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A GENERAL FIXED POINT THEOREM FOR PAIRS OF MAPPINGS IN ORBITALLY 0 - COMPLETE PARTIAL METRIC SPACES

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Abstract. The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying implicit relations in orbitally 0 - complete partial metric space, which include also a result of Hardy - Rogers type.

1. Introduction

In 1974, Ćirić [**7**] has first introduced orbitally complete metric spaces and orbitally continuous function. Let f be a self mapping of a metric spaces (X, d). If $x_0 \in X$, every Cauchy sequence of the orbit $O_{x_0}(f) = \{x_0, fx_0, f^2x_0, ...\}$ is convergent to a point $y \in X$, then X is said to be orbitally complete in x_0. If f is orbitally complete at each $x \in X$, then X is said to be f - orbitally complete. Every complete metric space is f - orbitally complete for every function f. An orbitally complete metric space may not be a complete metric space ([**21**], Example 4.5).

Some fixed point results for mappings in orbitally complete metric spaces are obtained in [**2**], [**8**], [**15**], [**16**] and in other papers.

In 1994, Matthews [**13**] introduced the concept of partial metric space as a part of the study of denotional semantics of dataflow networks and proved the Banach contraction principle in such spaces. Recently, in [**1**], [**4**], [**5**], [**11**], [**12**] and in other papers, some fixed point theorems under various contractive conditions are proved.

Romaguera [**20**] introduced the notion of 0 - Cauchy sequence, 0 - complete partial metric space and proved some characterizations of partial metric spaces in terms of completeness and 0 - completeness.

Some fixed point theorems for mappings in 0 - complete partial metric spaces are proved in [**3**], [**14**], [**22**].

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Several classical fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit relation in [**16**], [**17**].

Some fixed point results for mappings satisfying implicit relations in partial metric spaces are obtained in [**9**], [**10**], [**22**] and [**6**].

The purpose of this paper is to prove a general fixed point theorem for two pairs of mappings satisfying implicit relations in 0 - complete partial metric spaces, which include also a result of Hardy - Rogers type.

2. Preliminaries

Definition 2.1 ([**13**]). Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

$(P_1) : p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$,

$(P_2) : p(x, x) \leqslant p(x, y)$,

$(P_3) : p(x, y) = p(y, x)$,

$(P_4) : p(x, z) \leqslant p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called a partial metric space.

If $p(x, y) = 0$ then by (P_1) and (P_2), $x = y$, but the converse does not always hold.

Each partial metric p on X generates a T_0 - topology τ_p which has as base the family of open p - balls $\{B_p(x, \varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) \leqslant p(x, x) + \varepsilon$ for all $x \in X$ and $\varepsilon > 0\}$.

If p is a p - metric on X, then the function $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ ($x_n \to x$) with respect to τ_p if and only if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

Lemma 2.1 ([**1**], [**12**]). *Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X such that $x_n \to z$ as $n \to \infty$, where $p(z, z) = 0$. Then, $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Definition 2.2 ([**13**], [**19**]). a) A sequence $\{x_n\}$ in a partial metric space (X, p) is called Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.

b) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$.

c) A sequence $\{x_n\}$ in (X, p) is called 0 - Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

d) (X, p) is called 0 - complete if every 0 - Cauchy sequence in X converges with respect to τ_p to a point x such that $p(x, x) = 0$.

Lemma 2.2 ([**13**], [**19**], [**20**]). *Let (X, p) be a partial metric space and $\{x_n\}$ is a sequence in X.*

a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in metric space (X, d_p).*

b) *(X,p) is complete if and only if (X,d_p) is complete. Furthermore,* $\lim_{n\to\infty} d_p(x_n,x)=0$ *if and only if*

$$p(x,x)=\lim_{n\to\infty} p(x,x_n)=\lim_{n,m\to\infty} p(x_n,x_m).$$

c) *Every* 0 *- Cauchy sequence in (X,p) is Cauchy in (X,d_p).*

d) *If (X,p) os complete, then is* 0 *- complete.*

DEFINITION 2.3 ([**14**]). Let S and T be two self mappings on a partial metric space (X,p).

1) If for a point $x\in X$, a sequence $\{x_n\}$ in X such that

$$x_{2n+1}=Sx_{2n},$$
$$x_{2n+2}=Tx_{2n+1},\ n=0,1,2,...,$$

then the set $O_{x_0}(S,T)=\{x_n : n=0,1,2,...\}$ is called the orbit of (S,T) in x_0.

2) The space (X,p) is said to be (S,T) - orbitally 0 - complete at x_0 if every 0 - Cauchy sequence in $O_{x_0}(S,T)$ converges to a point $z\in X$ such that $p(z,z)=0$.

3. Implicit relations

DEFINITION 3.1. Let $\mathcal{F}_{R0}$ be the set of all continuous functions $F(t_1,...,t_6):\mathbb{R}_+^6\to\mathbb{R}$ satisfying:

(F_1) : F is nonincreasing in variables t_5 and t_6,

(F_2) : (F_{2a}) : There exists $h_1\in[0,1)$ such that for all $u,v\geqslant 0$ and $F(u,v,v,u,u+v,v)\leqslant 0$ implies $u\leqslant h_1 v$;

(F_{2b}) : There exists $h_2\in[0,1)$ such that for all $u,v\geqslant 0$ and $F(u,v,u,v,v,u+v)\leqslant 0$ implies $u\leqslant h_2 v$,

(F_3) : $F(t,t,0,0,t,t)>0,\ \forall t>0$.

In the following examples the property (F_1) is obviously.

EXAMPLE 3.1. $F(t_1,...,t_6)=t_1-at_2-bt_3-ct_4-dt_5-et_6$, where $a,b,c,d,e\geqslant 0$ and $a+b+c+2d+2e<1$.

(F_2) : Let $u,v\geqslant 0$ be such that $F(u,v,v,u,u+v,v)=u-av-bv-cu-d(u+v)-ev\leqslant 0$. Then $u\leqslant h_1 v$, where $0\leqslant h_1=\dfrac{a+b+d+e}{1-(c+d)}<1$.

Similarly, $u,v\geqslant 0$ and $F(u,v,u,v,v,u+v)\leqslant 0$ implies $u\leqslant h_2 v$, where $0\leqslant h_2=\dfrac{a+c+d+e}{c[1-(b+e)]}<1$.

(F_3) : $F(t,t,0,0,t,t)=t[1-(a+b+d+e)]>0,\ \forall t>0$.

EXAMPLE 3.2. $F(t_1,...,t_6)=t_1-k\max\{t_2,t_3,t_4,...,t_6\}$, where $k\in\left[0,\frac{1}{2}\right)$.

(F_2) : Let $u,v\geqslant 0$ be such that $F(u,v,v,u,u+v,v)=u-k(u+v)\leqslant 0$ which implies $u\leqslant h_1 v$, where $0\leqslant h_1=\dfrac{k}{1-k}<1$.

Similarly, $u,v\geqslant 0$ and $F(u,v,u,v,v,u+v)\leqslant 0$ implies $u\leqslant h_2 v$, where $0\leqslant h_2=h_1<1$.

(F_3) : $F(t,t,0,0,t,t)=t(1-k)>0,\ \forall t>0$.

EXAMPLE 3.3. $F(t_1,...,t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $c \in (0,1)$, $a, b \geqslant 0$ and $2a + 2b < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that $F(u, v, v, u, u+v, v) = u - \max\{cu, cv, a(u+v) + bv\} \leqslant 0$. If $u > v$, then $u[1 - \max\{c, 2a+b\}] \leqslant 0$, a contradiction. Hence $u \leqslant h_1 v$, where $0 \leqslant h_1 = \max\{c, 2a+b\} < 1$.

Similarly, $u, v \geqslant 0$ and $F(u, v, u, v, v, u+v) \leqslant 0$ implies $u \leqslant h_2 v$, where $0 \leqslant h_2 = \max\{c, a+2b\} < 1$.

(F_3) : $F(t, t, 0, 0, t, t) = t(1 - \max\{c, a+b\}) > 0$, $\forall t > 0$.

EXAMPLE 3.4. $F(t_1,...,t_6) = t_1^2 - a\max\left\{t_2^2, t_3^2, t_4^2\right\} - bt_5t_6$, where $a, b \geqslant 0$ and $a + 2b < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that $F(u, v, v, u, u+v, v) = u^2 - a\max\{u^2, v^2\} - bv(u+v) \leqslant 0$. If $u > v$, then $u^2[1 - (a+2b)] \leqslant 0$, a contradiction. Hence $u \leqslant v$ which implies $u \leqslant h_1 v$, where $0 \leqslant h_1 = \sqrt{a+2b} < 1$.

Similarly, $u, v \geqslant 0$ and $F(u, v, u, v, v, u+v) \leqslant 0$ implies $u \leqslant h_2 v$, where $0 \leqslant h_2 = h_1 < 1$.

(F_3) : $F(t, t, 0, 0, t, t) = t^2[1 - (a+b)] > 0$, $\forall t > 0$.

EXAMPLE 3.5. $F(t_1,...,t_6) = t_1^3 - at_2t_3t_4 - bt_3t_4t_5 - ct_4t_5t_6$, where $a, b, c \geqslant 0$ and $a + 2b + 2c < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that $F(u, v, v, u, u+v, v) = u^3 - auv^2 - buv(u+v) - cuv(u+v) \leqslant 0$. If $u > v$, then $u^3[1 - (a+2b+2c)] \leqslant 0$, a contradiction. Hence $u \leqslant v$ which implies $u \leqslant h_1 v$, where $0 \leqslant h_1 = \sqrt[3]{a+2b+2c} < 1$.

Similarly, $u, v \geqslant 0$ and $F(u, v, u, v, v, u+v) \leqslant 0$ implies $u \leqslant h_2 v$, where $0 \leqslant h_2 = h_1 < 1$.

(F_3) : $F(t, t, 0, 0, t, t) = t^3 > 0$, $\forall t > 0$.

EXAMPLE 3.6. $F(t_1,...,t_6) = t_1^2 + \dfrac{t_1}{t_5 + t_6} - \left(at_2^2 + bt_3^2 + ct_4^2\right)$, where $a, b, c \geqslant 0$ and $a + b + c < 1$.

(F_2) : Let $u, v \geqslant 0$ be such that $F(u, v, v, u, u+v, v) = u^2 + \dfrac{u}{u+2v} - \left(av^2 + bv^2 + cu^2\right) \leqslant 0$, which implies $u^2 - \left(av^2 + bv^2 + cu^2\right) \leqslant 0$. If $u > v$, then $u^2[1 - (a+b+c)] \leqslant 0$, a contradiction. Hence $u \leqslant v$ which implies $u \leqslant h_1 v$, where $0 \leqslant h_1 = \sqrt{a+b+c} < 1$.

Similarly, $u, v \geqslant 0$ and $F(u, v, u, v, v, u+v) \leqslant 0$ implies $u \leqslant h_2 v$, where $0 \leqslant h_2 = h_1 < 1$.

(F_3) : $F(t, t, 0, 0, t, t) = t^2 + \dfrac{1}{2} - at^2 = t^2(1-a) + \dfrac{1}{2} > 0$, $\forall t > 0$.

4. Main results

THEOREM 4.1. *Let (X, p) be a partial metric space and $T, S : X \to X$ be two mappings satisfying inequality*

$$F(p(Tx, Sy), p(x, y), p(x, Tx), p(y, Sy), p(x, Sy), p(y, Tx)) \leqslant 0, \tag{4.1}$$

for all $x, y \in \overline{O_{x_0}(S, T)}$ for some $x_0 \in X$ and $F \in \mathcal{F}_{R0}$. If (X, p) is (S, T) - orbitally 0 - complete at x_0, then T and S have a common fixed point z such that $p(z, z) = p(z, Tz) = p(z, Sz) = 0$.

If moreover, each common fixed point z of S and T in $O_{x_0}(S,T)$ satisfies $p(z,z)=0$, then the common fixed point of S and T in $O_{x_0}(S,T)$ is unique.

PROOF. First we prove that if $z = Sz$ and $p(z,z) = 0$, then z is a common fixed point of S and T.

By (4.1) we obtain

$$F(p(Tz,Sz), p(z,z), p(z,Tz), p(z,Sz), p(z,Sz), p(z,Tz)) \leqslant 0,$$

$$F(p(Tz,z), 0, p(z,Tz), 0, 0, p(z,Tz)) \leqslant 0.$$

By (F_{2a}) we obtain $p(z,Tz) = 0$ which implies $z = Tz$ and z is a common fixed point of S and T.

We define a sequence $\{x_n\}$ in X as follows

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \text{ for } n = 0,1,2,... \tag{4.2}$$

If there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, Sx_{n_0}) = 0$ or $p(x_{n_0}, Tx_{n_0}) = 0$ for $n_0 \in \mathbb{N}$, then S and T have a common fixed point. We suppose that $p(x_n, x_{n+1}) \neq 0$, for $n \in \mathbb{N}$.

By (4.1) and (4.2) for $x = x_{2n+1}$ and $y = x_{2n}$ we obtain

$$\begin{gathered} F(p(Tx_{2n+1}, Sx_{2n}), p(x_{2n+1}, x_{2n}), p(x_{2n+1}, Tx_{2n+1}), \\ p(x_{2n}, Sx_{2n}), p(x_{2n+1}, Sx_{2n}), p(x_{2n}, Tx_{2n+1})) \leqslant 0, \end{gathered}$$

$$\begin{gathered} F(p(x_{2n+2}, x_{2n+1}), p(x_{2n+1}, x_{2n}), p(x_{2n+1}, x_{2n+2}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+1}), p(x_{2n}, x_{2n+2})) \leqslant 0. \end{gathered} \tag{4.3}$$

By (P_2),

$$p(x_{2n+1}, x_{2n+1}) \leqslant p(x_{2n+1}, x_{2n})$$

and by (P_4)

$$p(x_{2n}, x_{2n+2}) \leqslant p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n}).$$

By (F_1) and (4.3) we obtain

$$\begin{gathered} F(p(x_{2n+2}, x_{2n+1}), p(x_{2n+1}, x_{2n}), p(x_{2n+1}, x_{2n+2}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})) \leqslant 0. \end{gathered}$$

By (F_{2b}) we obtain

$$p(x_{2n+2}, x_{2n+1}) \leqslant hp(x_{2n+1}, x_{2n}), \text{ where } h = \max\{h_1, h_2\}.$$

By (4.1) and (4.2) for $x = x_{2n-1}$ and $y = x_{2n}$, for $n = 1,2,...$ we obtain

$$\begin{gathered} F(p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, Tx_{2n-1}), \\ p(x_{2n}, Sx_{2n}), p(x_{2n-1}, Sx_{2n}), p(x_{2n}, Tx_{2n-1})) \leqslant 0, \end{gathered}$$

$$\begin{gathered} F(p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n}), p(x_{2n-1}, x_{2n}), \\ p(x_{2n}, x_{2n+1}), p(x_{2n-1}, x_{2n+1}), p(x_{2n}, x_{2n})) \leqslant 0. \end{gathered} \tag{4.4}$$

By (P_2)

$$p(x_{2n}, x_{2n}) \leqslant p(x_{2n-1}, x_{2n})$$

and by (P_4)

$$p(x_{2n-1}, x_{2n+1}) \leqslant p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}).$$

By (F_1) and (4.4) we obtain

$$\begin{gathered}F(p(x_{2n},x_{2n+1}),p\left(x_{2n-1},x_{2n}\right),p\left(x_{2n-1},x_{2n}\right),\\ p\left(x_{2n},x_{2n+1}\right),p\left(x_{2n-1},x_{2n}\right)+p\left(x_{2n},x_{2n+1}\right),p\left(x_{2n},x_{2n-1}\right))\leqslant 0.\end{gathered}$$

By (F_{2a}) we obtain

$$p\left(x_{2n},x_{2n+1}\right)\leqslant hp\left(x_{2n-1},x_{2n}\right).$$

Hence

$$p\left(x_n,x_{n+1}\right)\leqslant hp\left(x_{n-1},x_n\right)\leqslant ...\leqslant h^n p\left(x_0,x_1\right). \tag{4.5}$$

Then for each $m>n\in\mathbb{N}$, by (4.5) and (P_4) we have

$$\begin{aligned}p\left(x_n,x_{n+m}\right) &\leqslant p\left(x_n,x_{n+1}\right)+p\left(x_{n+1},x_{n+2}\right)+...+p\left(x_{m-1},x_m\right)\\ &\leqslant h^n\left(1+h+...+h^{m-1}\right)p\left(x_0,x_1\right)\\ &\leqslant \frac{h^n}{1-h}p\left(x_0,x_1\right).\end{aligned}$$

Thus $\lim_{n,m\to\infty}p\left(x_n,x_m\right)=0$. This implies that $\{x_n\}$ is a 0 - Cauchy sequence in the partial metric space $O_{x_0}\left(S,T\right)$. Since X is (S,T) - orbitally 0 - complete at x_0, then there exists $z\in X$ with $\lim_{n\to\infty}x_n=z$ and $p\left(z,z\right)=0$.

We prove that z is a fixed point for S.

By (4.1) for $x=x_{2n+1}$ and $y=z$ we obtain

$$\begin{gathered}F(p(Tx_{2n+1},Sz),p\left(x_{2n+1},z\right),p\left(x_{2n+1},Tx_{2n+1}\right),\\ p\left(z,Sz\right),p\left(x_{2n+1},Sz\right),p\left(z,Tx_{2n+1}\right))\leqslant 0,\end{gathered}$$

$$\begin{gathered}F(p(x_{2n+2},Sz),p\left(x_{2n+1},z\right),p\left(x_{2n+1},x_{2n+2}\right),\\ p\left(z,Sz\right),p\left(x_{2n+1},Sz\right),p\left(z,x_{2n+2}\right))\leqslant 0.\end{gathered}$$

Letting n tends to infinity, by Lemma 2.1 and (4.5) we obtain

$$F(p(z,Sz),0,0,p\left(z,Sz\right),p\left(z,Sz\right),0)\leqslant 0.$$

By (F_{2a}) we obtain $p\left(z,Sz\right)=0$ which implies $z=Sz$. By the first part of the proof we have $z=Tz$ and z is a common fixed point of S and T.

Now suppose that each common fixed point z of T and S in $O_{x_0}\left(S,T\right)$ satisfy $p\left(z,z\right)=0$. We claim that S and T have a unique common fixed point. Assume that $p\left(u,Su\right)=p\left(u,Tu\right)=0$ and $p\left(v,Tv\right)=p\left(Sv,v\right)=0$ but $u\neq v$. Then, by (4.1) for $x=u$ and $y=v$ we have

$$\begin{gathered}F(p(Tu,Sv),p\left(u,v\right),p\left(u,Tu\right),\\ p\left(v,Sv\right),p\left(u,Sv\right),p\left(v,Tu\right))\leqslant 0,\end{gathered}$$

$$F(p(u,v),p\left(u,v\right),0,0,p\left(u,v\right),p\left(u,v\right))\leqslant 0,$$

a contradiction of (F_3). Hence, $u=v$. □

Remark 4.1. By Theorem 4.1 and Example 3.1 we obtain a fixed point theorem of Hardy - Rogers type.

If $S=T$ by Theorem 4.1 we obtain

THEOREM 4.2. *Let (X, p) be a partial metric space such that X is T - orbitally 0 - complete at some $x_0 \in X$ and*

$$F(p(Tx, Ty), p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)) \leqslant 0,$$

for all $x, y \in \overline{O_{x_0}(T)}$ and F satisfies properties (F_1), (F_{2a}) and (F_3). Then T has a fixed point. If moreover, each fixed point $z \in X$ in $\overline{O_{x_0}(T)}$ satisfies $p(z, z) = 0$, then the fixed point is unique.

EXAMPLE 4.1. Let $X = [0, 1]$ be and $p(x, y) = max\{x, y\}$. Then (X, p) is a partial metric space. Consider the following mappings: $S(x) = \frac{1}{3} \cdot x$ and $T(x) = \frac{1}{5} \cdot x$. If $x_0 = 1$ then $O_1(S, T) = \left\{ \left(\frac{1}{3}\right)^k \cdot \left(\frac{1}{5}\right)^m : k, m \in \mathbb{N} \right\}$ and $\overline{O_1(S, T)} \subset O_1(S, T) \cup \{0\}$.
1) If $x > y$, then $p(Sx, Ty) = \dfrac{1}{3} \cdot x$ and $p(x, y) = x$. Hence $p(Sx, Ty) \leqslant k_1 \cdot p(x, y)$, for $k_1 \in \left[\dfrac{1}{3}, \dfrac{1}{2}\right)$, which implies

$$p(x, y) \leqslant k_1 max \{p(x, y), p(Sx, x), p(Ty, y) \cdot p(x, Ty), p(y, Ax)\}, \text{ for } k_1 \in \left[\frac{1}{3}, \frac{1}{2}\right).$$

2) If $\dfrac{3}{5} \cdot y < x < y$ then $p(Sx, Ty) = \dfrac{1}{3} \cdot x$ and $p(x, Sx) = x$. Hence $p(Sx, Ty) \leqslant k_1 p(x, Sx)$, which implies

$$p(Sx, Ty) \leqslant k_1 max \{p(x, y), p(x, Sx), p(y, Ty), p(x, Ty)p(y, Sx)\}, \text{ for } k_1 \in \left[\frac{1}{3}, \frac{1}{2}\right).$$

3) If $x \leqslant \dfrac{3}{5} \cdot y$ then $p(Sx, Ty) = \dfrac{1}{5} \cdot y$ and $p(y, Ty) = y$. Hence $p(Sx, Ty) \leqslant k_2 \cdot p(y, Ty)$, for $k_2 \in \left[\frac{1}{5}, \frac{1}{2}\right)$, which implies

$$p(Sx, Ty) \leqslant k_2 \max\{p(x, y), p(x, Sx) \cdot p(y, Ty), p(x, Ty), p(y, Sx)\}.$$

Hence

$$p(Sx, Ty) \leqslant kmax \{p(x, y), p(x, Sx), p(y, Ty), p(x, Ty), p(y, Sx)\}$$

where $k \in \left[\dfrac{1}{3}, \dfrac{1}{2}\right)$.

By Example 3.1 and Theorem 4.1, S and T have a unique common fixed point $z = 0$ and $p(z, z) = 0$.

References

[1] T. Abdeljawad, E. Karapinar and K. Tas. Existence and uniqueness of a common fixed point on a partial metric space. *Appl. Math. Lett.*, **24**(11)(2011), 1900–1904.

[2] J. Achari. Ćirić's non - unique fixed point. *Mat. Vesnik*, **13**(28)(1976), 255–257.

[3] A. E. B. Ahmad, Z. M. Fadail, V. C. Rajić and S. Radenović: Nonlinear contractions in 0 - complete partial metric spaces, *Abstr. Appl. Anal.*, Vol. **2012**(2012), Article ID 451239.

[4] I. Altun and H. Simsek. Some fixed point theorems on dualistic metric spaces. *J. Adv. Math. Stud.*, **1**(2008), 1–8.

[5] I. Altun, F. Sola and H. Simsek. Generalized contractions on partial metric spaces. *Topology Appl.*, **157**(18)(2010), 2778–2785.

[6] H. Aydi, M. Jellali and E. Karapinar. Common fixed points for generalized α - implicit contractions in partial metric spaces: consequences and application. *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas*, **109**(2)(2015), 367–384.
[7] L. Ćirić. On some maps with nonunique fixed points. *Publ. Inst. Math. New Ser.*, **17**(31)(1974), 52–58.
[8] L. Ćirić and N. Jotić. A further extension of maps with nonunique fixed points. *Mat. Vesnik*, **50**(1-2)(1998), 1–4.
[9] S. Gülyaz and E. Karapinar. Coupled fixed point results in partial ordered metric spaces through implicit functions. *Hacet. J. Math. Stat.*, **42**(4)(2013), 347–357.
[10] S. Gülyaz, E. Karapinar and S. Yüce. A coupled coincidence point theorem in partially ordered metric spaces with an implicit relation. *Fixed Point Theory Appl.*, 2013, **2013**:38.
[11] Z. Kadelburg, H. K. Nashine and S. Radenović. Fixed point results under various contractive conditions in partial metric spaces. *Revista de la Real Academia de Ciencias Exactas, Fsicas y Naturales. Serie A. Matemticas*, **107**(2)(2013), 241–256.
[12] E. Karapinar and I. M. Erhan. Fixed point theorems for operators on partial metric spaces. *Appl. Math. Lett.*, **24**(11)(2011), 1894–1899.
[13] S. G. Matthews. Partial metric topology. In S. Andima et.al. (Eds.) Proc. 8^{th} Summer Conference on General Topology and Applications, *Ann. New York Acad. Sci.*, **728**(1994), 183–197.
[14] H. K. Nashine, W. Sintunavarat, Z. Kadelburg and P. Kumam. Fixed point theorems in orbitally 0 - complete partial metric spaces via rational contractive conditions. *Afr. Math.* DOI 10110007/s13370-014-2097.
[15] H. K. Pathak. Some nonunique fixed point theorems for a new class of mappings, *Ranchi Univ. Math. J.*, **17**(1986), 65–70.
[16] H. K. Pathak. On some nonunique fixed point theorems for the maps of Dhage type. *Pure Appl. Math. Sci.*, **27**(1988), 41–47.
[17] V. Popa. Fixed point theorems for implicit contractive mappings, *Stud. Cercet. Ştiinţ., Ser. Mat., Univ. Bacău*, **7**(1997), 129–133.
[18] V. Popa. Some fixed point theorems for compatible mappings satisfying an implicit relation, *Demonstr. Math.*, **32**(1)(1999), 159–163.
[19] S. Romaguera. A Kirk type characterization of completeness for partial metric spaces. *Fixed Point Theory Appl.*, **2010**(2010), Article ID 493298.
[20] S. Romaguera. Matkowski's type theorems for generalized contractions on (ordered) partial metric spaces. *Appl. Gen. Topol.*, **12**(2)(2011), 213–220.
[21] D. Türkoğlu, O. Özer and B. Fisher. Fixed point theorems for T - orbitally complete metric spaces. *Stud. Cercet. Ştiinţ., Ser. Mat., Univ. Bacău*, **9**(1999), 211–218.
[22] C. Vetro and F. Vetro. Common fixed points of mappings satisfying implicit relations in partial metric spaces. *J. Nonlinear Sci. Appl.*, **6**(2013), 156–161.

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