow$) Let φ be a U-mappings. Then, if $(\varphi(x), \varphi(y)) \in \tau$, i.e. if $\varphi(y) \in U_\tau(\varphi(x))$, then $\varphi(y) \in \varphi(U_\sigma(x)) = U_\tau(\varphi(x))$. So, there exists an element t of A such that $\varphi(y) = \varphi(t)$ and $(x, t) \in \sigma$. Hence, from the implication $(x, y) \in \sigma \Longrightarrow (x, y) \in \sigma \vee (y, t) \in \sigma$ we conclude that $(x, y) \in \sigma$, because out $(y, t) \in \sigma$ we have

$(\varphi(y), \varphi(t)) \in \tau$ and $\varphi(y) \neq \varphi(t)$ since φ is an isotone mapping. On the other hand, out of $(x, y) \in \sigma$, i.e. out of $y \in U_\sigma(x)$, we have $\varphi(y) \in \varphi(U_\sigma(x))$. Thus, there exists an element s of A such that $\varphi(y) = \varphi(s)$ with $(x, s) \in \sigma$. Since φ is an isotone mapping, we have $(\varphi(x), \varphi(s)) \in \tau$, i.e. we have $(x, s) \in \varphi^{-1}(\tau)$. So, finally, we prove the inclusion $\sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau)$.

$(\Longleftarrow)$ Assume that φ is a reverse isotone and suppose that $\sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau)$ holds. Let a be an arbitrary element of $\varphi(U_\sigma(x))$. Then there exists an element t of A such that $a = \varphi(t)$ and $(x, t) \in \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau)$. Thus, there exists an element s of A such that $(x, s) \in \varphi^{-1}(\tau)$ and $\varphi(s) = \varphi(t) = a$. Therefore, out of $(\varphi(x), \varphi(s)) \in \tau$ and $\varphi(s) = \varphi(t) = a$ we have $a = \varphi(t) = \varphi(s) \in U_\tau(\varphi(x))$. So, the inclusion $\varphi(U_\sigma(x)) \subseteq U_\tau(\varphi(x))$ is valid. Since the opposite inclusion holds because φ is a reverse isotone mapping, finally, we have $\varphi(U_\sigma(x)) = U_\tau(\varphi(x))$. □

Now, in the following theorem we show a necessary condition for a U-mapping to be a strong mapping:

THEOREM 3.3. *Let (A, σ) and (B, τ) be quasi-antiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional surjective U-mapping and*

$$Ker\varphi \circ \varphi^{-1}(\tau) \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$$

holds. Then, the mapping φ is a strongly extensional reverse isotone strong mapping.

PROOF. Suppose that $\varphi : A \longrightarrow B$ is a strongly extensional U-mapping of co-quasiordered relational systems such that $Ker\varphi \circ \varphi^{-1}(\tau) \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$. Then, by Theorem 2, we have

$$\varphi^{-1}(\tau) \subseteq Ker\varphi \circ \varphi^{-1}(\tau).$$

In accordance with the hypothesis, out of

$$\varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \subseteq Ker\varphi \circ Ker\varphi \circ \varphi^{-1}(\tau)$$

we conclude that

$$\varphi^{-1}(\tau) \subseteq \sigma \subseteq Ker\varphi \circ \varphi^{-1}(\tau) \circ Ker\varphi.$$

So, the mapping $\varphi : A \longrightarrow B$ is a strongly extensional reverse isotone strong mapping. □

3.3. L **- mappings.** At the beginning of this subsection we pose the following two questions: What kind of a mapping $\varphi : (A, \sigma) \longrightarrow (B, \tau)$ between two co-quasiordered relational systems is such that:

(a) $\varphi^{-1}(\tau) \subseteq \sigma \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$?

(b) $\sigma \subseteq \varphi^{-1}(\tau) \subseteq \sigma \circ Ker\varphi$?

In this section we give a partial answer to the question (a).

Let (A, α) and (B, β) be relational systems. For a binary relation α on A and $a \in A$ we set $L_\alpha(a) = \{t \in A : (t, a) \in \alpha\}$. The set $L_\alpha(a)$ (the right class of α generated by a) is called *the lower class* generated by a. A mapping $\varphi : A \longrightarrow B$ is called *L-mapping* if

$$\varphi(L_\alpha(a)) = L_\beta(\varphi(a)) \text{ for each } a \in A.$$

REMARK 3.2. **(1)**. Let φ be a (strongly extensional) reverse isotone surjective mapping and let $z \in L_\beta(\varphi(a))$, i.e. let $(z, \varphi(a)) \in \beta$. Since φ is a surjective mapping, there exists an element t of A such that $z = \varphi(t)$ and $(\varphi(t), \varphi(a)) \in \beta$. Since φ is a reverse isotone mapping, we have $(t, a) \in \alpha$. Thus, $t \in L_\alpha(a)$ and $z = \varphi(t) \in \varphi(L_\alpha(a))$. So,

$$\varphi^{-1}(\beta) \subseteq \alpha \Longrightarrow L_\beta(\varphi(a)) \subseteq \varphi(L_\alpha(a)).$$

(2). If φ is a strongly extensional and isotone surjective mapping, then $L_\beta(\varphi(a)) \subseteq \varphi(L_\alpha(a))$. Indeed, let $z \in \varphi(L_\alpha(x))$. Then there exists an element t of A such that $z = \varphi(t)$ and $(t, a) \in \alpha$. Since φ is an isotone mapping, we have $(\varphi(t), \varphi(a)) \in \beta$. Thus, $z = \varphi(t) \in L_\beta(\varphi(a))$. Therefore, the following implication is valid

$$\alpha \subseteq \varphi^{-1}(\beta) \Longrightarrow \varphi(L_\alpha(a)) \subseteq L_\beta(\varphi(a)).$$

In the following theorem we give an answer to the question (a):

THEOREM 3.4. *Let (A, σ), (B, τ) be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional reverse isotone surjective mapping. Then, if φ is an L-mapping then the following holds $\varphi^{-1}(\tau) \subseteq \sigma \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$.*

PROOF. Let φ be a reverse isotone L-mapping. Then, if (x, y) is an element of σ, i.e. if $x \in U_\sigma(y)$, then $\varphi(x) \in \varphi(L_\sigma(y)) = L_\tau(\varphi(y))$. Thus, $(\varphi(x), \varphi(y)) \in \tau$ and it follows that φ is an isotone mapping. Further on, if $(\varphi(x), \varphi(y)) \in \tau$, i.e. if $\varphi(x) \in L_\tau(\varphi(y)) = \varphi(U_\sigma(y))$, then there exists an element t of A such that $\varphi(x) = \varphi(t)$ and $(t, y) \in \sigma$. Hence, out of the implication $(t, y) \in \sigma \Longrightarrow (t, x) \in \sigma \vee (x, y) \in \sigma$ we conclude $(x, y) \in \sigma$, because out $(t, x) \in \sigma$ we have $(\varphi(t), \varphi(x)) \in \tau$ and $\varphi(x) \neq \varphi(t)$ since φ is an isotone mapping. On the other hand, from $(x, y) \in \sigma$, i.e. from $x \in L_\sigma(y)$, we have $\varphi(x) \in \varphi(L_\sigma(y))$. Thus, there exists an element s of A such that $\varphi(x) = \varphi(s)$ with $(s, y) \in \sigma$. Since φ is an isotone mapping, we have $(\varphi(s), \varphi(y)) \in \tau$, i.e. we have $(s, y) \in \varphi^{-1}(\tau)$. So, finally, we proved the inclusion $\sigma \subseteq \varphi^{-1}(\tau) \circ Ker\varphi$. □

3.4. Some special strong mappings. In the article [**14**], Novotny studied a strong homomorphism between binary relational structures as a mapping $\varphi : A \longrightarrow B$ of (A, α) into (B, β) such that φ is isotone and $\varphi \circ \alpha = \beta \circ \varphi$ holds. By Lemma 1 in [**14**], it is equivalent with the following assertion: For any $a \in A$ and any $b' \in B$ the condition $(\varphi(a), b') \in \beta$ is equivalent to the existence of $a' \in A$ such that $(a, a') \in \alpha$ and $\varphi(a') = b'$. Developing this idea, we can analyze some results on such mappings in the following assertions.

REMARK 3.3. Let (A, α), (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a (strongly extensional) mapping. Then:

$$\beta \circ \varphi \subseteq \varphi \circ \alpha \Longrightarrow \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha. \tag{3.5}$$

If $(a, a') \in \varphi^{-1}(\beta)$, then $(\varphi(a), \varphi(a')) \in \beta$ and $(a, \varphi(a)) \in \varphi$. Thus, $(a, \varphi(a')) \in \beta \circ \varphi \subseteq \varphi \circ \alpha$. So, by definition of the composition of relations, there exists an element a'' of A such that $(a, a'') \in \alpha$ and $(a'', \varphi(a')) \in \varphi$, i.e. $(a'', a') \in Ker\varphi$. Finally, we have $(a, a') \in Ker\varphi \circ \alpha$.

So, from (2.3), (3.3) and (3.5) we conclude:

THEOREM 3.5. *Every strongly extensional Novotny strong mapping $\varphi : A \longrightarrow B$ between two co-quasiordered relational systems is an isotone U-mapping.*

In the following theorem we give the opposite assertion:

THEOREM 3.6. *Let (A, α), (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a (strongly extensional) surjective mapping. Then:*

$$\alpha \subseteq \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha \Longrightarrow \beta \circ \varphi = \varphi \circ \alpha. \tag{3.6}$$

PROOF. Let $a \in A$ and $b' \in B$ be elements such that $(a, b') \in \beta \circ \varphi$. Thus, there exists an element $b \in B$ with $(a, b) \in \varphi$ and $(b, b') \in \beta$. Since φ is a surjective mapping, there exists an element $a' \in A$ such that $b' = \varphi(a')$. So, we have $(\varphi(a), \varphi(a')) \in \beta$ and out of $(a, a') \in \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha$ we conclude that there exists an element $a'' \in A$ such that $(a, a'') \in \alpha$ and $\varphi(a'') = \varphi(a') = b'$. Hence, finally, we have $(a, b') \in \varphi \circ \alpha$. Combining this result with (2.3) we have proved the implication. □

Combining the comment before Theorem 3.2 and the result (3.6) of Theorem 3.6 we have:

COROLLARY 3.1. *Let (A, σ), (B, τ) be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional surjective mapping. Then φ is a Novotny strong mapping if and only if it is an isotone U-mapping.*

Following Novotny definition of strong mappings, let us analyze a new condition for mapping between binary relational structures: a mapping φ of (A, α) into (B, β) such that

$$\alpha \circ \varphi^{-1} = \varphi^{-1} \circ \beta$$

holds. When φ is surjective it is easy to see that the previous condition is equivalent with the following formula

$$(\forall b \in B)(\forall a' \in A)((b, \varphi(a')) \in \beta \Longleftrightarrow (\varphi^{-1}(b), a') \in \alpha).$$

Indeed, for (b, a') of $\varphi^{-1} \circ \beta$ there exists an element $b' \in B$ such that $(b, b') \in \beta$ and $(b', a') \in \varphi^{-1}$. Thus, $(b, \varphi(a')) \in \beta$. On the other hand, for $(b, a') \in \alpha \circ \varphi^{-1}$ there exists an element $a'' \in A$ with $(a'', a') \in \alpha$ and $(b, a'') \in \varphi^{-1}$. Finally, we have $(\varphi^{-1}(b), a') \in \alpha$.

Conversely, suppose that the above formula is valid. Then, we have

(i) $$\begin{aligned} (b, a') \in \varphi^{-1} \circ \beta &\Longleftrightarrow (\exists b' \in B)((b, b') \in \beta \wedge (b', a') \in \varphi^{-1}) \\ &\Longrightarrow (b, (\varphi(a')) \in \beta \\ &\Longleftrightarrow (\varphi^{-1}(b), a') \in \alpha \\ &\Longrightarrow (b, a') \in \alpha \circ \varphi^{-1}; \end{aligned}$$

(ii) $$\begin{aligned} (b, a') \in \alpha \circ \varphi^{-1} &\Longleftrightarrow (\exists a'' \in A)((b, a'') \in \varphi^{-1} \wedge (a'', a') \in \alpha) \\ &\Longrightarrow (\varphi^{-1}(b), a') \in \alpha \\ &\Longleftrightarrow (b, \varphi(a')) \in \beta \end{aligned}$$

$$\Longrightarrow (b, a') \in \varphi^{-1} \circ \beta.$$

Developing this idea, we have the following theorem:

THEOREM 3.7. *Let (A, α) and (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a (strongly extensional) mapping. Then*

$$\varphi^{-1} \circ \beta \subseteq \alpha \circ \varphi^{-1} \Longrightarrow \varphi^{-1}(\beta) \subseteq \alpha \circ Ker\varphi; \tag{3.7}$$

and

$$\alpha \circ \varphi^{-1} \subseteq \varphi^{-1} \circ \beta \Longrightarrow \alpha \subseteq \varphi^{-1}(\beta). \tag{3.8}$$

PROOF. Let $(\varphi(x), \varphi(x')) \in \beta$. Then, out of $(\varphi(x), \varphi(x')) \in \beta$ and $(\varphi(x'), x') \in \varphi^{-1}$ we conclude $(\varphi(x), x') \in \varphi^{-1} \circ \beta \subseteq \alpha \circ \varphi^{-1}$. Hence, there exists an element x'' of A such that $(\varphi(x), x'') \in \varphi^{-1}$ and $(x'', x') \in \alpha$. So, finally, we have $(x, x') \in \alpha \circ Ker\varphi$.

$$\begin{aligned}
(x, x') \in \alpha &\Longrightarrow (\varphi(x), x) \in \varphi^{-1} \wedge (x, x') \in \alpha \\
&\Longrightarrow (\varphi(x), x') \in \alpha \circ \varphi^{-1} \subseteq \varphi^{-1} \circ \beta \\
&\Longrightarrow (\exists y \in B)((\varphi(x), y) \in \beta \wedge (y, x') \in \varphi^{-1}) \\
&\Longrightarrow (\varphi(x), \varphi(x')) \in \beta \\
&\Longleftrightarrow (x, x') \in \varphi^{-1}(\beta).
\end{aligned}$$

Therefore, we have the following implication:

$$\varphi^{-1} \circ \beta = \alpha \circ \varphi^{-1} \Longrightarrow \alpha \subseteq \varphi^{-1}(\beta) \subseteq \alpha \circ Ker\varphi.$$

□

Similar to the case of Novotny strong mapping, we have the following:

THEOREM 3.8. *Let (A, α) and (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a (strongly extensional) mapping. Then*

$$\alpha \subseteq \varphi^{-1}(\beta) \Longrightarrow \alpha \circ \varphi^{-1} \subseteq \varphi^{-1} \circ \beta. \tag{3.9}$$

PROOF. Let (b', a) be an arbitrary element of $B \times A$ such that $(b', a) \in \alpha \circ \varphi^{-1}$. Thus, there exists an element a' of A with $(b', a') \in \varphi^{-1}$ and $(a', a) \in \alpha$. Since φ is an isotone mapping, we have $\varphi(a') = b'$, $(\varphi(a'), \varphi(a)) \in \beta$ and $(b', \varphi(a)) \in \beta$. Finally, out of $(b', \varphi(a)) \in \beta$ and $(\varphi(a), a) \in \varphi^{-1}$ we conclude that $(b, a) \in \varphi^{-1} \circ \beta$. □

Combining the results (2.3), (3.8) and (3.9) we have the equivalences

$$\varphi \circ \alpha \subseteq \beta \circ \varphi \Longleftrightarrow \alpha \subseteq \varphi^{-1}(\beta) \Longleftrightarrow \alpha \circ \varphi^{-1} \subseteq \varphi^{-1} \circ \beta.$$

THEOREM 3.9. *Let (A, σ) and (B, τ) be co-quasiordered relational systems and let $\varphi : A \longrightarrow B$ be a strongly extensional surjective mapping. Then:*

$$\varphi^{-1}(\tau) \subseteq \sigma \circ Ker\varphi \Longrightarrow \varphi^{-1} \circ \tau \subseteq \sigma \circ \varphi^{-1}. \tag{3.10}$$

Proof. If $(b', a) \in \varphi^{-1} \circ \tau$ then there exists an element $b \in B$ with $(b', b) \in \tau \wedge (b, a) \in \varphi^{-1}$ and since φ is surjective, there exists an element $a' \in A$ such that $(a', b') \in \varphi$. Thus, we have $(\varphi(a'), \varphi(a)) \in \tau$, i.e. we have $(a', a) \in \varphi^{-1}(\tau) \subseteq \sigma \circ Ker\varphi$. Therefore, there exists an element $a'' \in A$ with $(a'', a) \in \sigma$ and $\varphi(a') = \varphi(a'') = b'$, i.e. we have the following conjunction $(a'', a) \in \sigma \wedge (b', a'') \in \varphi^{-1}$. This means that $(b', a) \in \sigma \circ \varphi^{-1}$. □

Combining formulas (3.7)-(3.10) we have an answer to the question (b):

Corollary 3.2. *Let (A, σ), (B, τ) be co-quasiordered relational systems and $\varphi : A \longrightarrow B$ a strongly extensional surjective mapping. Then:*

$$\sigma \subseteq \varphi^{-1}(\tau) \subseteq \sigma \circ Ker\varphi \Longleftrightarrow \varphi^{-1} \circ \tau = \sigma \circ \varphi^{-1}.$$

Let (b', a) be an arbitrary element of $\varphi^{-1} \circ \tau$. Then, there exists an element b of B with $(b', b) \in \tau$ and $(b, a) \in \varphi^{-1}$. If the mapping φ is a surjective mapping, then there exists an element a' of A with $(a', b') \in \varphi$. $(a', a) \in \sigma$ follows from $(\varphi(a'), \varphi(a)) \in \tau$ if φ is a reverse isotone mapping. This gives $(b', a') \in \varphi^{-1}$ and $(a', a) \in \sigma$. Finally, we have $(b', a) \in \sigma \circ \varphi^{-1}$. Therefore, the following theorem is obtained:

Theorem 3.10. *Let (A, σ) and (B, τ) be relational systems and let $\varphi : A \longrightarrow B$ be a (strongly extensional) mapping. Then*

$$\varphi^{-1}(\tau) \subseteq \sigma \Longrightarrow \varphi^{-1} \circ \tau \subseteq \sigma \circ \varphi^{-1}. \tag{3.11}$$

Suppose $\sigma \subseteq \varphi^{-1}(\beta) \circ Ker\varphi$. Then $\alpha \circ \varphi^{-1} \subseteq \varphi^{-1} \circ \beta$. Indeed, if (b', a) is an arbitrary element of $\alpha \circ \varphi^{-1}$, then there exists an element a' of A such that $(b', a') \in \varphi^{-1}$ and $(a', a) \in \alpha \subseteq \varphi^{-1}(\beta) \circ Ker\varphi$. Thus, there exists an element a'' of A with $\varphi(a') = \varphi(a'') = b'$ and $(b', \varphi(a)) \in \beta$. Out of $(b', \varphi(a)) \in \beta$ and $(\varphi(a), a) \in \varphi^{-1}$ we have $(b', a) \in \varphi^{-1} \circ \beta$. Therefore, we have the following theorem:

Theorem 3.11. *Let (A, α) and (B, β) be relational systems and let $\varphi : A \longrightarrow B$ be a (strongly extensional) mapping. Then*

$$\alpha \subseteq \varphi^{-1}(\beta) \circ Ker\varphi \Longrightarrow \alpha \circ \varphi^{-1} \subseteq \varphi^{-1} \circ \beta. \tag{3.12}$$

Combining (3.11) and (3.12), at the end of this section, for co-quasiordered relational systems (A, σ) and (B, τ) we have the following implication

$$\varphi^{-1}(\tau) \subseteq \sigma \subseteq \varphi^{-1}(\tau) \circ Ker\varphi \Longrightarrow \sigma \circ \varphi^{-1} = \varphi^{-1} \circ \tau.$$

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(Home) 6, Kordunaška Street, 78000 Banja Luka, Bosnia and Herzegovina
E-mail address: : bato49@hotmail.com

(Work 1) Banja Luka University, Faculty of Mechanical Engineering, 71, Vojvoda Stepa Stepanović, 78000 Banja Luka, Bosnia and Herzegovina

(Work 2) East Sarajevo University, Faculty of Education, 76300 Bijeljina, b.b., Semberski Ratari Street, Bosnia and Herzegobina