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CONNECTED INJECTIVE DOMINATION OF GRAPHS

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Abstract. Let $G = (V, E)$ be a connected graph. A subset S of V is called injective dominating set (Inj-dominating set) if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $|\Gamma(u, v)| \geqslant 1$, where $|\Gamma(u, v)|$ is the number of common neighbors between the vertices u and v. In this research work, we introduce the connected injective domination of graphs. Exact values for some families of graphs, relations with the other domination parameters are obtained. Bounds and some interesting results are established.

1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of all vertices and edges of G, respectively. we use $\langle X \rangle$ to denote the subgraph of G induced by the set of vertices X. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The distance between two vertices u and v in G is the number of edges in a shortest path connecting them, this is also known as the geodesic distance. The eccentricity of a vertex v is the greatest geodesic distance between v and any other vertex and denoted by $e(v)$. For more terminologies and notations about graph we refer the reader to [**4, 5**].

A subset D of $V(G)$ is called dominating set if for every $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u. The minimum cardinality of a minimal dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set D of G is called connected dominating set if the induced subgraph $\langle D \rangle$ is connected. The connected domination number of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set. For more details about domination number and connected domination number, we refer to [**6**], [**8**], [**10**] and [**9**].

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The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with the vertex set $V(G)$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G. The common neighborhood (CN-neighborhood) of a vertex $u \in V(G)$ denoted by $N_{cn}(u)$ is defined as $N_{cn}(u) = \{v \in V(G) : uv \in E(G)$ and $|\Gamma(u,v)| \geqslant 1\}$, where $|\Gamma(u,v)|$ is the number of common neighborhood between the vertices u and v, [**3**].

The concept of Injective domination in graph has introduced in [**1**]. For a graph G, a subset S of $V(G)$ is called injective dominating set (Inj-dominating set) if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $|\Gamma(u,v)| \geqslant 1$. The minimum cardinality of such dominating set denoted by $\gamma_{in}(G)$ and is called injective domination number (Inj-domination number) of G. The Inj-neighborhood of a vertex $u \in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u,v)| \geqslant 1\}$. The cardinality of $N_{in}(u)$ is called injective degree of the vertex u and denoted by $deg_{in}(u)$ in G, and $N_{in}[u] = N_{in}(u) \cup \{u\}$. The maximum and minimum injective degree of a vertex in G are denoted respectively by $\Delta_{in}(G)$ and $\delta_{in}(G)$. That is $\Delta_{in}(G) = \max_{u \in V} |N_{in}(u)|$, $\delta_{in}(G) = \min_{u \in V} |N_{in}(u)|$. The injective complement of G denoted by $\overline{G}^{inj}$ is the graph with same vertex set $V(G)$ and any two vertices u and v in $\overline{G}^{inj}$ are adjacent if and only if they are not Inj-adjacent in G. A subset S of V is called an injective independent set (Inj-independent set), if for every $u \in S, v \notin N_{in}(u)$ for all $v \in S - \{u\}$. An injective independent set S is called maximal if any vertex set properly containing S is not Inj-independent set, the maximum cardinality of Inj-independent set is denoted by β_{in}, and the lower Inj-independence number i_{in} is the minimum cardinality of the Inj-maximal independent set.

Proposition 1.1 ([**1**]). *Let G be a graph with p vertices. Then $\gamma_{in}(G) = p$ if and only if G is a forest with $\Delta(G) \leqslant 1$.*

Proposition 1.2 ([**1**]). *Let G be a nontrivial connected graph. Then $\gamma_{in}(G) = 1$ if and only if there exists a vertex $v \in V(G)$ such that $N(v) = N_{cn}(v)$ and $e(v) \leqslant 2$.*

Theorem 1.1 ([**1**]). *For any graph G with p vertices, $\lceil \frac{p}{1+\Delta_{in}(G)} \rceil \leqslant \gamma_{in}(G)$. Further, the equality hold if and only if for every minimum Inj-dominating set D in G the following conditions are satisfied:*

1. *for any vertex v in D, $deg_{in}(v) = \Delta_{in}(G)$.*
2. *D is Inj-independent set in G.*
3. *every vertex in $V - D$ has common neighborhood with exactly one vertex in D.*

Throughout this paper, we denote $|V(G)| = p$, $|E(G)| = q$. Also, P_p, C_p, K_p and S_p are the path, cycle, complete and star graphs, respectively. $K_{r,m}$ is the complete bipartite graph on $r + m$ vertices and $G + H$, $G \square H$, $G \cdot H$ and $G \circ H$ are the Join, Cartesian product, Composition and Corona product graph of any two graphs G and H, respectively.

In this paper, we introduce the connected injective domination in graph. Exact values for some families of graphs, relations with the other domination parameters are obtained. Bounds and some interesting results are established.

2. Connected Injective Domination of Graphs

DEFINITION 2.1. *Let G be a connected graph. A subset $S \subseteq V(G)$ is called a connected injective dominating set (CInj-dominating set) of G if S is an injective dominating set and the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of such injective dominating set is called the connected injective domination number (CInj-domination number) of G and denoted by $\gamma_{cin}(G)$.*

Since any CInj-dominating set is also an Inj-dominating set, then it is easy to check the following propositions.

PROPOSITION 2.1. *For any connected graph G, $\gamma_{in}(G) \leqslant \gamma_{cin}(G)$.*

PROPOSITION 2.2. *Let G be a connected graph and H be any connected spanning subgraph of G. Then $\gamma_{cin}(G) \leqslant \gamma_{cin}(H)$.*

PROPOSITION 2.3. *For any connected graph G, $\gamma_{cin}(G) = 1$ if and only if $\gamma_{in}(G) = 1$.*

THEOREM 2.1. *Let G be a connected C_7-free graph with diameter two or three and $\gamma_{in}(G) \neq 1$. Then $\gamma_{in}(G) = \gamma_{cin}(G) = 2$.*

PROOF. We have two cases:

Case 1. Suppose $diam(G) = 2$. Since $\gamma_{in}(G) \neq 1$, then by Proposition 1.2, G is a triangle-free graph. Choose any edge uv in G, then all the other vertices of G are injective dominated by u or v. Hence, $\gamma_{cin}(G) = 2$.

Case 2. Suppose $diam(G) = 3$ and G does not contain an induced subgraph isomorphic to C_7. Then there exist at least one edge uv in G such that all the other vertices of G have distance less than or equal two from u or v. Thus $\gamma_{cin}(G) \leqslant 2$. But $\gamma_{in}(G) \neq 1$. Hence, $\gamma_{cin}(G) = 2$. □

PROPOSITION 2.4. *For any connected graph G without Inj-isolated vertices, $\gamma_{cin}(G) \leqslant \gamma_c(G)$.*

PROOF. Since G is a connected graph without Inj-isolated vertices, then G does not contain a vertex of full degree without a triangle. Suppose G contains a vertex of full degree and a triangle. Then $\gamma_{cin}(G) = \gamma_c(G) = 1$.

Suppose now $\Delta(G) \leqslant p - 2$. Let S be a γ_c-set of G, we have to prove that S is a CInj-dominating set of G. Suppose that $u \in V - S$ is arbitrary. Then there exists at least a vertex $v \in S$ such that $uv \in E(G)$. Since $\langle S \rangle$ is connected, then u is Inj-adjacent to a vertex in S. Thus S is a connected injective dominating set of G and hence, $\gamma_{cin}(G) \leqslant \gamma_c(G)$. □

REMARK 2.1. *If G is a connected graph with Inj-isolated vertices (Inj-isolated vertex here means that a vertex with a full degree in a triangle-free graph) or by more accurately, with an Inj-isolated vertex (since G is connected, then it must* *contain at most one vertex of full degree as well as it is a triangle-free graph), then G is a star. Hence, $\gamma_{cin}(G) = 2$.*

As we well know, a leaf of a tree T is a vertex of degree one and a support vertex of T is a vertex adjacent to a leaf. The support vertices in any tree can be divided to two types:

- Isolator support vertex: is a support vertex whose removal results a graph with at most one non-trivial connected component and isolated vertices.
- Semi-isolator support vertex: is a support vertex whose removal results a graph with at least two non-trivial connected components and isolated vertices.

Proposition 2.5. *Let T be a tree on p vertices with diameter greater than or equal six. Then $\gamma_{cin}(T) = p - t_1 - t_2$, where t_1 and t_2 are the number of leafs and isolator support vertices of T, respectively.*

Proof. Since T is a tree with $diam(T) \geqslant 6$, then T has no Inj-isolated vertices, so by Proposition 2.4, $\gamma_{cin}(T) \leqslant \gamma_c(T) = p - t_1$. But the $\gamma_c(T)$-set is not minimal CInj-dominating set because the $\gamma_c(T)$-set contains all the support vertices of T which some of them can be connected injective dominated by other vertices of it (those vertices are the isolator vertices of T). Thus $\gamma_{cin}(T) \leqslant p - t_1 - t_2$. Since T has only one minimal CInj-dominating set, then $\gamma_{cin}(T) = p - t_1 - t_2$. □

Proposition 2.6. *For any connected graph G with diameter greater than or equal six, $\gamma_{cin}(G) \leqslant p - 4$.*

Proof. Since G is a connected graph, then G contains a spanning tree say H, then by Proposition 2.2 and Proposition 2.5, $\gamma_{cin}(G) \leqslant p - t_1 - t_2$, where t_1 and t_2 are the leafs and the isolator support vertices of H, respectively. Since each of t_1 and t_2 at least equals to two, then the result. □

Let $m(t_1)_T$ and $m(t_2)_T$ be the maximum number of leafs and isolator support vertices of a spanning tree T of a connected graph G, respectively. In the following Theorem we determine the connected injective domination number of a graph G in terms of $m(t_1)_T$ and $m(t_2)_T$.

Theorem 2.2. *Let G be a connected graph which does not contain a spanning tree T of diameter less than five ($diam(T) \geqslant 5$). Then $\gamma_{cin}(G) = p - m(t_1)_T - m(t_2)_T$.*

Proof. Let T be a spanning tree of G with $m(t_1)_T$ leafs and $m(t_2)_T$ isolator support vertices. Then by Proposition 2.5 and Proposition 2.2, $\gamma_{cin}(G) \leqslant \gamma_{cin}(T) = p - m(t_1)_T - m(t_2)_T$.

Conversely, suppose S be a γ_{cin}-set of G. Since $\langle S \rangle$ is connected, then it has a spanning tree T_1. A spanning tree T of G is formed by adding the remaining $p - \gamma_{cin}(G)$ vertices of $V - S$ to T_1 and adding edges of G such that each vertex of $V - S$ is Inj-adjacent to exactly one vertex in S. Thus T has at least $p - \gamma_{cin}(G)$ leafs and isolator support vertices. Hence, $\gamma_{cin}(G) \geqslant p - m(t_1)_T - m(t_2)_T$.

Note that if $diam(T) \leqslant 4$, then any connected injective dominating set of T contains at least one isolator support vertex. $\square$

PROPOSITION 2.7. *Let G be a connected graph,*

(1) *If G isomorphic to P_p or C_p with $p \geqslant 6$, then $\gamma_{cin}(G) = p - 4$.*
(2) *If $G \cong K_{r,m}$, then $\gamma_{cin}(G) = 2$.*
(3) *If $G \cong G_1 + G_2$, where G_1 and G_2 are any two graphs, then*

$$\gamma_{cin}(G) = \begin{cases} 2, & \text{if } G \cong K_{r,m}; \\ 1, & \text{otherwise.} \end{cases}$$

PROPOSITION 2.8. *For any graph G isomorphic to $P_m \square P_2$ or $C_m \square P_2$,*

$$\gamma_{cin}(G) = \begin{cases} 2, & \text{if } m \leqslant 5; \\ m - 2, & \text{otherwise.} \end{cases}$$

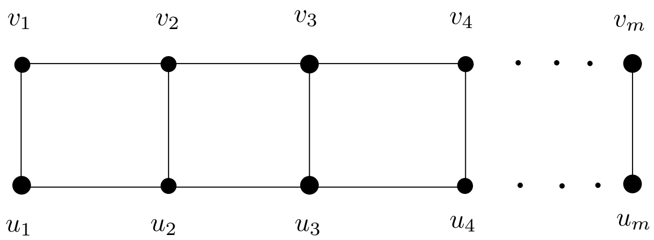

FIGURE 1. $P_m \square P_2$

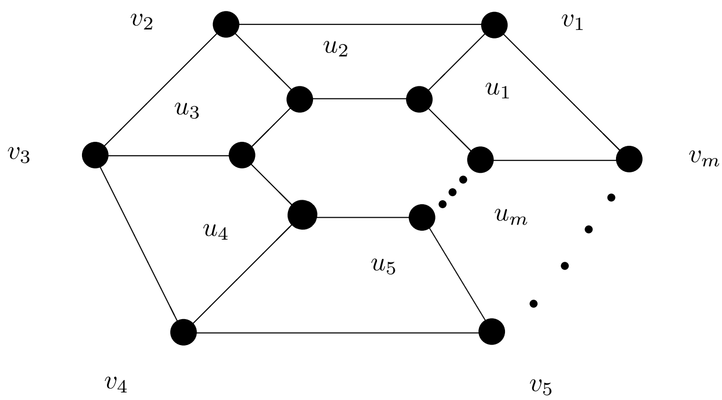

FIGURE 2. $C_m \square P_2$

PROOF. We have two cases:

Case 1. Suppose $m \leqslant 5$. Since G is a triangle-free graph, then $\gamma_{cin}(G) \neq 1$. So, if $m = 2$, then $diam(G) = 2$, then by Theorem 2.1, $\gamma_{cin}(G) = 2$. Also, if $3 \leqslant m \leqslant 5$, then by Figures 1 and 2, the set $S = \{v_3, u_3\}$ is a connected injective dominating set of G. Hence, $\gamma_{cin}(G) = 2$.

Case 2. Suppose now $m \geqslant 6$. From Figures 1 and 2, to obtain a minimal CInj-dominating set of G we have the following possibilities:

i. The set $S_1 = \{v_2, v_3, \ldots, v_{m-1}\}$ or $S_1 = \{u_2, u_3, \ldots, u_{m-1}\}$, which has cardinality $m - 2$.

ii. The set $S_2 = \{u_3, v_3, v_4, \ldots, v_{m-2}, u_{m-2}\}$ or $S_2 = \{v_3, u_3, u_4, \ldots, u_{m-2}, v_{m-2}\}$, which also has cardinality $m - 2$.

Without loss of generality, we can obtain minimal CInj-dominating sets of G containing vertices from S_1 and S_2, but all of them having cardinalities greater than or equal $m - 2$. Hence, $\gamma_{cin}(G) = m - 2$. □

PROPOSITION 2.9. *For any graph $G \cong C_n \circ H$, where $n \geqslant 4$ and H is any graph, $\gamma_{cin}(G) = n - 2$.*

PROOF. Consider $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. It is clear that, any minimal CInj-dominating set S of G has the form $S = \{v_i, v_{i+1}, \ldots, v_{n+i-3}\}$ of order $n - 2$. Hence, $\gamma_{cin}(G) = n - 2$. □

PROPOSITION 2.10. *For any graph $G \cong K_n \circ H$, where H is any graph,*

$$\gamma_{cin}(G) = \begin{cases} 2, & \text{if } \delta(H) = 0; \\ 1, & \text{otherwise.} \end{cases}$$

PROOF. Suppose H has no isolated vertices. Then any $v \in V(K_n)$ satisfies $N(v) = N_{cn}(v)$ and $e(v) \leqslant 2$. Hence by Proposition 1.2, $\gamma_{cin}(G) = 1$.

Suppose now $\delta(H) = 0$ (H has at least an isolated vertex). Then any vertex $v \in V(K_n)$ can Inj-dominates all the other vertices of G except the isolated vertices of the copy of H whose adjacent to v. So we have to choose one more vertex from the neighborhood of v. Hence, $\gamma_{cin}(G) = 2$. □

THEOREM 2.3. *Let T be a tree. Then $\gamma_{cin}(T) = 2$ if and only if T has diameter less than or equal five. Furthermore, in this case $\gamma_{cin}(T) = \gamma_{in}(T)$.*

PROOF. Suppose $diam(T) \leqslant 5$. Then there exists at least one edge uv in T such that all the other vertices of T lie in distance one or two from u or v. Thus $\gamma_{cin}(T) \leqslant 2$, but $\gamma_{cin}(T) \geqslant \gamma_{in}(T) \neq 1$. Hence, $\gamma_{cin}(T) = 2$.

Similarly, suppose $\gamma_{cin}(T) = 2$. Let $S = \{u, v\}$ be a minimum CInj-dominating set of T (uv is an edge in T). Then the vertices u and v can injective dominate all the vertices of T until the second neighborhood of each of them. Then T has diameter at most five. Finally, since $\gamma_{cin}(T) = 2$ and $\gamma_{in}(T) \neq 1$, then $\gamma_{cin}(T) = \gamma_{in}(T)$. □

PROPOSITION 2.11. *For any nontrivial connected graph G on p vertices, $\gamma_{in}(G) = \gamma_{cin}(G) = p$ if and only if $G \cong K_2$.*

PROOF. The proof comes immediately from Proposition 1.1. □

Proposition 2.12. *For any connected graph G on p vertices, $\gamma_{in}(G) = \gamma_{cin}(G) = p-1$ if and only if $G \cong P_3$.*

Proof. Let $\gamma_{cin}(G) = p-1$. Suppose $p \geqslant 4$. Since G is connected, then G contains a spanning tree say T, so by Proposition 2.2, $\gamma_{cin}(G) \leqslant \gamma_{cin}(T) \leqslant p-2$. Suppose now $p \leqslant 3$. If $p = 2$, then by Proposition 2.11, $\gamma_{cin}(G) = p$. Thus $p = 3$, which means that, either $G \cong C_3$ or $G \cong P_3$. But $\gamma_{cin}(C_3) = 1 = p-2$. Hence, $G \cong P_3$. The converse is obvious. □

Proposition 2.13. *For any connected graph G on p vertices, $\gamma_{cin}(G) = p-2$ if and only if G isomorphic to one of the following graphs C_3, C_4, S_4 and P_4.*

Proof. Let $\gamma_{cin}(G) = p-2$. Then we have two cases:
Case 1. Suppose G is a tree. Then, if G has Inj-isolated vertex, then by Remark 2.1, $G \cong S_p$. But, $\gamma_{cin}(G) = p-2$. Hence, $G \cong S_4$. Also, if G has no Inj-isolated vertex, then there are two possibilities:

(1) If $diam(G) \geqslant 6$, then by Proposition 2.6, $\gamma_{cin}(G) \leqslant p-4$ which contradicts to the hypothesis.
(2) If $diam(G) \leqslant 5$, then by Theorem 2.3, $\gamma_{cin}(G) = 2$. Thus p should be equal to 4. Hence, $G \cong P_4$.

Case 2. Suppose G is not a tree. Then G has no Inj-isolated vertex. Since G is connected, then it contains a spanning tree say T, and hence $\gamma_{cin}(G) \leqslant \gamma_{cin}(T)$. Thus as in Case 1:

(1) If $diam(T) \geqslant 6$, then $\gamma_{cin}(G) \leqslant \gamma_{cin}(T) \leqslant p-4$ which contradicts to the hypothesis.
(2) If $diam(T) \leqslant 5$, then $\gamma_{cin}(G) \leqslant \gamma_{cin}(T) = 2$. So, if $\gamma_{cin}(G) = 1$, then $p = 3$, and hence $G \cong C_3$. And if $\gamma_{cin}(G) = 2$, then $p = 4$. Since G is not a tree, then $diam(G) = 2$ and G is a triangle-free graph (Theorem 2.1). Hence, $G \cong C_4$.

The converse is obvious. □

Theorem 2.4. *Let T be a tree on p vertices. Then $\gamma(T) = \gamma_c(T) = \gamma_{cin}(T) = \gamma_{in}(T)$ if and only if $\gamma_c(T) = 2$.*

Proof. It is clear that, if $\gamma_c(T) = \gamma_{cin}(T) = 2$, then $\gamma(T) = \gamma_{in}(T) = 2$. Thus, it is enough if we prove that $\gamma_c(T) = \gamma_{cin}(T)$ if and only if $\gamma_c(T) = 2$. Assume that $\gamma_c(T) = \gamma_{cin}(T)$. Clearly that $\gamma_c(T) \neq 1$, because for any tree T, $\gamma_{cin}(T) \neq 1$. Suppose $\gamma_c(T) \geqslant 3$. Then, if $diam(T) \leqslant 5$, then by Proposition 2.3, $\gamma_c(T) \neq \gamma_{cin}(T)$, and if $diam(T) \geqslant 6$, then by Proposition 2.5, $\gamma_{cin}(T) = p-t_1-t_2$, where t_1 and t_2 are the number of leafs and isolator support vertices of T, respectively. Since for any tree $t_2 \neq 0$, then again $\gamma_c(T) \neq \gamma_{cin}(T)$. Hence, $\gamma_c(T) = 2$. Conversely, suppose $\gamma_c(T) = 2$. This means that there exist two adjacent vertices $u, v \in V(T)$ such that $N(u) \cup N(v) = V(T)$. Hence, $\gamma_{cin}(T) = 2$. □

Proposition 2.14. *Let G be a connected graph. Then $\gamma(G) = \gamma_c(G) = \gamma_{cin}(G) = \gamma_{in}(G) = 1$ if and only if G contains a full degree vertex and a triangle.*

Actually, the result in Theorem 2.4, can be generalized as following

THEOREM 2.5. *Let G be a connected graph on p vertices and $\gamma_c(G) \neq 1$. Then $\gamma(G) = \gamma_c(G) = \gamma_{cin}(G) = \gamma_{in}(G)$ if and only if $\gamma_c(G) = 2$.*

PROOF. If G is a tree, then by Theorem 2.4, the result holds. Suppose now G is not a tree. Then G contains a spanning tree say T(G is connected), so by Proposition 2.2, $\gamma_{cin}(G) \leqslant \gamma_{cin}(T)$. Hence again by Theorem 2.4, the result holds. □

THEOREM 2.6. *Let T be a tree with diameter greater than or equal six and let S be the γ_{cin}-set of T. Then $\gamma_{cin}(T) = \gamma_{in}(T)$ if and only if the following conditions are satisfied:*

(1) *Every vertex in S is adjacent to at least one isolator support vertex in T.*
(2) *If $v \in S$ is not adjacent to an isolator support vertex in T, then v is the unique vertex in S which is adjacent to a vertex of S satisfies (i).*

PROOF. Let $|S| = \gamma_{cin}(T) = \gamma_{in}(T)$. Suppose conditions (i) and (ii) do not hold. Then there exist a vertex $u \in S$ such that u is not adjacent to an isolator support vertex in T and it is not the unique vertex in S which is adjacent to an S-neighbor of an isolator support vertex in T. Thus the set $S - \{u\}$ is an injective dominating set of T, which contradicts to the hypothesis. The converse is obvious. □

THEOREM 2.7 ([**9**]). *For any connected (p, q)-graph G with maximum degree Δ,*

$$\lceil \frac{p}{\Delta + 1} \rceil \leqslant \gamma_c(G) \leqslant 2q - p.$$

The lower bound is attained if and only if G has a vertex of full degree, and the upper bound is attained if and only if G is a path.

As an immediately result from Proposition 2.4 and Theorem 2.7, is the following

COROLLARY 2.1. *For any connected (p, q)-graph G without Inj-isolated vertices, $\gamma_{cin}(G) \leqslant 2q - p$.*

THEOREM 2.8. *For any connected graph G with p vertices and maximum Inj-degree Δ_{in}, $\lceil \frac{p}{1+\Delta_{in}(G)} \rceil \leqslant \gamma_{cin}(G) \leqslant p - \Delta_{in}(G)$. Further, the equality of the lower bound holds if and only if G has a vertex with full Inj-degree (there exists a vertex $v \in V(G)$ such that $N(v) = N_{cn}(v)$ and $e(v) \leqslant 2$).*

PROOF. The proof of the lower bound is straightforward from Proposition 2.1 and Theorem 1.1. Now for the upper bound, suppose $v \in V(G)$ such that $deg_{in}(v) = \Delta_{in}(G)$. Then a spanning tree T of G can be formed in which v is Inj-adjacent to each its neighbors in G. Thus, if G is a triangle-free graph, then $\Delta_{in}(G) \leqslant m(t_1)_T$ and hence by Theorem 2.2, $\gamma_{cin}(G) \leqslant p - \Delta_{in}(G)$. Also, if G has triangles, then $\Delta_{in}(G) \leqslant m(t_1)_T + m(t_2)_T$ and again by Theorem 2.2, $\gamma_{cin}(G) \leqslant p - \Delta_{in}(G)$. □

THEOREM 2.9. *Let G be a connected (p, q)-graph with diameter greater than or equal six. Then $\gamma_{cin}(G) \leqslant 2(q-1)-p$, and the equality is attained if and only if G is a path.*

PROOF. To prove this bound we make use Proposition 2.6 and the fact that G is connected, as follows:

$$\begin{aligned}\gamma_{cin}(G) &\leqslant p-4 = 2(p-2)-p\\ &\leqslant 2(q-1)-p.\end{aligned}$$

For the equality, if $G \cong P_p$ with $diam(G) \geqslant 6$, then $\gamma_{cin}(G) = 2(p-2)-p = p-4$. Conversely, if $\gamma_{cin}(G) = 2(q-1)-p$ and $diam(G) \geqslant 6$, then by Proposition 2.6, $q \leqslant p-1$. Since G is connected, then $q = p-1$, so G must be a tree. But then by Proposition 2.5, $\gamma_{cin}(G) = p - t_1 - t_2$. If $t_1 > 2$ and $t_2 > 2$ or $t_1 > 2$ and $t_2 = 2$, then $\gamma_{cin}(G) = p - t_1 - t_2 < p-4 = 2(q-1)-p$ as above, which is a contradiction. Thus $t_1 \leqslant 2$ and $t_2 \leqslant 2$. But since G is a tree, then $t_1 \geqslant 2$ and $t_2 \geqslant 2$. Hence, $t_1 = t_2 = 2$. So G must be a path. □

THEOREM 2.10. *Let G be a connected graph and its injective complement $\overline{G}^{inj}$ is connected. Then*

$$\gamma_{cin}(G) + \gamma_{cin}(\overline{G}^{inj}) \leqslant p+1.$$

PROOF. Since G and $\overline{G}^{inj}$ are connected, then by Theorem 2.8, $\gamma_{cin}(G) \leqslant p - \Delta_{in}(G)$ and $\gamma_{cin}(\overline{G}^{inj}) \leqslant p - \Delta_{in}(\overline{G}^{inj})$. Thus

$$\begin{aligned}\gamma_{cin}(G) + \gamma_{cin}(\overline{G}^{inj}) &\leqslant p - \Delta_{in}(G) + p - \Delta_{in}(\overline{G}^{inj})\\ &= 2p - (\Delta_{in}(G) + \Delta_{in}(\overline{G}^{inj}))\\ &= 2p - (\Delta_{in}(G) + p - 1 - \delta_{in}(G))\\ &= p + 1 + \delta_{in}(G) - \Delta_{in}(G) \leqslant p+1.\end{aligned}$$

□

THEOREM 2.11 ([**9**]). *Let G be a connected graph of order $p \geqslant 4$ such that both G and $\overline{G}$ are connected. Then*

$$\gamma_c(G) + \gamma_c(\overline{G}) \leqslant p(p-3).$$

The bound is attained if and only if $G \cong P_4$.

PROPOSITION 2.15. *Let G be a connected graph without an Inj-isolated vertex and its complement $\overline{G}$ is connected. Then $\overline{G}$ has no Inj-isolated vertex.*

THEOREM 2.12. *Let G be a connected graph without an Inj-isolated vertex and its complement $\overline{G}$ is connected. Then*

$$2 \leqslant \gamma_{cin}(G) + \gamma_{cin}(\overline{G}) \leqslant p(p-3).$$

The lower bound is attained if and only if G has a vertex v satisfies $N(v) = N_{cn}(v)$ and $e(v) = 2$, and the upper bound is attained if and only if $G \cong P_4$.

PROOF. The proof of the upper bound and its equality is straightforward from Proposition 2.4 and Theorem 2.11.
The lower bound can be seen in Figure 3, where $\gamma_{cin}(G) = \gamma_{cin}(\overline{G}) = 1$. To prove the equality of the lower bound, if G has a vertex v satisfies $N(v) = N_{cn}(v)$ and $e(v) = 2$, then $\gamma_{cin}(G) = 1$. Suppose $u \in V(G)$ such that u is not adjacent to v in G. By Proposition 2.15, G and $\overline{G}$ are connected graphs and each of them has no a vertex of full degree. Then the vertex u satisfies $\overline{N}(u) = \overline{N}_{cn}(u)$ and $\overline{e}(u) = 2$, where $\overline{N}(u)$, $\overline{N}_{cn}(u)$ and $\overline{e}(u)$ are the neighborhood set, the common neighborhood set and the eccentricity of a vertex u in $\overline{G}$, respectively. Hence, $\gamma_{cin}(\overline{G}) = 1$.

Conversely, if $\gamma_{cin}(G) + \gamma_{cin}(\overline{G}) = 2$, then $\gamma_{cin}(G) = 1$, so by Proposition 1.2, there exists a vertex $v \in V(G)$ satisfies $N(v) = N_{cn}(v)$ and $e(v) \leqslant 2$. Now, if $e(v) = 1$, then v has a full degree in G, which a contradiction to the hypothesis. Hence, $e(v) = 2$. □

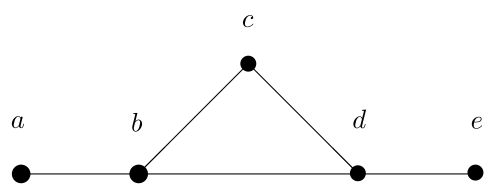

FIGURE 3. Graph G with $\gamma_{in}(G) = \gamma_{in}(\overline{G}) = 1$.

THEOREM 2.13 ([**4**]). *If G is a graph of diameter greater than or equal four, then $\overline{G}$ of diameter less than or equal two.*

THEOREM 2.14. *Let G be a connected graph with diameter greater than or equal six and $\overline{G}$ is connected. Then*

$$\gamma_{cin}(G) + \gamma_{cin}(\overline{G}) \leqslant 2q - p - 1.$$

The bound is attained if and only if G is a path.

PROOF. By Theorem 2.9, $\gamma_{cin}(G) \leqslant 2(q-1) - p$ and the bound is attained if and only if G is a path. By Theorem 2.13, $diam(\overline{G}) \leqslant 2$ and then by Proposition 1.2 and Theorem 2.1, $\gamma_{cin}(\overline{G}) \leqslant 2$ with the equality if and only if $\overline{G}$ is a triangle-free graph (the converse comes immediately from Proposition 1.2). Since $diam(G) \geqslant 6$, then there exists at least three vertices in G which they are not adjacent one to each others, so $\overline{G}$ has a triangle. Thus $\gamma_{cin}(\overline{G}) = 1$. Hence the Theorem. □

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