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THE RESTRAINED EDGE MONOPHONIC NUMBER OF A GRAPH

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Abstract. A set S of vertices of a connected graph G is a *monophonic set* if every vertex of G lies on an $x-y$ monophonic path for some elements x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G, denoted by $m(G)$. A set S of vertices of a graph G is an *edge monophonic set* if every edge of G lies on an $x-y$ monophonic path for some elements x and y in S. The minimum cardinality of an edge monophonic set of G is the *edge monophonic number* of G, denoted by $em(G)$. A set S of vertices of a graph G is a *restrained edge monophonic set* if either $V = S$ or S is an edge monophonic set with the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertices. The minimum cardinality of a restrained edge monophonic set of G is the *restrained edge monophonic number* of G and is denoted by $em_r(G)$. It is proved that, for the integers a, b and c with $3 \leqslant a \leqslant b < c$, there exists a connected graph G having the monophonic number a, the edge monophonic number b and the restrained edge monophonic number c.

1. Introduction

By a graph $G = (V, E)$ we mean a simple graph of order at least two. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology, we refer to Harary [**5**]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. A vertex v is a *semi-extreme vertex* of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex.

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For any two vertices x and y in a connected graph G, the *distance* $d(x,y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x,y)$ is called an $x-y$ *geodesic*. A vertex v is said to lie on an $x-y$ geodesic P if v is a vertex of P including the vertices x and y.

The *closed interval* $I[x,y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup\limits_{x,y\in S} I[x,y]$. A set S of vertices is a *geodetic set* if $I[S]=V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [**1, 6**] and further studied in [**2, 3, 4, 5**]. A set S of vertices of a graph G is an *edge geodetic set* if every edge of G lies on an $x-y$ geodesic for some elements x and y in S. The minimum cardinality of an edge geodetic set of G is the *edge geodetic number* of G, denoted by $eg(G)$. The edge geodetic number was introduced and studied in [**8**]. A set S of vertices of G is a *restrained edge geodetic set* of G if S is an edge geodetic set, and if either $S=V$ or the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertices. The minimum cardinality of a restrained edge geodetic set of G is the *restrained edge geodetic number*, denoted by $eg_r(G)$. The restrained edge geodetic number of a graph was introduced and studied in [**10**].

A *chord* of a path $u_1, u_2, \ldots, u_k$ in G is an edge u_iu_j with $j \geqslant i+2$. A u-v path P is called a *monophonic path* if it is a chordless path. A set S of vertices is a *monophonic set* if every vertex of G lies on a monophonic path joining some pair of vertices in S, and the minimum cardinality of a monophonic set is the *monophonic number* $m(G)$. A monophonic set of cardinality $m(G)$ is called an *m-set* of G. The monophonic number of a graph G was studied in [**9**]. A set S of vertices of a graph G is an *edge monophonic set* if every edge of G lies on an $x-y$ monophonic path for some elements x and y in S. The minimum cardinality of an edge monophonic set of G is the *edge monophonic number* of G, denoted by $em(G)$. A set S of vertices of a graph G is a *restrained monophonic set* if either $S=V$ or S is an monophonic set with the subgraph $G[V-S]$ induced by $V-S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the *restrained monophonic number* of G, and is denoted by $m_r(G)$. The restrained monophonic number of a graph was studied in [**11**].

The following theorems will be used in the sequel.

THEOREM 1.1. [**5**] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every $u-w$ path.*

(iii) *There exists a partition of the set of vertices $V-\{v\}$ into subsets U and W such that for any vertices $u\in U$ and $w\in W$, the vertex v is on every $u-w$ path.*

THEOREM 1.2. [**9**] *Each extreme vertex of a connected graph G belongs to every monophonic set of G.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Restrained Edge Monophonic Number

DEFINITION 2.1. *A set S of vertices of a graph G is a restrained edge monophonic set if either $V = S$ or S is an edge monophonic set with the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained edge monophonic set of G is the restrained edge monophonic number of G, and is denoted by $em_r(G)$.*

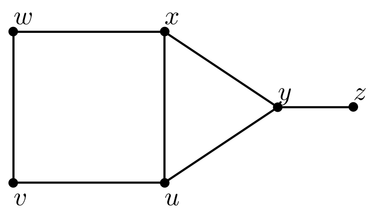

Figure 2.1: G

EXAMPLE 2.1. *For the graph G given in Figure 2.1, it is clear that $S_1 = \{z, w\}$, $S_2 = \{z, v\}$ are the minimum monophonic sets of G and so $m(G) = 2$; $S_3 = \{z, w, u\}$, $S_4 = \{z, v, x\}$ are the minimum edge monophonic sets of G and so $em(G) = 3$; and $S_5 = \{z, w, v, u\}$, $S_4 = \{z, w, v, x\}$ are the minimum restrained edge monophonic sets of G and so $em_r(G) = 4$. Thus the monophonic number, the edge monophonic number and the restrained edge monophonic number of a graph are all different.*

THEOREM 2.1. Each semi-extreme vertex of a graph G belongs to every restrained edge monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is an restrained edge monophonic set, then S is the unique minimum restrained edge monophonic set of G.

PROOF. Let S be the set of all semi-extreme vertices of G and let T be any restrained edge monophonic set of G. Suppose that there exists a vertex $u \in S$ such that $u \notin T$. Since $\Delta(< N(u) >) = |N(u)| - 1$, there exists a $v \in N(u)$ such that $deg_{<N(u)>}(v) = |N(u)| - 1$. Since T is a restrained edge monophonic set of G, the edge $e = uv$ lies on an $x - y$ monophonic path $P : x = x_0, x_1, \ldots, x_{i-1}, x_i = u, x_{i+1} = v, \ldots, x_n = y$ with $x, y \in T$. Since $u \notin T$, it is clear that u is an internal vertex of the path P. Since $deg_{<N(u)>}(v) = |N(u)| - 1$, we see that v is adjacent to x_{i-1}, which is a contradiction to the fact that P is an $x - y$ monophonic path. Hence S is contained in every restrained edge monophonic set of G. □

Every restrained edge monophonic set is an edge monophonic set and the converse need not be true. For the graph G given in Figure 2.1, S_3 is an edge monophonic set, however it is not a restrained edge monophonic set. Also, every edge monophonic set is a monophonic set and so every restrained edge monophonic set is a monophonic set of a graph G. Since every restrained edge monophonic set of G is an edge monophonic set, by Theorem 2.1, each semi-extreme vertex of a connected graph G belongs to every restrained edge monophonic set of G. Hence for the complete graph $K_p (p \geqslant 2)$, $em_r(K_p) = p$.

The next theorem follows from the respective definitions.

THEOREM 2.2. *For any connected graph* G, $2 \leqslant m(G) \leqslant em(G) \leqslant em_r(G) \leqslant p$.

If $em(G) = p$ or $p-1$, then $em_r(G) = p$. The converse need not be true. For the cycle C_4, $em(C_4) = 2 = p - 2$ and $em_r(C_4) = 4 = p$. Also, since every restrained edge monophonic set of G is an edge monophonic set of G and the complement of each restrained edge monophonic set has cardinality different from 1, we have $em_r(G) \neq p - 1$. Thus there is no graph G of order p with $em_r(G) = p - 1$.

THEOREM 2.3. If a graph G of order p has exactly one vertex of degree $p - 1$, then $em_r(G) = p$.

PROOF. Let G be a graph of order p with exactly one vertex of degree $p - 1$, and let it be u. Since the vertex u is adjacent to all other vertices in G, then any edge uv where $v \in V(G) - \{u\}$, is not an internal edge of any monophonic path joining two vertices of G other than u and v. Hence $em_r(G) = p$. □

REMARK 2.1. The converse of the Theorem 2.3 need not be true. For the cycle C_4, all the vertices of C_4 is the unique minimum restrained edge monophonic set of G, but it does not have a vertex of degree $p - 1 = 3$.

The following theorem is easy to verify.

THEOREM 2.4. *(i) If T is a tree with k end vertices, then*

$$em_r(T) = \begin{cases} p & \text{if } T \text{ is a star} \\ k & \text{if } T \text{ is not a star.} \end{cases}$$

(ii) For the cycle $C_p (p \geqslant 3)$,

$$em_r(C_p) = \begin{cases} p & \text{for } p < 6 \\ 2 & \text{for } p \geqslant 6. \end{cases}$$

(iii) For the wheel $W_p = K_1 + C_{p-1} (p \geqslant 5)$, $em_r(W_p) = p$.
(iv) For the complete bipartite graph $K_{m,n} (m, n \geqslant 2)$, $em_r(K_{m,n}) = m + n$.
(v) For the hyper cube Q_n, $em_r(Q_n) = 2$.

THEOREM 2.5. *Let G be a connected graph with every vertex of G is either a cut vertex or an extreme vertex. Then $em_r(G) = p$ if and only if $G = K_1 + \bigcup m_j K_j$.*

PROOF. Let $G = K_1 + \bigcup m_j K_j$. Then G has at most one cut vertex. Suppose that G has no cut vertex. Then $G = K_p$ and hence $em_r(G) = p$. Suppose that G has exactly one cut vertex. Then all the remaining vertices are extreme vertices and hence $em_r(G) = p$.

Conversely, suppose that $em_r(G) = p$. If $p = 2$, then $G = K_2 = K_1 + K_1$. If $p \geqslant 3$, there exists a vertex x, which is not a cut vertex of G. If G has two or more cut vertices, then the induced subgraph of the cut vertices is a non-trivial path.

Then the set of all extreme vertices is the minimum restrained edge monophonic set of G and so $em_r(G) \leqslant p-2$, which is a contradiction. Thus, the number of cut vertices k of G is at most one.

Case 1. If $k = 0$, then the graph G is a block. If $p = 3$, then $G = K_3 = K_1 + K_2$. If $p \geqslant 4$, we claim that G is complete. Suppose G is not complete. Then there exist two vertices x and y in G such that $d(x, y) \geqslant 2$. By Theorem 1.1, both x and y lie on a common cycle and hence x and y lie on a smallest cycle $C : x, x_1, ..., y, ..., x_n, x$ of length at least 4. Thus every vertex of C on G is neither a cut vertex nor an extreme vertex, which is a contradiction to the assumption. Hence G is the complete graph K_p and so $G = K_1 + K_{p-1}$.

Case 2. If $k = 1$, let x be the cut vertex of G. If $p = 3$, then $G = P_3 = K_1 + \bigcup m_j K_1$, where $\sum m_j = 2$. If $p \geqslant 4$, we claim that $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geqslant 2$. It is enough to prove that every block of G is complete. Suppose there exists a block B, which is not complete. Let u and v be two vertices in B such that $d(u, v) \geqslant 2$. Then by Theorem 1.1, both u and v lie on a common cycle and hence u and v lie on a smallest cycle of length at least 4. Hence every vertex of C on G is neither a cut vertex nor an extreme vertex, which is a contradiction. Thus every block of G is complete so that $G = K_1 + \bigcup m_j K_j$, where K_1 is the vertex x and $\sum m_j \geqslant 2$. □

A *caterpillar* is a tree for which the removal of all the end vertices gives a path.

THEOREM 2.6. *For every non-trivial tree T with diameter $d \geqslant 3$, $em_r(T) = p - d + 1$ if and only if T is a caterpillar.*

PROOF. Let T be any non-trivial tree with diameter $d \geqslant 3$. Let $P : u = v_0, v_1, ..., v_d = v$ be a diametral path. Let k be the number of end vertices of T and let l be the number of internal vertices of T other than $v_1, v_2, ..., v_{d-1}$. Then $d - 1 + l + k = p$. By Theorem 2.4(i), $em_r(T) = k$ and so $em_r(T) = p - d - l + 1$. Hence $em_r(T) = p - d + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is a caterpillar. □

The next theorem gives a realization result of the monophonic number, the edge monophonic number and the restrained edge monophonic number.

THEOREM 2.7. *For any integers a, b and c with $3 \leqslant a \leqslant b < c$, then there exists a connected graph G such that $m(G) = a$, $em(G) = b$ and $em_r(G) = c$.*

PROOF. **Case 1.** $3 \leqslant a = b < c$.

Let $K_{2,c-a+2}$ be the complete bipartite graph with bipartite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, ..., y_{c-a+2}\}$ and let $P_3 : u_1, u_2, u_3$ be a path of order 3. Let H be the graph obtained from $K_{2,c-a+2}$ and P_3 by identifying the vertex x_2 in $K_{2,c-a+2}$ with the vertex u_1 in P_3. Add $a-2$ new vertices $v_1, v_2, ..., v_{a-2}$ to H and join each vertex v_i $(1 \leqslant i \leqslant a-2)$ with the vertex u_3. The graph G is shown in Figure 2.2.

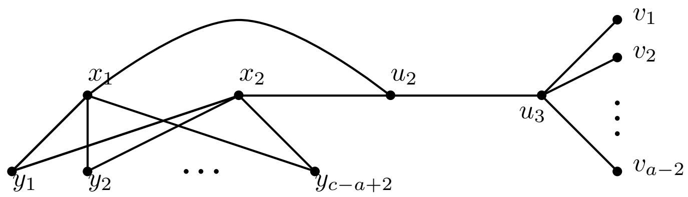

Figure 2.2: G

Let $S = \{v_1, v_2, , ..., v_{a-2}\}$ be the set of all extreme vertices of G. By Theorems 1.2 and 2.1, S is a subset of every monophonic set, edge monophonic set and restrained edge monophonic set of G. It is clear that $S_1 = S \cup \{x_1, x_2\}$ is both the unique minimum monophonic set and unique minimum edge monophonic set of G and so $m(G) = em(G) = a$. Also, $S_2 = S \cup \{y_1, y_2, ..., y_{c-a+2}\}$ is a minimum restrained edge monophonic set of G and so $em_r(G) = c$.

Case 2. $a + 1 = b < c$.

Let $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of order 5. Let G be the graph obtained from C_5 by adding $c - b + a - 1$ new vertices $u_1, u_2, ..., u_{a-1}, w_1, w_2, ..., w_{c-b}$ and joining each u_i $(1 \leqslant i \leqslant a-1)$ to the vertex v_1; joining each w_i $(1 \leqslant i \leqslant c-b)$ to both the vertices v_3, v_5; and joining the vertices v_2 and v_5. The graph G is shown in Figure 2.3.

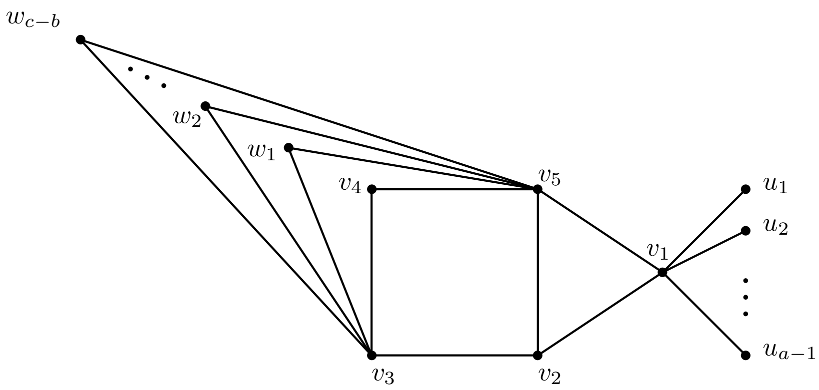

Figure 2.3: G

Let $S = \{u_1, u_2, , ..., u_{a-1}\}$ be the set of all extreme vertices of G. By Theorems 1.2 and 2.1, S is a subset of every monophonic set, edge monophonic set and restrained edge monophonic set of G. It is clear that S is not a monophonic set of G and so $m(G) > a$. It is clear that $S_1 = S \cup \{v_3\}$ is a monophonic set of G and so $m(G) = a$. Also, since the edge v_2v_5 does not lie on any $x - y$ monophonic path for some vertices $x, y \in S_1$, we have S_1 is not an edge monophonic set of G and so $em_r(G) > b$. Let $S_2 = S_1 \cup \{v_5\}$. Clearly, S_2 is an edge monophonic set of G and so $em(G) = |S_2| = a + 1$. Also, it is clear that $S_3 = S \cup \{v_2, v_4, w_1, w_2, ..., w_{c-b}\}$ is a minimum restrained edge monophonic set of G and so $em_r(G) = c$.

Case 3. $a+2 \leqslant b < c$.

Let $P_2 : x, y$ be a path of order 2 and let $P_{b-a+1} : u_1, u_2, , ..., u_{b-a+1}$ be a path of order $b-a+1$. Let H be the graph obtained from P_2 and P_{b-a+1} by joining the vertices u_i $(1 \leqslant i \leqslant b-a+1)$ with y and also joining the vertices x and u_{b-a+1}. Let G be the graph obtained from H by adding $c-b+a-1$ new vertices $v_1, v_2, ..., v_{a-1}, w_1, w_2, ..., w_{c-b}$ and joining each v_i $(1 \leqslant i \leqslant a-1)$ to the vertex x and joining each w_i $(1 \leqslant i \leqslant c-b)$ with the vertices u_1 and u_{b-a+1}. The graph G is shown in Figure 2.4.

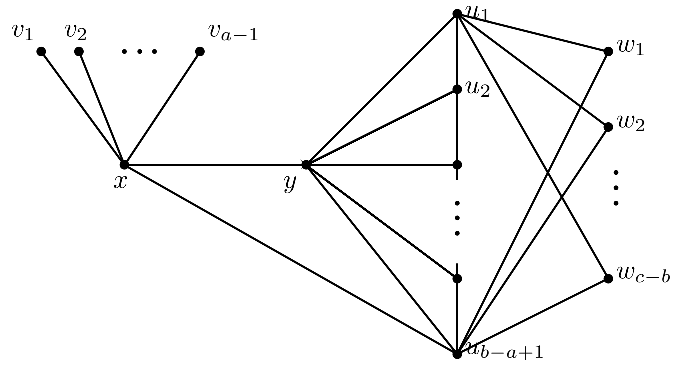

Figure 2.4: G

Let $S = \{v_1, v_2, , ..., v_{a-1}\}$ be the set of all extreme vertices of G. By Theorems 1.2 and 2.1, every monophonic set, edge monophonic set and restrained edge monophonic set contains S. Clearly, S is not a monophonic set of G and so $m(G) > a$. It is clear that $S_1 = S \cup \{u_1\}$ is a monophonic set of G and so $m(G) = a$. Let $S_2 = S \cup \{u_2, u_3, ..., u_{b-a}\}$ be the set of all semi-extreme vertices of G. By Theorem 2.1, S_2 is a subset of every edge monophonic set of G. Since the edge yu_{b-a+1} does not lie on any $x-y$ monophonic path for some vertices $x, y \in S_2$, we have S_2 is not an edge monophonic set of G and so $emG) > b-2$. It is clear that $S_3 = S_2 \cup \{u_1, u_{b-a+1}\}$ is an edge monophonic set of G and so $em(G) = b$. Also, it is clear that $S_4 = S_3 \cup \{w_1, w_2, ..., w_{c-b}\}$ is a minimum restrained edge monophonic set of G, we have $em_r(G) = c$. □

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