

A STUDY OF n -ARY SUBGROUPS WITH RESPECT TO t -CONORM

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ABSTRACT. In this paper, we introduce a notion of fuzzy n -ary subgroups with respect to t -conorm(s -fuzzy n -ary subgroups) in an n -ary groups (G, f) and have studied their related properties. The main contribution of this paper are studying the properties of s -fuzzy n -ary subgroups over s -level n -ary subgroup of (G, f) , n -ary homomorphism and $ret_a(G, f)$. Moreover some results of the S -product of s -fuzzy n -ary relations in an n -ary groups (G, f) are also obtained.

1. Introduction

The theory of fuzzy set was first developed by Zadeh [29] and has been applied to many branches in mathematics. Later fuzzification of the “group” concept into “fuzzy subgroup” was made by Rosenfeld [28]. This work was the first fuzzification of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests. The study of n -ary systems was initiated by Kasner [26] in 1904, but the important study on n -ary groups was done by Dörnte [3]. The theory of n -ary systems have many applications. For example, in the theory of automata [23], n -ary semigroup and n -ary groups are used. The n -ary groupoids are applied in the theory of quantum groups [27]. Also the ternary structures in physics are described by Kerner in [25]. The n -ary system dealt in detail [4-9,11,12,14-22]. The first fuzzification of n -ary system was introduced by Dudek [10]. Moreover Davvaz et. al [2] have studied fuzzy n -ary groups as a generalization of Rosenfeld’s fuzzy groups and have investigated their related properties. The notion of intuitionistic fuzzy sets, as a generalization of the notion of fuzzy set. Dudek [13] has introduced intuitionistic fuzzy sets idea’s in n -ary systems and has discussed in detail. Triangular norm(t -norm) and triangular conorm(t -conorm) are the most general

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families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. Thus, the t -norm generalizes the conjunctive (AND) operator and the t -conorm generalizes the disjunctive (OR) operator. In application, t -norm T and t -conorm S are two functions that map the unit square into the unit interval. To study more about t -conorm see [24]. In this paper, we introduce the notion of fuzzy n -ary subgroups with respect to t -conorm (s -fuzzy n -ary subgroup) in n -ary group (G, f) and have investigated their related properties.

2. Preliminaries

A non-empty set G together with one n -ary operation $f : G^n \rightarrow G$, where $n \geq 2$, is called an n -ary groupoid and is denoted by (G, f) . According to the general convention used in the theory of n -ary groupoids the sequence of elements x_i, x_{i+1}, \dots, x_j is denoted by x_i^j . In the case $j < i$, it denoted the empty symbol.

If $x_{i+1} = x_{i+2} = \dots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} and we write $x^{(t)}$. In this convention

$$f(x_1, \dots, x_n) = f(x_1^n)$$

and

$$f(x_1, \dots, x_i, \underbrace{x, \dots, x}_t, x_{i+t+1}, \dots, x_n) = f(x_1^i, x^{(t)}, x_{i+t+1}^n).$$

An n -ary groupoid (G, f) is called an (i, j) -associative if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

hold for all $x_1, \dots, x_{2n-1} \in G$. If this identity holds for all $1 \leq i \leq j \leq n$, then we say that the operation f is associative and (G, f) is called an n -ary semigroup. It is clear that an n -ary groupoid is associative if and only if it is $(1, j)$ -associative for all $j = 2, \dots, n$. In the binary case (i.e. $n=2$) it is usual semigroup. If for all $x_0, x_1, \dots, x_n \in G$ and fixed $i \in \{1, \dots, n\}$ there exists an element $z \in G$ such that

$$f(x_1^{i-1}, z, x_{i+1}^n) = x_0 \tag{1}$$

then we say that this equation is i -solvable or solvable at the place i . If the solution is unique, then we say that (1) is uniquely i -solvable. An n -ary groupoid (G, f) uniquely solvable for all $i = 1, \dots, n$ is called an n -ary quasigroup. An associative n -ary quasigroup is called an n -ary group.

Fixing an n -ary operation f , where $n \geq 3$, the elements a_2^{n-2} we obtain the new binary operation $x \diamond y = f(x, a_2^{n-2}, y)$. If (G, f) is an n -ary group then (G, \diamond) is a group. Choosing different elements a_2^{n-2} we obtain different groups. All these groups are isomorphic [8]. So, we can consider only group of the form

$$\text{ret}_a(G, f) = (G, \circ), \text{ where } x \circ y = f(x, a^{(n-2)}, y).$$

In this group $e = \bar{a}$, $x^{-1} = f(\bar{a}, a^{(n-3)}, \bar{x}, \bar{a})$.

In the theory of n -ary groups, the following Theorem plays an important role.

THEOREM 2.1. For any n -ary group (G, f) there exist a group (G, \circ) , its automorphism φ and an element $b \in G$ such that

$$f(x_1^n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \dots \circ \varphi^{n-1}(x_n) \circ b \tag{2}$$

holds for all $x_1^n \in G$.

In what follows, G is a non-empty set and (G, f) is an n -ary group unless otherwise specified.

DEFINITION 2.1. By a t -norm, a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions is meant:

- (T1) $T(x, 1) = x$;
 - (T2) $T(x, y) \leq T(x, z)$ if $y \leq z$;
 - (T3) $T(x, y) = T(y, x)$;
 - (T4) $T(x, T(y, z)) = T(T(x, y), z)$;
- for all $x, y, z \in [0, 1]$.

DEFINITION 2.2. By a t -conorm, a function $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions is meant:

- (S1) $S(x, 0) = x$;
 - (S2) $S(x, y) \leq S(x, z)$ if $y \leq z$;
 - (S3) $S(x, y) = S(y, x)$;
 - (S4) $S(x, S(y, z)) = S(S(x, y), z)$;
- for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition (S1), we obtain the concept of t -norm T .

DEFINITION 2.3. Given a t -norm T and a t -conorm S , T and S are dual (with respect to the negation ι) if and only if $(T(x, y))' = S(x', y')$.

Now we generalize the domain of S to $\prod_{i=1}^n [0, 1]$ as follows:

DEFINITION 2.4. The function $S_n : \prod_{i=1}^n [0, 1] \rightarrow [0, 1]$ is defined by:

$$S_n(\alpha_1^n) = S_n(\alpha_1, \alpha_2, \dots, \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n)) \tag{3}$$

for all $1 \leq i \leq n, n \geq 2$.

For a t -conorm S on $\prod_{i=1}^n [0, 1]$, it is denoted by

$$\Delta_t = \{\alpha \in [0, 1] \mid S(\alpha, \alpha, \dots, \alpha) = \alpha\}.$$

It is clear that every t -conorm has the following property:

$$S(\alpha_1^n) \geq \max\{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

for all $\alpha_1^n \in [0, 1]$.

3. s -fuzzy n -ary subgroups

DEFINITION 3.1. A fuzzy set μ in G is called a s -fuzzy n -ary subgroup of (G, f) if the following axioms holds:

$$(SF_nS1) (\forall x_1^n \in G), (\mu(f(x_1^n)) \leq S\{\mu(x_1), \dots, \mu(x_n)\}),$$

$$(SF_nS2) (\forall x \in G), (\mu(\bar{x}) \leq \mu(x)).$$

EXAMPLE 3.1. Let (\mathbb{Z}_4, f) be a 4-ary subgroup derived from additive group \mathbb{Z}_4 . Define a fuzzy subset μ in \mathbb{Z}_4 as follows:

$$\mu(x) = \begin{cases} 0.1 & \text{if } x = 0, \\ 0.7 & \text{if } x = 1, 2, 3. \end{cases}$$

and let $S : \prod_{i=1}^4 [0, 1] \rightarrow [0, 1]$ be a function defined by as follows:

$$S(x_1^4) = \min \{x_1 + x_2 + x_3 + x_4, 1\}$$

for all $x_1^4 \in [0, 1]$ and a function f is defined by

$$f(x_1^4) = x_1 +_4 x_2 +_4 x_3 +_4 x_4, \forall x_1^4 \in \mathbb{Z}_4.$$

By routine calculations, we know that μ is a s -fuzzy 4-ary subgroup of (\mathbb{Z}_4, f) .

THEOREM 3.1. If $\{\mu_i | i \in I\}$ is an arbitrary family of s -fuzzy n -ary subgroup of (G, f) then $\bigcup \mu_i$ is s -fuzzy n -ary subgroup of (G, f) , where $\bigcup A_i = \bigvee \mu_i$, where $\bigvee \mu_i(x) = \sup\{\mu_i(x) | x \in G \text{ and } i \in I\}$.

PROOF. The proof is trivial. \square

THEOREM 3.2. If μ is a fuzzy set in G is a s -fuzzy n -ary subgroup of (G, f) , then so is μ' , where $\mu' = 1 - \mu$.

PROOF. It is sufficient to show that μ' satisfies conditions (SF_nS1) and (SF_nS2). Let $x_1^n \in G$. Then

$$\begin{aligned} \mu'(f(x_1^n)) &= 1 - \mu(f(x_1^n)) \\ &\leq 1 - S\{\mu(x_1), \dots, \mu(x_n)\} \\ &\leq S\{1 - \mu(x_1), \dots, 1 - \mu(x_n)\} \\ &= S\{\mu'(x_1), \dots, \mu'(x_n)\}. \end{aligned}$$

and

$$\mu'(\bar{x}) = 1 - \mu(\bar{x}) \leq 1 - \mu(x) = \mu'(x).$$

Hence μ' is a s -fuzzy n -ary subgroup of (G, f) . \square

The following Lemma gives the relation between T and S .

LEMMA 3.1. Let T be a t -norm. Then the t -conorm S can be defined as

$$S(x_1^n) = 1 - T(1 - x_1, 1 - x_2, \dots, 1 - x_n), \forall x_1^n \in G.$$

PROOF. Straightforward. \square

The following Theorem gives the relation between t -fuzzy n -ary subgroup and s -fuzzy n -ary subgroup of G .

THEOREM 3.3. *A fuzzy set μ of G is a t -fuzzy n -ary subgroup of (G, f) if and only if its complement μ' is a s -fuzzy n -ary subgroup of (G, f) .*

PROOF. Let μ be a t -fuzzy n -ary subgroup of (G, f) . For all $x_1^n \in G$, we have

$$\begin{aligned}\mu'(f(x_1^n)) &= 1 - \mu(f(x_1^n)) \\ &\leq 1 - T\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\} \\ &= 1 - T\{1 - \mu'(x_1), 1 - \mu'(x_2), \dots, 1 - \mu'(x_n)\} \\ &= S\{\mu'(x_1), \mu'(x_2), \dots, \mu'(x_n)\}.\end{aligned}$$

For all $x \in G$, we have

$$\mu'(\bar{x}) = 1 - \mu(\bar{x}) \leq 1 - \mu(x) = \mu'(x).$$

The converse is proved similarly. \square

DEFINITION 3.2. *Let μ be a fuzzy set in G and let $t \in [0, 1]$. Then the set*

$$L(\mu; t) := \{x \in G \mid \mu(x) \leq t\}$$

is called anti-level subset μ of G .

The following Theorem is a consequence of the Transfer Principle described in [26].

THEOREM 3.4. *A fuzzy set μ in G , is a s -fuzzy n -ary subgroup of (G, f) if and only if the anti-level subset $L(\mu; t)$ of G is an n -ary subgroup of (G, f) for every $t \in [0, 1]$, which is called s -level n -ary subgroup of (G, f) .*

PROOF. Let μ be a s -fuzzy n -ary subgroup of (G, f) . If $x_1^n \in G$ and $t \in [0, 1]$, then $\mu(x_i) \leq t$ for all $i = 1, 2, \dots, n$. Thus

$$\mu(f(x_1^n)) \leq S\{\mu(x_1), \dots, \mu(x_n)\} \leq t,$$

which implies $f(x_1^n) \in L(\mu; t)$. Moreover, for some $x \in L(\mu; t)$, we have

$$\mu(\bar{x}) \leq \mu(x) \leq t,$$

which implies $\bar{x} \in L(\mu; t)$. Thus $L(\mu; t)$ is an n -ary subgroup of (G, f) .

Conversely, assume that $L(\mu; t)$ is an n -ary subgroup of (G, f) . Let us define

$$t_0 = S\{\mu(x_1), \dots, \mu(x_n)\},$$

for some $x_1^n \in G$. Then obviously $x_1^n \in L(\mu; t_0)$, consequently $f(x_1^n) \in L(\mu; t_0)$. Thus

$$\mu(f(x_1^n)) \leq t_0 = S\{\mu(x_1), \dots, \mu(x_n)\}.$$

Now, let $x \in L(\mu; t)$. Then $\mu(x) = t_0 \leq t$. Thus $x \in L(\mu; t_0)$. Since, by the assumption, $\bar{x} \in L(\mu; t_0)$. Whence $\mu(\bar{x}) \leq t_0 = \mu(x)$. This complete the proof. \square

Using the above theorem, we can prove the following characterization of s -fuzzy n -ary subgroups.

THEOREM 3.5. *A fuzzy set μ in G , is a s -fuzzy n -ary subgroup of (G, f) if and only if the anti-level subset $L(\mu; t)$ of G is an n -ary subgroup of (G, f) for all $i = 1, 2, \dots, n$ and all $x_1^n \in G$, μ satisfies the following conditions:*

- (i) $\mu(f(x_1^n)) \leq S\{\mu(x_1), \dots, \mu(x_n)\}$,
- (ii) $\mu(x_i) \leq S\{\mu(x_1), \dots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i-1}), \dots, \mu(x_n)\}$.

PROOF. Assume that μ is a s -fuzzy n -ary subgroup of (G, f) . Similarly as in the proof of Theorem 3.4, we can prove the non-empty level subset $L(\mu; t)$ under the operation f , that is $x_1^n \in L(\mu; t)$ implies $f(x_1^n) \in L(\mu; t)$.

Now let $x_0, x_1^{i-1}, x_{i+1}^n$, where $x_0 = f(x_1^{i-1}, z, x_{i+1}^n)$ for some $i = 1, 2, \dots, n$ and $z \in G$ which implies $x_0 \in L(\mu; t)$. Then, according to (ii), we have $\mu(z) \leq t$. So, the the equation (1) has a solution $z \in \mu(t)$. This mean that level subset $L(\mu; t)$ is an n -ary subgroups.

Conversely, assume that level subset $L(\mu; t)$ is an n -ary subgroups of (G, f) . Then it is easy to prove the condition (i).

For $x_1^n \in G$, we define

$$t_0 = S\{\mu(x_1), \dots, \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i-1}), \dots, \mu(x_n)\}.$$

Then $x_1^{i-1}, x_{i+1}^n, f(x_1^n) \in L(\mu, t_0)$. Whence, according to the definition of n -ary group, we conclude $x_i \in L(\mu, t_0)$. Thus $\mu(x_i) \leq t_0$. This proves the conditions (ii). \square

DEFINITION 3.3. *Let (G, f) and (G', f) be an n -ary groups. A mapping $g : G \rightarrow G'$ is called an n -ary homomorphism if $g(f(x_1^n)) = f(g^n(x_1^n))$, where $g^n(x_1^n) = (g(x_1), \dots, g(x_n))$ for all $x_1^n \in G$.*

For any fuzzy set μ in G' , we define the *preimage* of μ under g , denoted by $g^{-1}(\mu)$, is a fuzzy set in G defined by

$$g^{-1}(\mu) = \mu_{g^{-1}}(x) = \mu(g(x)), \forall x \in G.$$

For any fuzzy set μ in G , we define the *image* of μ under g , denoted by $g(\mu)$, is a fuzzy set in G' defined by

$$g(\mu)(y) = \begin{cases} \inf_{x \in g^{-1}(y)} \mu(x), & \text{if } g^{-1}(y) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in G$ and $y \in G'$.

THEOREM 3.6. *Let g be a n -ary homomorphism mapping from G into G' with $g(\bar{x}) = g(x)$ for all $x \in G$ and μ is a s -fuzzy n -ary subgroup of (G', f) . Then $g^{-1}(\mu)$ is a s -fuzzy n -ary subgroup of (G, f) .*

PROOF. Let $x_1^n \in G$, we have

$$\begin{aligned} \mu_{g^{-1}}(f(x_1^n)) &= \mu(g(f(x_1^n))) = \mu(f(g^n(x_1^n))) \\ &\leq S\{\mu(g(x_1), \dots, \mu(g(x_n)))\} \\ &= S\{\mu_{g^{-1}}(x_1), \dots, \mu_{g^{-1}}(x_n)\}. \end{aligned}$$

and

$$\mu_{g^{-1}}(\bar{x}) = \mu(g(\bar{x})) \leq \mu(g(x)) = \mu_{g^{-1}(\mu)}(x).$$

This completes the proof. \square

If we strengthen the condition of g , then we can construct the converse of Theorem 3.6 as follows.

THEOREM 3.7. *Let g be a n -ary homomorphism from G into G' and $g^{-1}(\mu)$ is a s -fuzzy n -ary subgroup of (G, f) . Then μ is a s -fuzzy n -ary subgroup of (G', f) .*

PROOF. For any $x_1 \in G'$, there exists $a_1 \in G$ such that $g(a_1) = x_1$ and for any $f(x_1^n) \in (G', f)$, there exists $f(a_1^n) \in (G, f)$ such that $g(f(a_1^n)) = f(x_1^n)$. Then

$$\begin{aligned} \mu(f(x_1^n)) &= \mu(g(f(a_1^n))) = \mu_{g^{-1}}(f(a_1^n)) \\ &\leq S\{\mu_{g^{-1}}(a_1), \mu_{g^{-1}}(a_2), \dots, \mu_{g^{-1}}(a_n)\} \\ &= S\{\mu(g(a_1), \dots, \mu(g(a_n)))\} \\ &= S\{\mu(x_1), \dots, \mu(x_n)\}. \end{aligned}$$

For any $\bar{x} \in G'$, there exists $\bar{a} \in G$ such that $g(\bar{a}) = \bar{x}$, we have

$$\mu(\bar{x}) = \mu(g(\bar{a})) = \mu_{g^{-1}}(\bar{a}) \leq \mu_{g^{-1}}(a) = \mu(a) = \mu(x).$$

This completes the proof. \square

THEOREM 3.8. *Let $g : G \rightarrow G'$ be an onto mapping. If μ is a s -fuzzy n -ary subgroup of (G, f) , then $g(\mu)$ is a s -fuzzy n -ary subgroup of (G', f) .*

PROOF. Let g be a mapping from G onto G' and let $x_1^n \in G$, $y_1^n \in G'$. Noticing that

$$\begin{aligned} &\{x_i (i = 1, 2, \dots, n) | x_i \in g^{-1}(f(y_1^n))\} \subseteq \\ &\{f(x_1^n) \in G | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), \dots, x_n \in g^{-1}(y_n)\}. \end{aligned}$$

we have

$$\begin{aligned} &g(\mu)(f(y_1^n)) \\ &= \inf\{\mu(x_1^n) | x_i \in g^{-1}(f(y_1^n))\} \\ &\leq \inf\{\mu(f(x_1^n)) | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), \dots, x_n \in g^{-1}(y_n)\} \\ &\leq \inf\{\max\{\mu(x_1), \mu(x_2), \dots, \mu(x_n)\} | x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), \dots, x_n \in g^{-1}(y_n)\} \\ &= \max\{\inf\{\mu(x_1) | x_1 \in g^{-1}(y_1)\}, \inf\{\mu(x_2) | x_1 \in g^{-1}(y_2)\}, \dots, \inf\{\mu(x_n) | x_1 \in g^{-1}(y_n)\}\} \\ &\leq S\{g(\mu)(y_1), g(\mu)(y_2), \dots, g(\mu)(y_n)\}. \end{aligned}$$

and

$$g(\mu)(\bar{x}) = \inf\{\mu(\bar{x}) | \bar{x} \in g^{-1}(f(\bar{y}))\} \leq \inf\{\mu(x) | x \in g^{-1}(f(\bar{y}))\} = g(\mu)(\bar{x}).$$

This completes the proof. \square

COROLLARY 3.1. A fuzzy subset μ defined on group (G, \cdot) is a s -fuzzy subgroup if and only if

- (1) $\mu(xy) \leq S\{\mu(x), \mu(y)\}$,
 - (2) $\mu(x) \leq S\{\mu(y), \mu(xy)\}$,
 - (3) $\mu(y) \leq S\{\mu(x), \mu(xy)\}$.
- holds for all $x, y \in G$.

THEOREM 3.9. Let μ be a s -fuzzy subgroup of (G, \cdot) . If there exists an element $a \in G$ such that $\mu(a) \leq \mu(x)$ for every $x \in G$, then μ is a s -fuzzy subgroup of a group $ret_a(G, f)$.

PROOF. For all $x, y, a \in G$, let if possible μ is not a s -fuzzy subgroup of a group $ret_a(G, f)$. Then we have $\mu(x \circ y) > S\{\mu(x), \mu(y)\}$. That is

$$\begin{aligned} S\{\mu(x), \mu(x)\} &< \mu(x \circ x) \\ &= \mu(f(x, \overset{(n-2)}{a}, x)) \\ &\leq S\{\mu(x), \mu(\overset{(n-2)}{a}), \mu(x)\} \\ S\{\mu(x), \mu(x)\} &< S\{\mu(x), \mu(a)\}. \end{aligned}$$

This holds only if $\mu(a) > \mu(x)$, which is contradiction to our assumption $\mu(a) \leq \mu(x)$.

Also, we have μ is a s -fuzzy subgroup of (G, \cdot) . Thus $\mu(x^{-1}) \leq \mu(x)$ is obvious for all $x \in G$.

which complete the proof. \square

In Theorem 3.9, the assumption that $\mu(a) \leq \mu(x)$ cannot be omitted.

EXAMPLE 3.2. Let (\mathbb{Z}_4, f) be a 4-ary group from Example 3.1. Define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.4, & \text{if } x = 0, \\ 1, & \text{if } x = 1, 2, 3. \end{cases}$$

Clearly, μ is a s -fuzzy 4-ary subgroup of (\mathbb{Z}_4, f) . For $ret_1(\mathbb{Z}_4, f)$, define

$$S(x, y) = \begin{cases} \max(x, y) & \text{if } x = y, \\ \min(x + y, 1) & \text{if } x \neq y. \end{cases}$$

we have

$$\mu(0 \circ 0) = \mu((f(0, 1, 1, 0))) = \mu(2) = 1 \not\leq 0.4 = \mu(0) = S\{\mu(0), \mu(0)\}.$$

Hence the assumptions $\mu(a) \leq \mu(x)$ cannot be omitted.

THEOREM 3.10. Let (G, f) be an n -ary group. If μ is a s -fuzzy n -ary subgroup of a group $ret_a(G, f)$ and $\mu(a) \leq \mu(x)$ for all $a, x \in G$, then μ is a s -fuzzy n -ary subgroup of (G, f) .

PROOF. According to Theorem 2.1, any n -ary group can be represented of the form (2), where $(G, \circ) = \text{ret}_a(G, f)$, $\varphi(x) = f(\bar{a}, x, \overset{(n-2)}{a})$ and $b = f(\bar{a}, \dots, \bar{a})$. Then we have

$$\mu(\varphi(x)) = \mu(f(\bar{a}, x, \overset{(n-2)}{a})) \leq S\{\mu(\bar{a}), \mu(x), \mu(a)\} \leq \mu(x).$$

$$\mu(\varphi^2(x)) = \mu(f(\bar{a}, \varphi(x), \overset{(n-2)}{x})) \leq S\{\mu(\bar{a}), \mu(\varphi(x)), \mu(a)\} \leq S\{\mu(\bar{a}), \mu(x), \mu(a)\} \leq \mu(x).$$

Consequently, $\mu(\varphi^k(x)) \leq \mu(x)$. for all $x \in G$ and $k \in \mathbb{N}$.

Similarly, for all $x \in G$ we have

$$\mu(b) = \mu(f(\bar{a}, \dots, \bar{a})) \leq \mu(\bar{a}) \leq \mu(x).$$

Thus

$$\begin{aligned} \mu(f(x_1^n)) &= \mu(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ \dots \circ \varphi^{n-2}(x_n) \circ b) \\ &\leq S\{\mu(x_1), \mu(\varphi(x_2)), \mu(\varphi^2(x_3)), \dots, \mu(\varphi^{n-2}(x_n)), \mu(b)\} \\ &\leq S\{\mu(x_1), \mu(x_2), \mu(x_3), \dots, \mu(x_n), \mu(b)\} \\ &\leq S\{\mu(x_1), \mu(x_2), \mu(x_3), \dots, \mu(x_n)\}. \end{aligned}$$

From (4) and (7) of [3], we have

$$\bar{x} = (\mu(\bar{a} \circ \varphi(x) \circ \varphi^2(x) \circ \dots \circ \varphi^{n-2}(x) \circ b))^{-1}$$

Thus

$$\begin{aligned} \mu(\bar{x}) &= \mu\left((\bar{a} \circ \varphi(x) \circ \varphi^2(x) \circ \dots \circ \varphi^{n-2}(x) \circ b)^{-1}\right) \\ &\leq \mu(\bar{a} \circ \varphi(x) \circ \varphi^2(x) \circ \dots \circ \varphi^{n-2}(x) \circ b) \\ &\leq S\{\mu(\bar{a}), \mu(\varphi(x)), \mu(\varphi^2(x)), \dots, \mu(\varphi^{n-2}(x)), \mu(b)\} \\ &\leq S\{\mu(x), \mu(b)\} = \mu(x). \end{aligned}$$

This completes the proof. □

COROLLARY 3.2. *If (G, f) is a ternary group, then any s -fuzzy subgroup of $\text{ret}_a(G, f)$ is a s -fuzzy ternary subgroup of (G, f) .*

PROOF. Since \bar{a} is a neutral element of a group $\text{ret}_a(G, f)$ then $\mu(\bar{a}) \leq \mu(x)$, for all $x \in G$. Thus $\mu(\bar{a}) \leq \mu(a)$. But in ternary group $\bar{a} = a$ for any $a \in G$, whence $\mu(a) = \mu(\bar{a}) \leq \mu(\bar{a}) \leq \mu(x)$. So, $\mu(a) = \mu(\bar{a}) \leq \mu(x)$, for all $x \in G$. This means that the assumption of Theorem 3.10 is satisfied. Hence $\text{ret}_a(G, f)$ is a s -fuzzy ternary subgroup of (G, f) . This completes the proof. □

EXAMPLE 3.3. Consider the ternary group (\mathbb{Z}_{12}, f) , derived from the additive group \mathbb{Z}_{12} . Let μ be a s -fuzzy subgroup of the group of $\text{ret}_1(G, f)$ induced by subgroups $S_1 = \{11\}$, $S_2 = \{5, 11\}$ and $S_3 = \{1, 3, 5, 7, 9, 11\}$. Define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.1 & \text{if } x = 11, \\ 0.3 & \text{if } x = 5, \\ 0.5 & \text{if } x = 1, 3, 7, 9, \\ 0.9 & \text{if } x \notin S_3. \end{cases}$$

Then

$$\mu(\bar{5}) = \mu(7) = 0.5 \not\leq 0.3 = \mu(5).$$

Hence μ is not a s -fuzzy ternary subgroup of (\mathbb{Z}_{12}, f) .

Observations. From the above Example 3.3 it follows that:

(1) There are s -fuzzy subgroups of $ret_a(G, f)$ which are not s -fuzzy n -ary subgroups of (G, f) .

(2) In Theorem 3.10 the assumption $\mu(a) \leq \mu(x)$ can not be omitted. In the above example we have $\mu(1) = 0.5 \not\leq 0.3 = \mu(5)$.

(3) The assumption $\mu(a) \leq \mu(x)$ cannot be replaced by the natural assumption $\mu(\bar{a}) \leq \mu(x)$. (\bar{a} is the identity of $ret_a(G, f)$). In the above example $\bar{1} = 11$, then $\mu(11) \leq \mu(x)$ for all $x \in \mathbb{Z}_{12}$.

THEOREM 3.11. *Let (G, f) be an n -ary group of b -derived from the group (G, \circ) . Any fuzzy set μ of (G, \circ) such that $\mu(b) \leq \mu(x)$ for every $x \in G$ is a s -fuzzy n -ary subgroup of (G, f) .*

PROOF. The condition (SFnS1) is obvious. To prove (SFnS2), we have n -ary group (G, f) b -derived from the group (G, \circ) , which implies

$$\bar{x} = (x^{n-2} \circ b)^{-1},$$

where x^{n-2} is the power of x in (G, \circ) [4].

Thus, for all $x \in G$

$$\mu(\bar{x}) = \mu((x^{n-2} \circ b)^{-1}) \leq \mu(x^{n-2} \circ b)^{-1} \leq S\{\mu(x^{n-2}), \mu(b)\} = \mu(x).$$

This complete the proof. \square

COROLLARY 3.3. *Any s -fuzzy subgroup of a group (G, \circ) is a s -fuzzy n -ary subgroup of an n -ary group (G, f) derived from (G, \circ) .*

PROOF. If n -ary group (G, f) is derived from the group (G, \circ) then $b = e$. Thus $\mu(e) \leq \mu(x)$ for all $x \in G$. \square

4. S -product of s -fuzzy n -ary relations

DEFINITION 4.1. *A fuzzy n -ary relation on any set G is a fuzzy set*

$$\mu : G^n = G \times G \times \dots \times G \text{ (} n \text{ times)} \rightarrow [0, 1].$$

DEFINITION 4.2. *Let μ be fuzzy n -ary relation on any set G and ν is a fuzzy set on G . Then μ is called s -fuzzy n -ary relation on ν if*

$$\mu(x_1^n) \leq S(\nu(x_1), \nu(x_2), \dots, \nu(x_n)),$$

for all $x_1^n \in G$.

DEFINITION 4.3. *Let $\mu_1^n = \mu_1, \mu_2, \dots, \mu_n$ be a fuzzy sets in G . Then direct S -product of μ_1^n is defined by*

$$(\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_1^n) = S(\mu_1(x_1), \mu_2(x_2), \dots, \mu_n(x_n)), \forall x_1^n \in G.$$

LEMMA 4.1. *Let S be a function induced by t -conorm and let μ_1^n be a fuzzy sets in G . Then*

- (i) $\mu_1 \times_S \mu_2 \times_S \dots \times_S \mu_n$ is a s -fuzzy n -ary relation on G ,
- (ii) $L(\mu_1 \times \mu_2 \times \dots \times \mu_n; t) = L(\mu_1; t) \times L(\mu_2; t) \times \dots \times L(\mu_n; t), \forall t \in [0, 1]$.

PROOF. The proof is obvious. □

DEFINITION 4.4. *Let S be a function induced by t -conorm. If ν is a fuzzy set in G , the strongest s -fuzzy n -ary relation on G that is a s -fuzzy n -ary relation on ν is μ_ν , given by*

$$\mu_\nu(x_1^n) = S(\nu(x_1), \nu(x_2), \dots, \nu(x_n)), \forall x_1^n \in G.$$

LEMMA 4.2. *For a given fuzzy set ν in G , let μ be the strongest s -fuzzy n -ary relation of G . Then for $t \in [0, 1]$, $L(\mu_\nu; t) = L(\nu; t) \times L(\nu; t) \times \dots \times L(\nu; t)$.*

PROOF. The proof is obvious. □

PROPOSITION 4.1. *Let S be a function induced by t -conorm and let $\mu_1, \mu_2, \dots, \mu_n$ be s -fuzzy n -ary subgroup of (G, f) . Then, $\mu_1 \times \mu_2 \times \dots \times \mu_n$ is a s -fuzzy n -ary subgroup of (G^n, f) .*

PROOF. For $x_1^n \in G$ and $f(x_1^n) = (f_1(x_1^n), \dots, f_n(x_1^n)) \in (G^n, f)$, we have

$$\begin{aligned} & (\mu_1 \times \mu_2 \times \dots \times \mu_n)(f(x_1^n)) \\ &= (\mu_1 \times \mu_2 \times \dots \times \mu_n)(f_1(x_1^n), \dots, f_n(x_1^n)) \\ &= S\{\mu_1(f(x_1^n)), \mu_2(f(x_1^n)), \dots, \mu_n(f(x_1^n))\} \\ &\leq S\{S\{\mu_1(x_1), \mu_1(x_2), \dots, \mu_1(x_n)\}, \dots, S\{\mu_n(x_1), \mu_n(x_2), \dots, \mu_n(x_n)\}\} \\ &= S\{(\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_1, \dots, x_1), \dots, (\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_n, \dots, x_n)\} \\ &= S\{(\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_1), \dots, (\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_n)\}. \end{aligned}$$

and for all $x = x_1^n, \bar{x} = \bar{x}_1^n \in G^n$, we have

$$\begin{aligned} (\mu_1 \times \mu_2 \times \dots \times \mu_n)(\bar{x}) &= (\mu_1 \times \mu_2 \times \dots \times \mu_n)(\bar{x}_1, \dots, \bar{x}_n) \\ &= S\{\mu_1(\bar{x}_1), \dots, \mu_n(\bar{x}_n)\} \\ &\leq S\{(\mu_1(x_1), \dots, \mu_n(x_n))\} \\ &= (\mu_1 \times \mu_2 \times \dots \times \mu_n)(x_1^n) \\ &= (\mu_1 \times \mu_2 \times \dots \times \mu_n)(x). \end{aligned}$$

This completes the proof. □

The following Corollary is the immediate consequence of Proposition 4.1.

COROLLARY 4.1. *Let S be a function induced by t -conorm and let $\prod_{i=1}^n (G_i, f)$ be the finite collection of n -ary subgroups and $G = \prod_{i=1}^n G_i$ the S -product of G_i . Let*

μ_i be a s -fuzzy n -ary subgroup of (G_i, f) , where $1 \leq i \leq n$. Then, $\mu = \prod_{i=1}^n \mu_i$ defined by

$$\mu(x_1^n) = \prod_{i=1}^n \mu_i(x_1^n) = S(\mu(x_1), \mu(x_2), \dots, \mu(x_n)).$$

Then μ is a s -fuzzy n -ary subgroup of (G, f) . \square

DEFINITION 4.5. Let μ_1^n be fuzzy sets in G . Then, the S -product of μ_1^n , written as $[\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_S$, is defined by:

$$[\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_S(x) = S(\mu_1(x), \mu_2(x), \dots, \mu_n(x)) \quad \forall x \in G.$$

THEOREM 4.1. Let μ_1^n be s -fuzzy n -ary subgroup of (G, f) . If S^* is a function induced by t -conorm dominates S , that is,

$$S^*(S(x_1^n), S(y_1^n), \dots, S(z_1^n)) \leq S(S^*(x_1, y_1, \dots, z_1), \dots, S^*(x_n, y_n, \dots, z_n))$$

for all $x_1^n, y_1^n, \dots, z_1^n \in [0, 1]$. Then S^* -product of μ_1^n , $[\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}$, is a s -fuzzy n -ary subgroup of (G, f) .

PROOF. Let $x_1^n \in G$, we have

$$\begin{aligned} & [\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}(f(x_1^n)) \\ &= S^*(\mu_1(f(x_1^n)), \mu_2(f(x_1^n)), \dots, \mu_n(f(x_1^n))) \\ &\leq S^*(S(\mu_1(x_1), \mu_1(x_2), \dots, \mu_1(x_n)), \dots, S(\mu_n(x_1), \mu_n(x_2), \dots, \mu_n(x_n))) \\ &\leq S(S^*(\mu_1(x_1), \mu_2(x_1), \dots, \mu_n(x_1)), \dots, S^*(\mu_1(x_n), \mu_2(x_n), \dots, \mu_n(x_n))) \\ &= S([\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}(x_1), \dots, [\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}(x_n)) \end{aligned}$$

and for all $x \in G$, we have

$$\begin{aligned} [\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}(\bar{x}) &= S^*(\mu_1(\bar{x}), \mu_2(\bar{x}), \dots, \mu_n(\bar{x})) \\ &\leq S^*(\mu_1(x), \mu_2(x), \dots, \mu_n(x)) \\ &= [\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}(x). \end{aligned}$$

Hence, $[\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}$ is a s -fuzzy n -ary subgroup of (G, f) . This completes the proof. \square

Let (G, f) and (G', f) be an n -ary groups. A mapping $g : G \rightarrow G'$ is an onto homomorphism. Let S and S^* be functions induced by t -conorm such that S^* dominates S . If μ_1^n are s -fuzzy n -ary subgroup of (G, f) , then the S^* -product of μ_1^n , $[\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}$ is a s -fuzzy n -ary subgroup. Since every onto homomorphic inverse image of a s -fuzzy n -ary subgroup, the inverse images $g^{-1}(\mu_1), g^{-1}(\mu_2), \dots, g^{-1}(\mu_n)$ and $g^{-1}([\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*})$ are s -fuzzy n -ary subgroup (G, f) .

The following theorem provides the relation between $g^{-1}([\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*})$ and S^* -product $([g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \dots \cdot g^{-1}(\mu_n)]_{S^*})$ of $g^{-1}(\mu_1), g^{-1}(\mu_2)$ and $g^{-1}(\mu_n)$.

THEOREM 4.2. *Let (G, f) and (G', f) be n -ary groups. A mapping $g : G \rightarrow G'$ is an onto n -ary homomorphism. Let S^* be a function induced by t -conorm such that S^* dominates S . Let μ_1^n be s -fuzzy n -ary subgroups of (G, f) . If $[\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}$ and is the S^* -product of μ_1^n , and $([g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \dots \cdot g^{-1}(\mu_n)]_{S^*})$ is the S^* -product of $g^{-1}(\mu_1), g^{-1}(\mu_2), \dots, g^{-1}(\mu_n)$ then*

$$g^{-1}([\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*}) = [g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \dots \cdot g^{-1}(\mu_n)]_{S^*}.$$

PROOF. Let $x \in G$, we have

$$\begin{aligned} g^{-1}([\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*})(x) &= ([\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_n]_{S^*})(g(x)) \\ &= S^*(\mu_1(g(x)) \cdot \mu_2(g(x)) \cdot \dots \cdot \mu_n(g(x))) \\ &= S^*(g^{-1}(\mu_1)(x) \cdot g^{-1}(\mu_2)(x) \cdot \dots \cdot g^{-1}(\mu_n)(x)) \\ &= [g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot \dots \cdot g^{-1}(\mu_n)]_{S^*}. \end{aligned}$$

This completes the proof. \square

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