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A STUDY OF n-ARY SUBGROUPS WITH RESPECT TO t-CONORM

D.R. Prince Williams

Abstract. In this paper, we introduce a notion of fuzzy n-ary subgroups with respect to t-conorm(s-fuzzy n-ary subgroups) in an n-ary groups (G, f) and have studied their related properties. The main contribution of this paper are studying the properties of s-fuzzy n-ary subgroups over s-level n-ary subgroup of (G, f), n-ary homomorphism and $ret_a(G, f)$. Moreover some results of the S-product of s-fuzzy n-ary relations in an n-ary groups (G, f) are also obtained.

1. Introduction

The theory of fuzzy set was first developed by Zadeh [29] and has been applied to many branches in mathematics. Later fuzzification of the “group” concept into “fuzzy subgroup” was made by Rosenfeld [28]. This work was the first fuzzification of any algebraic structure and thus opened a new direction, new exploration, new path of thinking to mathematicians, engineers, computer scientists and many others in various tests.The study of n-ary systems was initiated by Kasner [26] in 1904, but the important study on n-ary groups was done by Dörnte [3]. The theory of n-ary systems have many applications. For example, in the theory of automata [23], n-ary semigroup and n-ary groups are used. The n-ary groupoids are applied in the theory of quantum groups [27]. Also the ternary structures in physics are described by Kerner in [25]. The n-ary system dealt in detail [4-9,11,12,14-22]. The first fuzzification of n-ary system was introduced by Dudek [10]. Moreover Davvaz et. al [2] have studied fuzzy n-ary groups as a generalization of Rosenfeld’s fuzzy groups and have investigated their related properties. The notion of intuitionistic fuzzy sets, as a generalization of the notion of fuzzy set. Dudek [13] has introduced intuitionistic fuzzy sets idea’s in n-ary systems and has discussed in detail.
Triangular norm(t-norm) and triangular conorm(t-conorm) are the most general

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families of binary operations that satisfy the requirement of the conjunction and disjunction operators, respectively. Thus, the t-norm generalizes the conjunctive (AND) operator and the t-conorm generalizes the disjunctive (OR) operator. In application, t-norm T and t-conorm S are two functions that map the unit square into the unit interval. To study more about t-conorm see [24]. In this paper,we introduce the notion of fuzzy n-ary subgroups with respect to t-conorm (s-fuzzy n-ary subgroup) in n-ary group (G, f) and have investigated their related properties.

2. Preliminaries

A non-empty set G together with one n-ary operation $f : G^n \to G$,where $n \geqslant 2$, is called an *n-ary groupoid* and is denoted by (G, f). According to the general convention used in the theory of n-ary groupoids the sequence of elements $x_i, x_{i+1}, ..., x_j$ is denoted by x_i^j. In the case $j < i$, it denoted the empty symbol. If $x_{i+1} = x_{i+2} = ... = x_{i+t} = x$, then instead of x_{i+1}^{i+t} and we write $\overset{(t)}{x}$. In this convention

$$f(x_1, ..., x_n) = f(x_1^n)$$

and

$$f(x_1, ..., x_i, \underbrace{x, ..., x}_{t}, x_{i+t+1}, ..., x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n-ary groupoid (G, f) is called an (i, j)-*associative* if

$$f\left(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}\right)$$

hold for all $x_1, ..., x_{2n-1} \in G$.If this identity holds for all $1 \leqslant i \leqslant j \leqslant n$,then we say that the operation f is associative and (G, f) is called an *n-ary semigroup*. It is clear that an n-ary groupoid is associative if and only if it is $(1, j)$-associative for all $j = 2, .., n$. In the binary case (i.e. n=2)it is usual semigroup.If for all $x_0, x_1, ..., x_n \in G$ and fixed $i \in \{1, ..., n\}$ there exists an element $z \in G$ such that

$$f\left(x_1^{i-1}, z, x_{i+1}^n\right) = x_0 \tag{1}$$

then we say that this equation is *i-solvable* or *solvable at the place i*. If the solution is unique,then we say that (1) is *uniquely i-solvable*. An n-ary groupoid (G, f) uniquely solvable for all $i = 1, ..., n$ is called an *n-ary quasigroup* . An associative n-ary quasigroup is called an *n-ary group* .

Fixing an n-ary operation f, where $n \geqslant 3$, the elements a_2^{n-2} we obtain the new binary operation $x \diamond y = f(x, a_2^{n-2}, y)$. If (G, f) is an n-ary group then $(G, \diamond)$ is a group. Choosing different elements a_2^{n-2} we obtain different groups. All these groups are isomorphic[8]. So, we can consider only group of the form

$$ret_a(G, f) = (G, \circ), \text{ where } x \circ y = f(x, \overset{(n-2)}{a}, y).$$

In this group $e = \overline{a}, x^{-1} = f(\overline{a}, \overset{(n-3)}{a}, \overline{x}, \overline{a})$.

In the theory of n-ary groups, the following Theorem plays an important role.

THEOREM 2.1. *For any n-ary group (G, f) there exist a group $(G, \circ)$, its automorphism φ and an element $b \in G$ such that*

$$f(x_1^n) = x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ ... \circ \varphi^{n-1}(x_n) \circ b \tag{2}$$

holds for all $x_1^n \in G$.

In what follows, G is a non-empty set and (G, f) is an n-ary group unless otherwise specified.

DEFINITION 2.1. *By a t-norm , a function $T : [0, 1] \times [0, 1] \to [0, 1]$satisfying the following conditions is meant:*
(T1) $T(x, 1) = x$;
(T2) $T(x, y) \leqslant T(x, z)$ if $y \leqslant z$;
(T3) $T(x, y) = T(y, x)$;
(T4) $T(x, T(y, z)) = T(T(x, y), z)$;
for all $x, y, z \in [0, 1]$.

DEFINITION 2.2. *By a t-conorm , a function $S : [0, 1] \times [0, 1] \to [0, 1]$satisfying the following conditions is meant:*
(S1) $S(x, 0) = x$;
(S2) $S(x, y) \leqslant S(x, z)$ if $y \leqslant z$;
(S3) $S(x, y) = S(y, x)$;
(S4) $S(x, S(y, z)) = S(S(x, y), z)$;
for all $x, y, z \in [0, 1]$.

Replacing 0 by 1 in condition (S1), we obtain the concept of t-norm T.

DEFINITION 2.3. *Given a t-norm T and a t-conorm S, T and S are dual (with respect to the negation $\prime$) if and only if $(T(x, y))' = S(x', y')$.*

Now we generalize the domain of S to $\prod_{i=1}^{n} [0, 1]$ as follows:

DEFINITION 2.4. *The function $S_n : \prod_{i=1}^{n} [0, 1] \to [0, 1]$ is defined by:*

$$S_n(\alpha_1^n) = S_n(\alpha_1, \alpha_2, ..., \alpha_n) = S(\alpha_i, S_{n-1}(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n)) \tag{3}$$

for all $1 \leqslant i \leqslant n$,$n \geqslant 2$.

For a t-conorm S on $\prod_{i=1}^{n} [0, 1]$, it is denoted by

$$\triangle_t = \{\alpha \in [0, 1] \,|\, S(\alpha, \alpha, ..., \alpha) = \alpha\}.$$

It is clear that every t-conorm has the following property:

$$S(\alpha_1^n) \geqslant \max\{\alpha_1, \alpha_2, ..., \alpha_n\}$$

for all $\alpha_1^n \in [0, 1]$.

3. s-fuzzy n-ary subgroups

Definition 3.1. *A fuzzy set μ in G is called a s-fuzzy n-ary subgroup of (G, f) if the following axioms holds:*
(SFnS1) $(\forall x_1^n \in G), (\mu(f(x_1^n)) \leqslant S\{\mu(x_1), ..., \mu(x_n)\})$,
(SFnS2) $(\forall x \in G), (\mu(\overline{x}) \leqslant \mu(x))$.

Example 3.1. Let $(\mathbb{Z}_4, f)$ be a 4-ary subgroup derived from additive group $\mathbb{Z}_4$. Define a fuzzy subset μ in $\mathbb{Z}_4$ as follows:

$$\mu(x) = \begin{cases} 0.1 & if\ x = 0, \\ 0.7 & if\ x = 1, 2, 3. \end{cases}$$

and let $S : \prod_{i=1}^{4} [0, 1] \longrightarrow [0, 1]$ be a function defined by as follows:

$$S(x_1^4) = \min\{x_1 + x_2 + x_3 + x_4, 1\}$$

for all $x_1^4 \in [0, 1]$ and a function f is defined by

$$f(x_1^4) = x_1 +_4 x_2 +_4 x_3 +_4 x_4, \forall x_1^4 \in \mathbb{Z}_4.$$

By routine calculations, we know that μ is a s-fuzzy 4-ary subgroup of $(\mathbb{Z}_4, f)$.

Theorem 3.1. *If $\{\mu_i | i \in I\}$ is an arbitrary family of s-fuzzy n-ary subgroup of (G, f) then $\bigcup \mu_i$ is s-fuzzy n-ary subgroup of (G, f), where $\bigcup A_i = \bigvee \mu_i$, where $\bigvee \mu_i(x) = sup\{\mu_i(x) | x \in G$ and $i \in I\}$.*

Proof. The proof is trivial. □

Theorem 3.2. *If μ is a fuzzy set in G is a s-fuzzy n-ary subgroup of (G, f), then so is μ', where $\mu' = 1 - \mu$.*

Proof. It is sufficient to show that μ' satisfies conditions (SFnS1) and (SFnS2). Let $x_1^n \in G$. Then

$$\begin{aligned} \mu'(f(x_1^n)) &= 1 - \mu(f(x_1^n)) \\ &\leqslant 1 - S\{\mu(x_1), ..., \mu(x_n)\} \\ &\leqslant S\{1 - \mu(x_1), ..., 1 - \mu(x_n)\} \\ &= S\{\mu'(x_1), ..., \mu'(x_n)\}. \end{aligned}$$

and

$$\mu'(\overline{x}) = 1 - \mu(\overline{x}) \leqslant 1 - \mu(x) = \mu'(x).$$

Hence μ' is a s-fuzzy n-ary subgroup of (G, f). □

The following Lemma gives the relation between T and S.

Lemma 3.1. *Let T be a t-norm. Then the t-conorm S can be defined as*

$$S(x_1^n) = 1 - T(1 - x_1, 1 - x_2, ..., 1 - x_n), \forall x_1^n \in G.$$

Proof. Straightforward. □

The following Theorem gives the relation between t-fuzzy n-ary subgroup and s-fuzzy n-ary subgroup of G.

THEOREM 3.3. *A fuzzy set μ of G is a t-fuzzy n-ary subgroup of (G, f) if and only if its complement μ' is a s-fuzzy n-ary subgroup of (G, f).*

PROOF. Let μ be a t-fuzzy n-ary subgroup of (G, f). For all $x_1^n \in G$, we have

$$\begin{array}{rcl} \mu'(f(x_1^n) & = & 1 - \mu(f(x_1^n)) \\ & \leqslant & 1 - T\{\mu(x_1), \mu(x_2), ..., \mu(x_n)\} \\ & = & 1 - T\{1 - \mu'(x_1), 1 - \mu'(x_2), ..., 1 - \mu'(x_n)\} \\ & = & S\{\mu'(x_1), \mu'(x_2), ..., \mu'(x_n)\}. \end{array}$$

For all $x \in G$, we have

$$\mu'(\overline{x}) = 1 - \mu(\overline{x}) \quad \leqslant \quad 1 - \mu(x) = \mu'(x).$$

The converse is proved similarly. □

DEFINITION 3.2. *Let μ be a fuzzy set in G and let $t \in [0, 1]$. Then the set*

$$L(\mu; t) := \{x \in G | \mu(x) \leqslant t\}$$

is called anti-level subset μ of G.

The following Theorem is a consequence of the Transfer Principle described in [26].

THEOREM 3.4. *A fuzzy set μ in G, is a s-fuzzy n-ary subgroup of (G, f) if and only if the anti-level subset $L(\mu; t)$ of G is an n-ary subgroup of (G, f) for every $t \in [0, 1]$, which is called s-level n-ary subgroup of (G, f).*

PROOF. Let μ be a s-fuzzy n-ary subgroup of (G, f). If $x_1^n \in G$ and $t \in [0, 1]$,then $\mu(x_i) \leqslant t$ for all $i = 1, 2, ..., n$. Thus

$$\mu(f(x_1^n) \leqslant S\{\mu(x_1), ..., \mu(x_n)\} \leqslant t,$$

which implies $f(x_1^n) \in L(\mu; t)$. Moreover, for some $x \in L(\mu; t)$, we have

$$\mu((\overline{x}) \leqslant \mu(x)) \leqslant t,$$

which implies $\overline{x} \in L(\mu; t)$. Thus $L(\mu; t)$ is an n-ary subgroup of (G, f).

Conversely, assume that $L(\mu; t)$ is an n-ary subgroup of (G, f). Let us define

$$t_0 = S\{\mu(x_1), ..., \mu(x_n)\},$$

for some $x_1^n \in G$.Then obviously $x_1^n \in L(\mu; t_0)$, consequently $f(x_1^n) \in L(\mu; t_0)$. Thus

$$\mu(f(x_1^n)) \leqslant t_0 = S\{\mu(x_1), ..., \mu(x_n)\}.$$

Now, let $x \in L(\mu; t)$. Then $\mu(x) = t_0 \leqslant t$. Thus $x \in L(\mu; t_0)$. Since ,by the assumption, $\overline{x} \in L(\mu; t_0)$. Whence $\mu(\overline{x}) \leqslant t_0 = \mu(x)$. This complete the proof. □

Using the above theorem, we can prove the following characterization of s-fuzzy n-ary subgroups.

THEOREM 3.5. *A fuzzy set μ in G, is a s-fuzzy n-ary subgroup of (G, f) if and only if the anti-level subset $L(\mu;t)$ of G is an n-ary subgroup of (G, f) for all $i = 1, 2, ..., n$ and all $x_1^n \in G$, μ satisfies the following conditions:*

(i) $\mu(f(x_1^n) \leqslant S\{\mu(x_1), ..., \mu(x_n)\}$,

(ii) $\mu(x_i) \leqslant S\{\mu(x_1), ..., \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i-1}), ..., \mu(x_n)\}$.

PROOF. Assume that μ is a s-fuzzy n-ary subgroup of (G, f). Similarly as in the proof of Theorem 3.4, we can prove the non-empty level subset $L(\mu;t)$ under the operation f, that is $x_1^n \in L(\mu;t)$ implies $f(x_1^n) \in L(\mu;t)$.

Now let $x_0, x_1^{i-1}, x_{i+1}^n$, where $x_0 = f(x_1^{i-1}, z, x_{i+1}^n)$ for some $i = 1, 2, ..., n$ and $z \in G$ which implies $x_0 \in L(\mu;t)$. Then, according to (ii), we have $\mu(z) \leqslant t$. So, the the equation (1) has a solution $z \in \mu(t)$. This mean that level subset $L(\mu;t)$ is an n-ary subgroups.

Conversely, assume that level subset $L(\mu;t)$ is an n-ary subgroups of (G, f). Then it is easy to prove the condition (i).
For $x_1^n \in G$, we define

$$t_0 = S\{\mu(x_1), ..., \mu(x_{i-1}), \mu(f(x_1^n)), \mu(x_{i-1}), ..., \mu(x_n)\}.$$

Then $x_1^{i-1}, x_{i+1}^n, f(x_1^n) \in L(\mu, t_0)$. Whence, according to the definition of n-ary group, we conclude $x_i \in L(\mu, t_0)$. Thus $\mu(x_i) \leqslant t_0$. This proves the conditions (ii). □

DEFINITION 3.3. *Let (G, f) and (G', f) be an n-ary groups. A mapping $g : G \to G'$ is called an n-ary homomorphism if $g(f(x_1^n)) = f(g^n(x_1^n))$, where $g^n(x_1^n) = (g(x_1), ..., g(x_n))$ for all $x_1^n \in G$.*

For any fuzzy set μ in G', we define the *preimage* of μ under g, denoted by $g^{-1}(\mu)$, is a fuzzy set in G defined by

$$g^{-1}(\mu) = \mu_{g^{-1}}(x) = \mu(g(x)), \forall x \in G.$$

For any fuzzy set μ in G, we define the *image* of μ under g, denoted by $g(\mu)$, is a fuzzy set in G' defined by

$$g(\mu)(y) = \begin{cases} \underset{x \in g^{-1}(y)}{inf} \mu(x), & if \ g^{-1}(y) \neq \phi, \\ 0, & otherwise. \end{cases}$$

for all $x \in G$ and $y \in G'$.

THEOREM 3.6. *Let g be a n-ary homomorphism mapping from G into G' with $g(\overline{x}) = g(x)$ for all $x \in G$ and μ is a s-fuzzy n-ary subgroup of (G', f). Then $g^{-1}(\mu)$ is a s-fuzzy n-ary subgroup of (G, f).*

PROOF. Let $x_1^n \in G$, we have

$$\begin{aligned} \mu_{g^{-1}}(f(x_1^n)) &= \mu(g(f(x_1^n)) = \mu(f(g^n(x_1^n))) \\ &\leqslant S\{\mu(g(x_1), ..., \mu(g(x_n))\} \\ &= S\{\mu_{g^{-1}}(x_1), ..., \mu_{g^{-1}}(x_n)\}. \end{aligned}$$

and

$$\mu_{g^{-1}}(\overline{x}) \quad = \quad \mu(g(\overline{x})) \leqslant \mu(g(x)) = \mu_{g^{-1}(\mu)}(x).$$

This completes the proof. □

If we strengthen the condition of g, then we can construct the converse of Theorem 3.6 as follows.

THEOREM 3.7. *Let g be a n-ary homomorphism from G into G' and $g^{-1}(\mu)$ is a s-fuzzy n-ary subgroup of (G, f). Then μ is a s-fuzzy n-ary subgroup of (G', f).*

PROOF. For any $x_1 \in G'$, there exists $a_1 \in G$ such that $g(a_1) = x_1$ and for any $f(x_1^n) \in (G', f)$, there exists $f(a_1^n) \in (G, f)$ such that $g(f(a_1^n)) = f(x_1^n)$. Then

$$\begin{aligned} \mu(f(x_1^n)) &= \mu(g(f(a_1^n)) = \mu_{g^{-1}}(f(a_1^n)) \\ &\leqslant S\{\mu_{g^{-1}}(a_1), \mu_{g^{-1}}(a_2), ..., \mu_{g^{-1}}(a_n)\} \\ &= S\{\mu(g(a_1), ..., \mu(g(a_n)\} \\ &= S\{\mu(x_1), ..., \mu(x_n)\}. \end{aligned}$$

For any $\overline{x} \in G'$, there exists $\overline{a} \in G$ such that $g(\overline{a}) = \overline{x}$, we have

$$\mu(\overline{x}) \quad = \quad \mu(g(\overline{a})) = \mu_{g^{-1}}(\overline{a}) \leqslant \mu_{g^{-1}}(a) = \mu(a) = \mu(x).$$

This completes the proof. □

THEOREM 3.8. *Let $g : G \longrightarrow G'$ be an onto mapping. If μ is a s-fuzzy n-ary subgroup of (G, f), then $g(\mu)$ is a s-fuzzy n-ary subgroup of (G', f).*

PROOF. Let g be a mapping from G onto G' and let $x_1^n \in G$, $y_1^n \in G'$. Noticing that

$$\{x_i(i = 1, 2, ..., n)|x_i \in g^{-1}(f(y_1^n))\} \subseteq$$
$$\{f(x_1^n) \in G|x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\}.$$

we have

$$\begin{aligned} & g(\mu)(f(y_1^n) \\ & = inf\{\mu(x_1^n)|x_i \in g^{-1}(f(y_1^n))\} \\ & \leqslant inf\{\mu(f(x_1^n)|x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\} \\ & \leqslant inf\{\max\{\mu(x_1), \mu(x_2), ..., \mu(x_n)\}|x_1 \in g^{-1}(y_1), x_2 \in g^{-1}(y_2), ..., x_n \in g^{-1}(y_n))\} \\ & = \max\{inf\{\mu(x_1)|x_1 \in g^{-1}(y_1)\}, inf\{\mu(x_2)|x_1 \in g^{-1}(y_2)\}, ..., inf\{\mu(x_n)|x_1 \in g^{-1}(y_n)\}\} \\ & \leqslant S\{g(\mu)(y_1), g(\mu)(y_2), ..., g(\mu)(y_n)\}. \end{aligned}$$

and

$$g(\mu)(\overline{x}) \quad = \quad inf\{\mu(\overline{x})|\overline{x} \in g^{-1}(f(\overline{y}))\} \leqslant inf\{\mu(x)|x \in g^{-1}(f(y))\} = g(\mu)(x).$$

This completes the proof. □

COROLLARY 3.1. *A fuzzy subset μ defined on group $(G,.)$ is a s-fuzzy subgroup if and only if*
(1) $\mu(xy) \leqslant S\{\mu(x), \mu(y)\}$,
(2) $\mu(x) \leqslant S\{\mu(y), \mu(xy)\}$,
(3) $\mu(y) \leqslant S\{\mu(x), \mu(xy)\}$.
holds for all $x, y \in G$.

THEOREM 3.9. *Let μ be a s-fuzzy subgroup of $(G,.)$. If there exists an element $a \in G$ such that $\mu(a) \leqslant \mu(x)$ for every $x \in G$, then μ is a s-fuzzy subgroup of a group $ret_a(G, f)$.*

PROOF. For all $x, y, a \in G$, let if possible μ is not a s-fuzzy subgroup of a group $ret_a(G, f)$. Then we have $\mu(x \circ y) > S\{\mu(x), \mu(y)\}$. That is

$$\begin{aligned} S\{\mu(x), \mu(x)\} &< \mu(x \circ x) \\ &= \mu(f(x, \overset{(n-2)}{a}, x)) \\ &\leqslant S\{\mu(x), \overset{(n-2)}{\mu}(a), \mu(x)\} \\ S\{\mu(x), \mu(x)\} &< S\{\mu(x), \mu(a)\}. \end{aligned}$$

This holds only if $\mu(a) > \mu(x)$, which is contradiction to our assumption $\mu(a) \leqslant \mu(x)$.

Also, we have μ is a s-fuzzy subgroup of $(G,.)$. Thus $\mu(x^{-1}) \leqslant \mu(x)$ is obvious for all $x \in G$.
which complete the proof. □

In Theorem 3.9, the assumption that $\mu(a) \leqslant \mu(x)$ cannot be omitted.

EXAMPLE 3.2. Let $(\mathbb{Z}_4, f)$ be a 4-ary group from Example 3.1.
Define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.4, & if\ x = 0, \\ 1, & if\ x = 1, 2, 3. \end{cases}$$

Clearly, μ is a s-fuzzy 4-ary subgroup of $(\mathbb{Z}_4, f)$. For $ret_1(\mathbb{Z}_4, f)$, define

$$S(x, y) = \begin{cases} max(x, y) & if\ x = y, \\ min(x + y, 1) & if\, x \neq y. \end{cases}$$

we have

$$\mu(0 \circ 0) = \mu((f(0, 1, 1, 0)) = \mu(2) = 1 \nleq 0.4 = \mu(0) = S\{\mu(0), \mu(0)\}.$$

Hence the assumptions $\mu(a) \leqslant \mu(x)$ cannot be omitted.

THEOREM 3.10. *Let (G, f) be an n-ary group. If μ is a s-fuzzy n-ary subgroup of a group $ret_a(G, f)$ and $\mu(a) \leqslant \mu(x)$ for all $a, x \in G$, then μ is a s-fuzzy n-ary subgroup of (G, f).*

PROOF. According to Theorem 2.1, any n-ary group can be represented of the form (2), where $(G,\circ) = ret_a(G,f), \varphi(x) = f(\overline{a}, x, \overset{(n-2)}{a})$ and $b = f(\overline{a},...,\overline{a})$. Then we have

$$\mu(\varphi(x)) = \mu(f(\overline{a}, x, \overset{(n-2)}{a})) \leqslant S\{\mu(\overline{a}), \mu(x), \mu(a)\} \leqslant \mu(x).$$

$$\mu(\varphi^2(x)) = \mu(f(\overline{a}, \varphi(x), \overset{(n-2)}{x})) \leqslant S\{\mu(\overline{a}), \mu(\varphi(x)), \mu(a)\} \leqslant S\{\mu(\overline{a}), \mu(x), \mu(a)\} \leqslant \mu(x).$$

Consequently, $\mu(\varphi^k(x)) \leqslant\leqslant \mu(x)$. for all $x \in G$ and $k \in \mathbb{N}$.

Similarly, for all $x \in G$ we have

$$\mu(b) \quad = \quad \mu(f(\overline{a},...,\overline{a})) \leqslant \mu(\overline{a}) \leqslant \mu(x).$$

Thus

$$\begin{aligned} \mu(f(x_1^n)) &= \mu(x_1 \circ \varphi(x_2) \circ \varphi^2(x_3) \circ ... \circ \varphi^{n-2}(x_n) \circ b) \\ &\leqslant S\{\mu(x_1), \mu(\varphi(x_2)), \mu(\varphi^2(x_3)), ..., \mu(\varphi^{n-2}(x_n)), \mu(b)\} \\ &\leqslant S\{\mu(x_1), \mu(x_2), \mu(x_3), ..., \mu(x_n), \mu(b)\} \\ &\leqslant S\{\mu(x_1), \mu(x_2), \mu(x_3), ..., \mu(x_n)\}. \end{aligned}$$

From (4) and (7) of [3], we have

$$\overline{x} = \left(\mu(\overline{a} \circ \varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right)^{-1}$$

Thus

$$\begin{aligned} \mu(\overline{x}) &= \mu\left(\left(\overline{a} \circ \varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right)^{-1}\right) \\ &\leqslant \mu\left(\overline{a} \circ \varphi(x) \circ \varphi^2(x) \circ ... \circ \varphi^{n-2}(x) \circ b\right) \\ &\leqslant S\{\mu(\overline{a}, \mu(\varphi(x)), \mu(\varphi^2(x)), ..., \mu(\varphi^{n-2}(x)), \mu(b)\} \\ &\leqslant S\{\mu(x), \mu(b)\} = \mu(x). \end{aligned}$$

This completes the proof. □

COROLLARY 3.2. *If (G,f) is a ternary group, then any s-fuzzy subgroup of $ret_a(G,f)$ is a s-fuzzy ternary subgroup of (G,f).*

PROOF. Since $\overline{a}$ is a neutral element of a group $ret_a(G,f)$ then $\mu(\overline{a}) \leqslant \mu(x)$, for all $x \in G$. Thus $\mu(\overline{a}) \leqslant \mu(a)$. But in ternary group $\overline{\overline{a}} = a$ for any $a \in G$, whence $\mu(a) = \mu(\overline{\overline{a}}) \leqslant \mu(\overline{a}) \leqslant \mu(x)$. So, $\mu(a) = \mu(\overline{\overline{a}}) \leqslant \mu(x)$, for all $x \in G$. This means that the assumption of Theorem 3.10 is satisfied. Hence $ret_a(G,f)$ is a s-fuzzy ternary subgroup of (G,f). This completes the proof. □

EXAMPLE 3.3. Consider the ternary group $(\mathbb{Z}_{12}, f)$, derived from the additive group $\mathbb{Z}_{12}$. Let μ be a s-fuzzy subgroup of the group of $ret_1(G,f)$ induced by subgroups $S_1 = \{11\}$, $S_2 = \{5, 11\}$ and $S_3 = \{1,3,5,7,9,11\}$. Define a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.1 & if \;\; x = 11, \\ 0.3 & if \;\; x = 5, \\ 0.5 & if \;\; x = 1,3,7,9, \\ 0.9 & if \;\; x \notin S_3. \end{cases}$$

Then

$$\mu(\overline{5}) = \mu(7) = 0.5 \quad \nleq \quad 0.3 = \mu(5).$$

Hence μ is not a s-fuzzy ternary subgroup of $(\mathbb{Z}_{12}, f)$.

Observations. From the above Example 3.3 it follows that:

(1) There are s-fuzzy subgroups of $ret_a(G, f)$ which are not s-fuzzy n-ary subgroups of (G, f).

(2) In Theorem 3.10 the assumption $\mu(a) \leqslant \mu(x)$ can not be omitted. In the above example we have $\mu(1) = 0.5 \nleq 0.3 = \mu(5)$.

(3) The assumption $\mu(a) \leqslant \mu(x)$ cannot be replaced by the natural assumption $\mu(\overline{a}) \leqslant \mu(x)$. ($\overline{a}$ is the identity of $ret_a(G, f)$). In the above example $\overline{1} = 11$, then $\mu(11) \leqslant \mu(x)$ for all $x \in \mathbb{Z}_{12}$.

THEOREM 3.11. *Let (G, f) be an n-ary group of b-derived from the group $(G, \circ)$. Any fuzzy set μ of $(G, \circ)$ such that $\mu(b) \leqslant \mu(x)$ for every $x \in G$ is a s-fuzzy n-ary subgroup of (G, f).*

PROOF. The condition (SFnS1) is obvious. To prove (SFnS2), we have n-ary group (G, f) b-derived from the group $(G, \circ)$, which implies

$$\overline{x} = (x^{n-2} \circ b)^{-1},$$

where x^{n-2} is the power of x in $(G, \circ)$[4] .

Thus, for all $x \in G$

$$\mu(\overline{x}) \quad = \quad \mu((x^{n-2} \circ b)^{-1}) \leqslant \mu(x^{n-2} \circ b)^{-1} \leqslant S\{\mu(x^{n-2}), \mu(b)\} = \mu(x).$$

This complete the proof. □

COROLLARY 3.3. *Any s-fuzzy subgroup of a group $(G, \circ)$ is a s-fuzzy n-ary subgroup of an n-ary group (G, f) derived from $(G, \circ)$.*

PROOF. If n-ary group (G, f) is derived from the group $(G, \circ)$ then $b = e$. Thus $\mu(e) \leqslant \mu(x)$ for all $x \in G$. □

4. S-product of s-fuzzy n-ary relations

DEFINITION 4.1. *A fuzzy n-ary relation on any set G is a fuzzy set*

$$\mu : G^n = G \times G \times ... \times G \text{ (n times)} \to [0, 1].$$

DEFINITION 4.2. *Let μ be fuzzy n-ary relation on any set G and ν is a fuzzy set on G. Then μ is called s-fuzzy n-ary relation on ν if*

$$\mu(x_1^n) \leqslant S(\nu(x_1), \nu(x_2), ..., \nu(x_n)),$$

for all $x_1^n \in G$.

DEFINITION 4.3. *Let $\mu_1^n = \mu_1, \mu_2, ..., \mu_n$ be a fuzzy sets in G. Then direct S-product of μ_1^n is defined by*

$$(\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1^n) = S(\mu_1(x_1), \mu_2(x_2), ..., \mu_n(x_n)), \forall x_1^n \in G.$$

LEMMA 4.1. *Let S be a function induced by t-conorm and let μ_1^n be a fuzzy sets in G. Then*
(i) $\mu_1 \times_S \mu_2 \times_S ... \times_S \mu_n$ is a s-fuzzy n-ary relation on G,
(ii) $L(\mu_1 \times \mu_2 \times ... \times \mu_n; t) = L(\mu_1; t) \times L(\mu_2; t) \times ... \times L(\mu_n; t), \forall t \in [0, 1]$.

PROOF. The proof is obvious. □

DEFINITION 4.4. *Let S be a function induced by t-conorm. If ν is a fuzzy set in G, the strongest s-fuzzy n-ary relation on G that is a s-fuzzy n-ary relation on ν is μ_ν, given by*

$$\mu_\nu(x_1^n) = S(\nu(x_1), \nu(x_2), ..., \nu(x_n)), \forall x_1^n \in G.$$

LEMMA 4.2. *For a given fuzzy set ν in G, let μ be the strongest s-fuzzy n-ary relation of G. Then for $t \in [0, 1]$, $L(\mu_\nu; t) = L(\nu; t) \times L(\nu; t) \times ... \times L(\nu; t)$.*

PROOF. The proof is obvious. □

PROPOSITION 4.1. *Let S be a function induced by t-conorm and let $\mu_1, \mu_2, ..., \mu_n$ be s-fuzzy n-ary subgroup of (G, f). Then, $\mu_1 \times \mu_2 \times ... \times \mu_n$ is a s-fuzzy n-ary subgroup of (G^n, f).*

PROOF. For $x_1^n \in G$ and $f(x_1^n) = (f_1(x_1^n), ..., f_n(x_1^n)) \in (G^n, f)$,we have

$$\begin{aligned}
&(\mu_1 \times \mu_2 \times, ..., \times \mu_n)(f(x_1^n)) \\
&= (\mu_1 \times \mu_2 \times, ..., \times \mu_n)(f_1(x_1^n), ..., f_n(x_1^n)) \\
&= S\{\mu_1(f(x_1^n)), \mu_2(f(x_1^n)) ..., \mu_n(f(x_1^n))\} \\
&\leqslant S\{S\{\mu_1(x_1), \mu_1(x_2), ..., \mu_1(x_n)\}, ..., S\{\mu_n(x_1), \mu_n(x_2), ..., \mu_n(x_n)\}\} \\
&= S\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1, ..., x_1), ..., (\mu_1 \times \mu_2 \times ... \times \mu_n)(x_n, ..., x_n)\} \\
&= S\{(\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1), ..., (\mu_1 \times \mu_2 \times ... \times \mu_n)(x_n)\}.
\end{aligned}$$

and for all $x = x_1^n, \overline{x} = \overline{x}_1^n \in G^n$, we have

$$\begin{aligned}
(\mu_1 \times \mu_2 \times, ..., \times \mu_n)(\overline{x}) &= (\mu_1 \times \mu_2 \times, ..., \times \mu_n)(\overline{x}_1, ..., \overline{x}_n) \\
&= S\{\mu_1(\overline{x}_1), ..., \mu_n(\overline{x}_n)\} \\
&\leqslant S\{(\mu_1(x_1), ..., \mu_n(x_n))\} \\
&= (\mu_1 \times \mu_2 \times ... \times \mu_n)(x_1^n) \\
&= (\mu_1 \times \mu_2 \times ... \times \mu_n)(x).
\end{aligned}$$

This completes the proof. □

The following Corollary is the immediate consequence of Proposition 4.1.

COROLLARY 4.1. *Let S be a function induced by t-conorm and let $\prod_{i=1}^{n} (G_i, f)$ be the finite collection of n-ary subgroups and $G = \prod_{i=1}^{n} G_i$ the S-product of G_i. Let*

μ_i be a s-fuzzy n-ary subgroup of (G_i, f), where $1 \leqslant i \leqslant n$. Then,$\mu = \prod_{i=1}^{n} \mu_i$ defined by

$$\mu(x_1^n) = \prod_{i=1}^{n} \mu_i(x_1^n) = S(\mu(x_1), \mu(x_2), ..., \mu(x_n)).$$

Then μ is a s-fuzzy n-ary subgroup of (G, f). □

Definition 4.5. *Let μ_1^n be fuzzy sets in G. Then, the S-product of μ_1^n, written as $[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_S$, is defined by:*

$$[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_S(x) = S(\mu_1(x), \mu_2(x), ..., \mu_n(x)) \ \forall x \in G.$$

Theorem 4.1. *Let μ_1^n be s-fuzzy n-ary subgroup of (G, f). If S^* is a function induced by t-conorm dominates S, that is,*

$$S^*(S(x_1^n), S(y_1^n), ..., S(z_1^n)) \leqslant S(S^*(x_1, y_1, ..., z_1), ..., S^*(x_n, y_n, ..., z_n))$$

for all $x_1^n, y_1^n, ..., z_1^n \in [0, 1]$. Then S^*-product of μ_1^n, $[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*_n}$, is a s-fuzzy n-ary subgroup of (G, f).

Proof. Let $x_1^n \in G$, we have

$$\begin{aligned}
&[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}(f(x_1^n)) \\
&= S^*(\mu_1(f(x_1^n)), \mu_2(f(x_1^n)), ..., \mu_n(f(x_1^n))) \\
&\leqslant S^*(S(\mu_1(x_1), \mu_1(x_2), ..., \mu_1(x_n)), ..., S(\mu_n(x_1), \mu_n(x_2), ..., \mu_n(x_n)) \\
&\leqslant S(S^*(\mu_1(x_1), \mu_2(x_1), ..., \mu_n(x_1)), ..., S^*(\mu_1(x_n), \mu_2(x_n), ..., \mu_n(x_n)) \\
&= S([\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}(x_1), ..., [\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}(x_n))
\end{aligned}$$

and for all $x \in G$, we have

$$\begin{aligned}
[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}(\overline{x}) &= S^*(\mu_1(\overline{x}), \mu_2(\overline{x}), ..., \mu_n(\overline{x})) \\
&\leqslant S^*(\mu_1(x), \mu_2(x), ..., \mu_n(x)) \\
&= [\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}(x).
\end{aligned}$$

Hence, $[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}$ is a s-fuzzy n-ary subgroup of (G, f). This completes the proof. □

Let (G, f) and (G', f) be an n-ary groups. A mapping $g : G \to G'$ is an onto homomorphism. Let S and S^* be functions induced by t-conorm such that S^* dominates S. If μ_1^n are s-fuzzy n-ary subgroup of (G, f), then the S^*-product of $\mu_1^n, [\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}$ is a s-fuzzy n-ary subgroup. Since every onto homomorphic inverse image of a s-fuzzy n-ary subgroup, the inverse images $g^{-1}(\mu_1), g^{-1}(\mu_2), ..., g^{-1}(\mu_n)$ and $g^{-1}([\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*})$ are s-fuzzy n-ary subgroup (G, f).

The following theorem provides the relation between $g^{-1}([\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*})$ and S^*-product $([g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot ... \cdot g^{-1}(\mu_n)]_{S^*})$ of $g^{-1}(\mu_1), g^{-1}(\mu_2)$ and $g^{-1}(\mu_n)$.

THEOREM 4.2. *Let (G, f) and (G', f) be an n-ary groups. A mapping $g : G \to G'$ is an onto n-ary homomorphism . Let S^* be a function induced by t-conorm such that S^* dominates S. Let μ_1^n be s-fuzzy n-ary subgroups of (G, f). If $[\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}$ and is the S^*- product of μ_1^n, and $([g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot ... \cdot g^{-1}(\mu_n)]_{S^*})$ is the S^*-product of $g^{-1}(\mu_1), g^{-1}(\mu_2), ...g^{-1}(\mu_n)$ then*

$$g^{-1}([\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*}) = [g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot ... \cdot g^{-1}(\mu_n)]_{S^*}.$$

PROOF. Let $x \in G$, we have

$$\begin{aligned} g^{-1}([\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*})(x) &= ([\mu_1 \cdot \mu_2 \cdot ... \cdot \mu_n]_{S^*})(g(x)) \\ &= S^*(\mu_1(g(x)) \cdot \mu_2(g(x)) \cdot ... \cdot \mu_n(g(x))) \\ &= S^*(g^{-1}(\mu_1)(x) \cdot g^{-1}(\mu_2)(x) \cdot ... \cdot g^{-1}(\mu_n)(x)) \\ &= [g^{-1}(\mu_1) \cdot g^{-1}(\mu_2) \cdot ... \cdot g^{-1}(\mu_n)]_{S^*}. \end{aligned}$$

This completes the proof. □

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Department of Information Technology, College of Applied Sciences, Ministry of Higher Education, PB:135, Sohar-311, Sultanate of Oman.
E-mail address: princeshree1@gmail.com