BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(2017), 43-52

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

DECOMPOSITION OF A C-ALGEBRA THROUGH PARTIAL ORDERINGS

P. Sundarayya, Ramesh Sirisetti, and V. Sriramani

Abstract. In this paper, we recall two partial orderings on a C-algebra. Two types of decompositions are derived by using these partial orderings, which are not dual to each other. Two sufficient conditions for a C-algebra to become a Boolean algebra in terms of C-algebras L_a, R_a are obtained.

1. Introduction

In [**2**], Guzman and Squier introduced the variety of C-algebra as the variety generated by the three element algebra $C = \{T, F, U\}$ with the operations $\wedge, \vee$ and $'$ of type $(2, 2, 1)$, which is the algebraic form of the three valued conditional logic. They proved that the two element Boolean algebra B and C are the only subdirectly irreducible C-algebras and that the variety of C-algebras is a minimal cover of the variety of Boolean algebras. These C-algebras are of interest to logic and theoretical computer science.

In [**3, 5**], Rao and Sundarayya introduced two partial orderings on a C-algebra and derived a number of equivalent conditions for a C-algebra to become a Boolean algebra in terms of these partial orderings. In [**6**], Swamy, Rao and Ravi kumar introduced the centre of C-algebra and proved that it is a Boolean algebra.

In this paper, we recall two partial orderings $\leqslant_l$ and $\leqslant_r$ on a C-algebra. We obtain two C-algebraic structures on subsets of a C-algebra A (with T), which are not subalgebras of A. It is well known that if B is a Boolean algebra and $a \in B$, then B is isomorphic to $B \restriction a \times B \restriction a'$ (see [**1**]). We obtain a version of this decomposition for a C-algebra corresponding to these partial orderings. They are not symmetric to each other because C-algebras have no commutative property.

2010 *Mathematics Subject Classification.* 03G25, 06E99.
Key words and phrases. C-algebra, Central element, Boolean algebra.

2. Preliminaries

In this section, we collect some necessary definitions, examples and results of C-algebras from [**3, 5, 6**].

Definition 2.1. ([**2**]) By a C-algebra we mean algebra $(A, \wedge, \vee, ')$ of type $(2, 2, 1)$ satisfies the following identities;

(a) $x'' = x$
(b) $(x \wedge y)' = x' \vee y'$
(c) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
(d) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(e) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$
(f) $x \vee (x \wedge y) = x$
(g) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$

for all $x, y, z \in A$.

Example 2.1. ([**2**]) The three element set $\{T, F, U\}$ with operations $\wedge, \vee$ and $'$ given by

$\wedge$	T	F	U
T	T	F	U
F	F	F	F
U	U	U	U

$\vee$	T	F	U
T	T	T	T
F	T	F	U
U	U	U	U

x	x'
T	F
F	T
U	U

is a C-algebra . We denote this C-algebra by C and the two element C-algebra $\{T, F\}$ by B.

Example 2.2. ([**3**]) Let $G = \{g_1, g_2, g_3, g_4, g_5\}$ where $g_1 = (T, U), g_2 = (F, U)$, $g_3 = (U, F), g_4 = (U, T), g_5 = (U, U)$. Then G is a C-algebra with respect to the pointwise operations given in the following;

$\wedge$	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_2	g_5	g_5	g_5
g_2	g_2	g_2	g_2	g_2	g_2
g_3	g_5	g_5	g_3	g_4	g_5
g_4	g_4	g_4	g_4	g_4	g_4
g_5	g_5	g_5	g_5	g_5	g_5

$\vee$	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_1	g_1	g_1	g_1
g_2	g_1	g_2	g_5	g_5	g_5
g_3	g_3	g_3	g_3	g_3	g_3
g_4	g_5	g_5	g_3	g_4	g_5
g_5	g_5	g_5	g_5	g_5	g_5

x	x'
g_1	g_2
g_2	g_1
g_3	g_4
g_4	g_3
g_5	g_5

Example 2.3. ([**3**]) Let $C \times C = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}$ where $f_1 = (T, U)$, $f_2 = (F, U)$, $f_3 = (U, T)$, $f_4 = (U, F)$, $f_5 = (U, U)$, $f_6 = (T, T)$, $f_7 = (F, F)$, $f_8 = (T, F)$, $f_9 = (F, T)$. Then $C \times C$ is a C-algebra with respect to the pointwise operations.

It can be observe that the identities $(a), (b)$ imply that the variety of all C-algebras satisfies the dual statements (b) to (g). In general $\wedge$ and $\vee$ are not commutative in C and the ordinary right distributivity of $\wedge$ over $\vee$ fails in C.

Lemma 2.1. ([**2, 6**]) *If A is a C-algebra, then for any $x, y \in A$,*

(i) $x \wedge x = x$
(ii) $x \wedge (x' \vee y) = (x' \vee y) \wedge x = x \wedge (y \vee x') = x \wedge y$

(iii) $x \vee (x' \wedge x) = (x' \wedge x) \vee x = x$
(iv) $(x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y)$
(v) $x \vee x' = x' \vee x$
(vi) $x \vee y \vee x = x \vee y$
(vii) $x \wedge x' \wedge y = x \wedge x'$.

If A has an identity for $\wedge$, then it is unique and denoted by T (that is $x \wedge T = T \wedge x = x$ for all $x \in A$). In this case, we say that A is a C-algebra with T. If we write F for T', then F is the identity for $\vee$ (that is $x \vee F = F \vee x = x$ for all $x \in A$). Now, we have the following;

Lemma 2.2. ([**2**]) *If A is a C-algebra with T, then for any $x \in A$,*
(i) $T \vee x = x$ *and* $F \wedge x = F$
(ii) $x \vee T = x \vee x'$ *and* $x \wedge F = x \wedge x'$.

If there exists an element $x \in A$ such that $x' = x$, then it is unique and we denoted it by U (U is called the uncertain element of A).

Definition 2.2. ([**6**]) An element x of a C-algebra A with T is called a central element of A if $x \vee x' = T$.

If A is a C-algebra with T, then the set $\{x \in A \mid x \vee x' = T\}$ of central elements of A is called the centre of A and denoted by $B(A)$ [**6**]. It can be observed that $B(A)$ is a Boolean algebra with induced operations on A.

3. The partial orderings $\leqslant_l$ and $\leqslant_r$

In this section, we recall two partial orderings $\leqslant_l$ and $\leqslant_r$, and present that two C-algebra structures corresponding to $\leqslant_l$ and $\leqslant_r$ in a C-algebra which are not sub C-algebras of A. Some of the properties of these C-algebras are given in the following.

Lemma 3.1. ([**5**]) *A relation $\leqslant_l$ on a C-algebra A defined by $x \leqslant_l y$ if $x \wedge y = x$ and $x \vee y = y$, is a partial ordering on A.*

Example 3.1. ([**5**]) The Hasse diagrams of $(C, \leqslant_l), (G, \leqslant_l)$ are given in the following figure 1.

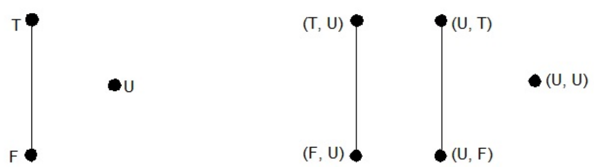

Figure 1

Lemma 3.2. ([**5**]) *A relation $\leqslant_r$ on a C-algebra A defined by $x \leqslant_r y$ if $x \wedge y = x$ and $y \vee x = y$, is a partial ordering on A.*

EXAMPLE 3.2. ([**5**]) The Hasse diagrams of $(C, \leqslant_r), (G, \leqslant_r)$ are given in the following figure 2.

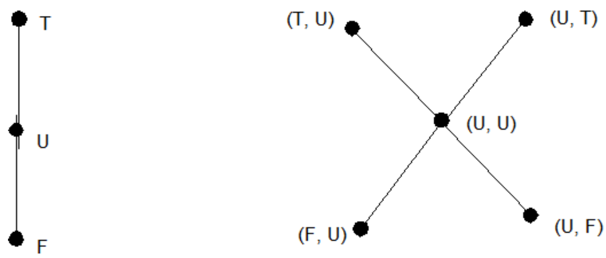

FIGURE 2

REMARK 3.1. " $\leqslant_r$ " and " $\leqslant_l$ " are not dual to each other, for example, in G(see Example 2.2), we have $g_3 \leqslant_r g_1$ but $g_1 \nleqslant_l g_3$ and $g_2 \leqslant_l g_4$ but $g_4 \nleqslant_r g_2$

Now, we prove the following.

LEMMA 3.3. *If A is a C-algebra with T, then, for any $x, y \in A$, $x \leqslant_l y$ implies $y \vee x = x \vee y = y$.*

PROOF. Let $x, y \in A$ such that $x \leqslant_l y$. Then $x \wedge y = x$ and $x \vee y = y$. Now,

$$\begin{array}{rcll} y \vee x & = & (x \vee y) \vee (x \wedge y) & \\ & = & (x \vee y \vee x) \wedge (x \vee y \vee y) & \text{(by the dual of Def. 2.1(d))} \\ & = & (x \vee y) \wedge (x \vee y) & \text{(by Lemma 2.1(vi))} \\ & = & x \vee y & \\ & = & y & \end{array}$$

Therefore $y \vee x = x \vee y = y$. □

In the following, it is defined that a C-algebra corresponding to the partial ordering $\leqslant_l$

THEOREM 3.1. *If A is a C-algebra with T and $a \in A$, then the set*

$$L_a = \{x \in A \mid a \leqslant_l x\}$$

is a C-algebra with the induced operations $\wedge$, $\vee$ and the complementation $*$ is defined by $x^* = a \vee x'$, for any $x \in L_a$.

PROOF. Let $x,\ y \in L_a$. Then $a \leqslant_l x,\ y$. That is $a \wedge x = a \wedge y = a$, $a \vee x = x \vee a = x$ and $a \vee y = y \vee a = a$ (see Lemma 3.3). Now,

$$\begin{array}{rcll} a \wedge (x \vee y) & = & (a \wedge x) \vee (a \wedge y) & \text{(by Def. 2.1(d))} \\ & = & a \vee a & \\ & = & a & \end{array}$$

$$\begin{array}{rcll} a \vee (x \vee y) & = & a \vee x \vee a \vee y & \text{(by Lemma 2.1(vi))} \\ & = & x \vee y & \end{array}$$

Therefore $a \leqslant_l x \vee y$ and hence $x \vee y \in L_a$. Similarly,

$$\begin{aligned}
a \wedge (x \wedge y) &= a \wedge x \wedge a \wedge y \quad \text{(by the dual of Lemma 2.1(vi))} \\
&= a \wedge a \\
&= a
\end{aligned}$$

$$\begin{aligned}
a \vee (x \wedge y) &= (a \vee x) \wedge (a \vee y) \quad \text{(by the dual of Def. 2.1(d))} \\
&= x \wedge y
\end{aligned}$$

Therefore $a \leqslant_l x \wedge y$ and hence $x \wedge y \in L_a$. Now, $a \wedge x^* = a \wedge (a \vee x') = a$ (by the dual of Def. 2.1(f)) and $a \vee x^* = a \vee (a \vee x') = a \vee x' = x^*$. Therefore $a \leqslant_l x^*$ and hence $x^* \in L_a$. Thus L_a is closed under $\wedge$, $\vee$ and $*$. Now,
for $x \in L_a$,

$$\begin{aligned}
x^{**} &= (x^*)^* \\
&= (a \vee x')^* \\
&= a \vee (a \vee x')' \\
&= a \vee (a' \wedge x) \quad \text{(by the dual of Def 2.1(a, b))} \\
&= a \vee x \quad \text{(by Lemma 2.1(ii))} \\
&= x.
\end{aligned}$$

For $x,\ y \in L_a$,

$$\begin{aligned}
(x \wedge y)^* &= a \vee (x \wedge y)' \\
&= a \vee (x' \vee y') \quad \text{(by Def. 2.1(b))} \\
&= (a \vee x') \vee (a \vee y') \quad \text{(by Lemma 2.1(vi))} \\
&= x^* \vee y^*.
\end{aligned}$$

For $x,\ y,\ z \in L_a$,

$$\begin{aligned}
(x \vee y) \wedge z &= a \vee ((x \vee y) \wedge z) && \text{(since } a \leqslant_l (x \vee y) \wedge z) \\
&= a \vee ((x \wedge z) \vee (x' \wedge y \wedge z)) && \text{(by Def. 2.1(e))} \\
&= (x \wedge z) \vee a \vee (x' \wedge y \wedge z) && \text{(since } a \leqslant_l x \wedge z) \\
&= (x \wedge z) \vee ((a \vee x') \wedge (a \vee y) \wedge (a \vee z)) && \text{(by Def. 2.1(d))} \\
&= (x \wedge z) \vee (x^* \wedge y \wedge z) && \text{(since } a \leqslant_l y,\ z)
\end{aligned}$$

The remaining identities hold in L_a, since they hold in A. Thus $(L_a, \wedge, \vee, *)$ is a C-algebra. □

We observe that L_a is itself a C-algebra but not a sub C-algebra of A, since the unary operation $*$ is not the restriction of $'$ to L_a. It can be easily prove that a is the join identity in L_a.

LEMMA 3.4. *Let A be a C-algebra with T. If $x \in A$ and $a \in B(A)$, then $x \leqslant_r a$ implies $x \wedge a = a \wedge x = x$.*

PROOF. Let $x \in A$, $a \in B(A)$ such that $x \leqslant_r a$. Then $x \wedge a = x$ and $a \vee x = a$. Now,

$$\begin{aligned} a \wedge x &= (a \vee x) \wedge x && \text{(since } a \vee x = a\text{)} \\ &= (a \wedge x) \vee (a' \wedge x \wedge x) && \text{(by Def. 2.1}(e)\text{)} \\ &= (a \wedge x) \vee (a' \wedge x) \\ &= (a \vee a') \wedge x && \text{(by Lemma 2.1(iv))} \\ &= T \wedge x && \text{(since } a \in B(A)\text{)} \\ &= x. \end{aligned}$$

Therefore $x \wedge a = a \wedge x = x$. □

In the following, it is defined that a C-algebra corresponding to the partial ordering $\leqslant_r$

THEOREM 3.2. *If A is a C-algebra with T and $a \in B(A)$, then the set*

$$R_a = \{x \in A \mid x \leqslant_r a\}$$

is a C-algebra with the induced operations $\wedge$, $\vee$ and the complementation $*$ is defined by $x^* = a \wedge x'$, for any $x \in R_a$

In the above lemma, if $a \notin B(A)$, then $a \wedge x$ need not be equal to x. For example, in $C \times C$ (see Example 2.3), we have $f_7 \leqslant_r f_3$ but $f_4 = f_3 \wedge f_7 \neq f_7$, where $f_3 \notin B(C \times C)$.

We observe that R_a is itself a C-algebra but not a sub C-algebra of A, since the unary operation $*$ is not the restriction of $'$ to R_a. It can be easily prove that a is the meet identity in R_a. Moreover, if a is not a central element, then the set R_a need not be a C-algebra. For example, in $C \times C$(see Example 2.3), $f_9 \in R_{f_3}$, $f_9 \neq f_9^{**}$ where f_3 is not a central element. Thus to become R_a is a C-algebra, it is necessary that a must be a central element.

4. Decompositions through $\leqslant_l$, and $\leqslant_r$

In this section, we obtain decompositions of a C-algebra with T corresponding to the partial orderings $\leqslant_l$ and $\leqslant_r$ and any decompositions of A is in the same form. We derive some sufficient conditions for a C-algebra to become a Boolean algebra.

DEFINITION 4.1. Let $a \in A$. Define the mapping $\alpha_a : A \to L_a$ is defined by $\alpha_a(x) = a \vee x$, for all $x \in A$.

For any $a \in A$, the set $\varphi_a = \{(x, y) \in A \times A \mid \alpha_a(x) = \alpha_a(y)\}$ is a congruences relation on A. Now, we have the following.

THEOREM 4.1. *Let A be a C-algebra with T and $a \in A$. Then α_a is a homomorphism from A onto L_a with kernel φ_a and hence $\dfrac{A}{\varphi_a} \cong L_a$.*

PROOF. Let $x, y \in A$. Then

$$\begin{array}{rcll} \alpha_a(x \wedge y) & = & a \vee (x \wedge y) & \\ & = & (a \vee x) \wedge (a \vee y) & \text{(by the dual of Def. 2.1(d))} \\ & = & \alpha_a(x) \wedge \alpha_a(y) & \\ \alpha_a(x \vee y) & = & a \vee (x \vee y) & \\ & = & (a \vee x) \vee (a \vee y) & \text{(by Lemma 2.1(vi))} \\ & = & \alpha_a(x) \vee \alpha_a(y). & \end{array}$$

and $\alpha_a(x') = a \vee x' = x^*$. Therefore α_a is a homomorphism from A onto L_a. □

LEMMA 4.1. *Let A be a C-algebra with T and $a \in B(A)$. Then, for $x, y \in A$, $\alpha_a(x) = \alpha_a(y)$ and $\alpha_{a'}(x) = \alpha_{a'}(y)$ if and only if $x = y$.*

PROOF. (i) Suppose that $\alpha_a(x) = \alpha_a(y)$ and $\alpha_{a'}(x) = \alpha_{a'}(y)$. Then $a \vee x = a \vee y$ and $a' \vee x = a' \vee y$. Now,

$$\begin{array}{rcll} x & = & F \vee x & \text{(since } F \text{ is the join identity)} \\ & = & (a \wedge a') \vee x & \text{(since } a \in B(A)) \\ & = & (a \vee x) \wedge (a' \vee x) & \text{(by the dual of Lemma 2.1(iv))} \\ & = & (a \vee y) \vee (a' \vee y) & \\ & = & (a \wedge a') \vee y & \text{(by the dual of Lemma 2.1(iv))} \\ & = & F \vee y & \text{(since } a \in B(A)) \\ & = & y & \text{(since } F \text{ is the join identity)} \end{array}$$

Therefore $x = y$. Other hand is trivial. □

THEOREM 4.2. *Let A be a C-algebra with T and $a \in B(A)$. Then $A \cong L_a \times L_{a'}$*

PROOF. Define $\alpha : A \to L_a \times L_{a'}$ by $\alpha(x) = (\alpha_a(x), \alpha_{a'}(x))$ for all $x \in A$. Then α is well-defined and homomorphism (See Theorems 4.1). By Lemma 4.1, α is one to one. Now, we will prove α is onto. For, let $(x, y) \in L_a \times L_{a'}$. Then $a \leqslant_l x$ and $a' \leqslant_l y$. Therefore $a \wedge x = a, a' \wedge y = a', a \vee x = x \vee a = x$ and $a' \vee y = y \vee a' = y$ (See Lemma 3.3). Now, for this $x \wedge y \in A$,

$$\begin{array}{rcll} \alpha(x \wedge y) & = & (\alpha_a(x \wedge y), \alpha_{a'}(x \wedge y)) & \\ & = & (a \vee (x \wedge y), a' \vee (x \wedge y)) & \\ & = & ((a \vee x) \wedge (a \vee y), (a' \vee x) \wedge (a' \vee y)) & \text{(by dual of 2.1(d))} \\ & = & (x \wedge (a \vee a' \vee y), (a' \vee a \vee x) \wedge y) & \\ & = & (x \wedge (T \vee y), (T \vee x) \wedge y) & \text{(since } a \in B(A)) \\ & = & ((x \wedge T) \vee (x \wedge y), (T \wedge y) \vee (F \wedge x \wedge y)) & \text{(by Def. 2.1(d, e))} \\ & = & (x \vee (x \wedge y), y \vee F) & \text{(by Lemma 2.2(i))} \\ & = & (x, y) & \text{(by Def. 2.1(f))} \end{array}$$

Therefore α is onto and hence α is an isomorphism from A onto $L_a \times L_{a'}$. □

THEOREM 4.3. *Let A, A_1, A_2 be three C-algebras with T such that $A \cong A_1 \times A_2$. Then there exists $a \in B(A)$ such that $A_1 \cong L_a$ and $A_2 \cong L_{a'}$.*

PROOF. Let $f : A_1 \times A_2 \to A$ be an isomorphism. Take $a = f(F_1, T_2)$, where T_1 & T_2 are the meet identities of A_1 & A_2 respectively and F_1 & F_2 are the join identities of A_1 & A_2 respectively. Then $f^{-1}(a) \in B(A_1) \times B(A_2) = B(A_1 \times A_2) = B(A)$ [**6**]. Define $\gamma : A_1 \to L_a$ by $\gamma(x_1) = f(x_1, T_2)$, for all $x_1 \in A_1$. Now,

$$\begin{aligned}
a \wedge f(x_1, T_2) &= f(F_1, T_2) \wedge f(x_1, T_2) \\
&= f(F_1 \wedge x_1, T_2 \wedge T_2) \qquad \text{(since } f \text{ is homomorphism)} \\
&= f(F_1, T_2) \\
&= a \\
a \vee f(x_1, T_2) &= f(F_1, T_2) \vee f(x_1, T_2) \\
&= f(F_1 \vee x_1, T_2 \vee T_2) \qquad \text{(since } f \text{ is homomorphism)} \\
&= f(x_1, T_2)
\end{aligned}$$

Then $a \leqslant_l f(x_1, T_2)$. Therefore $f(x_1, T_2) \in L_a$ and γ is well-defined. It is easy to prove that γ preserves $\wedge, \vee$ and γ is one to one. Let $x_1 \in A_1$. Then

$$\begin{aligned}
\gamma(x_1') &= f(x_1', T_2) \\
&= f(F_1 \vee x_1', T_2 \vee T_2) \\
&= f(F_1, T_2) \vee f(x_1', F_2') \quad \text{(since } f \text{ is homomorphism)} \\
&= a \vee (f(x_1, F_2))' \qquad \text{(since } f \text{ is homomorphism)} \\
&= a \vee (\gamma(x_1))' \\
&= (\gamma(x_1))^*
\end{aligned}$$

Therefore γ is a homomorphism. Since f is isomorphism, γ is one to one. Finally we will prove γ is onto. Let $x \in L_a$. Then, by Lemma 3.3, $a \vee x = x \vee a = x$ and $a \wedge x = a$. Since f is onto, there exist $x_1 \in A_1, x_2 \in A_2$, such that $f(x_1, x_2) = x$. Now,

$$\begin{aligned}
(x_1, x_2) &= f^{-1}(x) \qquad \text{(since } f^{-1} \text{ is bijective)} \\
&= f^{-1}(a \vee x) \\
&= f^{-1}(a) \vee f^{-1}(x) \quad \text{(since } f^{-1} \text{ is homomorphism)} \\
&= (F_1, T_2) \vee (x_1, x_2) \\
&= (F_1 \vee x_1, T_2 \vee x_2) \\
&= (x_1, T_2) \qquad \text{(by Lemma 2.2(i))}
\end{aligned}$$

Therefore $x_2 = T_2$ and $\gamma(x_1) = f(x_1, T_2) = f(x_1, x_2) = x$. Hence γ is onto. Thus γ is isomorphism. Similarly, we can prove $A_2 \cong L_{a'}$. □

From [**4**], it is observed that for any $a \in B(A)$, $R_a = A_a$ where $A_a = \{a \wedge x \mid x \in A\}$. Therefore we restate some results in the following;

THEOREM 4.4. *Let A be a C-algebra with T and $a \in B(A)$. Then $\beta_a : A \to R_a$ defined by $\beta_a(x) = a \wedge x$ for all $x \in A$, is an onto homomorphism with kernel θ_a, where $\theta_a = \{(x, y) \in A \times A \mid \beta_a(x) = \beta_a(y)\}$ and hence $\dfrac{A}{\theta_a} \cong R_a$.*

LEMMA 4.2. *Let A be a C-algebra with T and $a \in B(A)$. Then, for any $x, y \in A$, $\beta_a(x) = \beta_a(y)$ and $\beta_{a'}(x) = \beta_{a'}(y)$ if and only if $x = y$.*

THEOREM 4.5. *Let A be a C-algebra with T and $a \in B(A)$. Then $A \cong R_a \times R_{a'}$.*

THEOREM 4.6. *Let A, A_1, A_2 be three C-algebras with T such that $A \cong A_1 \times A_2$. Then there exists $a \in B(A)$ such that $A_1 \cong R_a$ and $A_2 \cong R_{a'}$.*

Now, we prove the following;

Lemma 4.3. *Let A be a C-algebra with T. If $x, y \in A$ and $a \in B(A)$ are such that $a \leqslant_l x$ and $a' \leqslant_l y$, then $x \wedge y = y \wedge x$.*

Proof. Let $x, y \in A$ such that $a \leqslant_l x$, $a' \leqslant_l y$, where $a \in B(A)$. Then $a \wedge x = a$, $a \vee x = x$, $a' \wedge y = a'$ and $a' \vee y = y$. Now,

$$\begin{aligned}
\alpha_a(x \wedge y) &= a \vee (x \wedge y) \\
&= (a \vee x) \wedge (a \vee y) && \text{(by Def. 2.1(d))} \\
&= x \wedge (a \vee a' \vee y) \\
&= x \wedge (T \vee y) && \text{(since } a \in B(A)) \\
&= (x \wedge T) \vee (x \wedge y) && \text{(by Def. 2.1(d))} \\
&= x \vee (x \wedge y) \\
&= x && \text{(by Def. 2.1(f))} \\
&= x \vee F \\
&= (T \wedge x) \vee (F \wedge y \wedge x) \\
&= (T \wedge x) \vee (T' \wedge y \wedge x) \\
&= (T \vee y) \wedge x && \text{(by Def. 2.1(d))} \\
&= (a \vee a' \vee y) \wedge x && \text{(since } a \in B(A)) \\
&= (a \vee y) \wedge x \\
&= (a \vee y) \wedge (a \vee x) \\
&= a \vee (y \wedge x) && \text{(by Def. 2.1(d))} \\
&= \alpha_a(y \wedge x)
\end{aligned}$$

Similarly, $\alpha_a(x \wedge y) = \alpha_{a'}(y \wedge x)$. Therefore $x \wedge y = y \wedge x$ (see Lemma 4.1). □

Lemma 4.4. *Let A be a C-algebra with T. If $x, y \in A$ and $a \in B(A)$ are such that $x \leqslant_r a$ and $y \leqslant_r a'$, then $x \vee y = y \vee x$.*

Proof. Let $x, y \in A$ $a \in B(A)$ such that $x \leqslant_r a$ and $y \leqslant_r a'$. Then $a \vee x = a$, $a' \vee y = a'$, $x \wedge a = a \wedge x = x$, $y \wedge a' = a' \wedge y = y$ (see Lemma 3.6). Now,

$$\begin{aligned}
\beta_a(x \vee y) &= a \wedge (x \vee y) \\
&= (a \wedge x) \vee (a \wedge y) && \text{(by Def. 2.1(d))} \\
&= x \vee (a \wedge a' \wedge y) \\
&= x \vee (F \wedge y) && \text{(since } a \in B(A)) \\
&= x \vee F \\
&= F \vee x \\
&= (F \wedge y) \vee (a \wedge x) \\
&= (a \wedge a' \wedge y) \vee (a \wedge x) && \text{(since } a \in B(A)) \\
&= (a \wedge y) \vee (a \wedge x) \\
&= a \wedge (y \vee x) && \text{(by Def. 2.1(d))} \\
&= \beta_a(y \vee x)
\end{aligned}$$

Similarly, $\beta_{a'}(x \vee y) = \beta_{a'}(y \vee x)$. Therefore $x \vee y = y \vee x$ (see Lemma 4.2). □

From Theorem 4.6 and Lemma 4.3, we have the following.

Theorem 4.7. *Let A be C-algebra with T.*

(i) *For any* $x, y \in A$, *there exists* $a \in B(A)$ *such that* $a \leqslant_l x$ *and* $a' \leqslant_l y$.
(ii) *For any* $x, y \in A$, *there exists* $a \in B(A)$ *such that* $x \leqslant_r a$ *and* $y \leqslant_r a'$
(iii) A *is a Boolean algebra*

Then (i) $\Rightarrow$ *(iii)* $\Leftarrow$ *(ii).*

References

[1] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer Verlag, New York, 1981.

[2] F. Guzman and C. Squier, The algebra of conditional logic, *Algebra Universalis*, **27**(1)(1990), 88–110.

[3] G. C. Rao and P. Sundarayya, C-algebra as a poset, *Int. J. Math. Sci.*, **4**(2)(2005), 225–236.

[4] G. C. Rao and P. Sundarayya, Decompositions of a C-algebra, *Int. J. Math. Math. Sci.*, Volume 2006, Article ID 78981, Pages 18.

[5] G. C. Rao and P. Sundarayya, Two partial orders on a C-algebra, *Int. J. Computational Congnition*, **7**(3)(2009), 40–43.

[6] U. M. Swamy, G. C. Rao and R. V. G. Ravi kumar, Centre of a C-algebra, *Southeast Asian Bull. Math.*, **27**(2)(2003), 357–368.

Received by editors 24.06.2016; Available online 12.09.2016.

Department of Mathematics, GITAM University, Visakhapatnam, INDIA
E-mail address: psundarayya@yahoo.co.in

Department of Mathematics, GITAM University, Visakhapatnam, INDIA
E-mail address: ramesh.sirisetti@gmail.com - Corresponding author

Department of Mathematics, Vasavi College of Engineering, Hyderabad, INDIA
E-mail address: ramaniv80@gmail.com