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A NUMERICAL METHOD FOR THE SOLUTION OF GENERAL THIRD ORDER BOUNDARY VALUE PROBLEM IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this article we have considered general third order boundary value problems and proposed an efficient difference method for numerical solution of the problems. We have shown under appropriate conditions that proposed method is convergent and second order accurate. The numerical results in experiment on test problems show the simplicity and efficiency of the method.

1. Introduction

The occurrences of differential equations are of common in modeling and studies of physical phenomena in natural and applied sciences.Third order BVPs arise in the study of aeroelasticity, sandwich beam analysis and beam deflection theory, electromagnetic waves, theory of thin film flow and incompressible flows are some specific subject in natural and applied science.

In this article we consider a direct method for the numerical solution of the third order boundary value problems of the following form

$$u'''(x) = f(x, u, u', u''), \quad a \leqslant x \leqslant b, \tag{1.1}$$

subject to the boundary conditions

$$u(a) = \alpha, \quad u'(a) = \beta \quad and \qquad u'(b) = \gamma$$

where α, β and γ are real constant.

The theoretical concepts of existence, uniqueness and convergence of the solution of problem (1.1) can be found in the literature [**1, 2, 3, 4, 5, 6**]. The specific

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assumption further to ensure existence and uniqueness of the solution problem (1.1) will not be considered. Thus the existence and uniqueness of the solution to problem (1.1) is assumed. Further we assume that problem (1.1) is well pose. In general analytical solution of model problems of this class is not available, we depend on numerical solution of these BVPs. The emphasis in this article will be on the development of an efficient numerical method to deal with approximate numerical solution of the third order boundary value problem.

Some efficient and accurate numerical methods for solving higher order boundary value problems are available in literature. Some researchers have studied and solved third order boundary value problems with different boundary conditions using different methods for instance some literary work in Finite Difference Method [**7**], Quintic Splines [**8**], Quartic splines[**9**] Non polynomial spline [**10**], Quartic B-splines [**11**], Haar wavelets method[**12**], Collocation quantic spline [**13**], Reproducing Kernel Method [**14**] and references therein can be found. With advent of computers it gained important to develop more accurate numerical methods to solve higher order boundary value problems. Hence, the purpose of this article is to develop an efficient numerical method for solution of third order boundary value problems (1.1).

We have presented our work in this article as follows. In the next section we proposed a finite difference method. We have discussed derivation and convergence of the proposed method under appropriate condition in Section 3 and Section 4 respectively. The application of the proposed method on the test problems and illustrative numerical results so produced to show the efficiency in Section 5. Discussion and conclusion on the performance of the proposed method are presented in Section 6.

2. The Difference Method 1

We define N finite numbers of nodal points of the domain $[a,b]$, in which the solution of the problem (1.1) is desired, as $a \leqslant x_0 < x_1 < x_2 < ...... < x_N = b$ using uniform step length h such that $x_i = a + ih, \quad i = 0, 1, 2, ..., N$. Suppose that we wish to determine the numerical approximation of the theoretical solution $u(x)$ of the problem (1.1) at the nodal point $x_i, \quad i = 1, 2, ..., N$. We denote the numerical approximation of $u(x)$ at node $x = x_i$ as u_i . Let us denote f_i as the approximation of the theoretical value of the source function $f(x, u(x), u'(x), u''(x))$ at node $x = x_i, \quad i = 0, 1, 2, ..., N$. Thus the boundary value problem (1.1) at node $x = x_i$ may be written as

$$u_i''' = f_i \quad , \quad a < x_i < b, \tag{2.1}$$

subject to the boundary conditions

$$u_0 = \alpha, \quad u_0' = \beta \quad \text{and} \qquad u_N' = \gamma.$$

Let we define nodes $x_{i\pm\frac{1}{2}} = x_i \pm \frac{h}{2}$, $\quad i = 1, 2, ..., N-1$ and denote $u_{i\pm\frac{1}{2}}$ the solution of the problem (1.1) at these nodes. Further let us define following approximations,

$$\overline{u}'_{i-\frac{1}{2}} = \begin{cases} \frac{4(u_{i-\frac{1}{2}} - u_{i-1}) - hu'_{i-1}}{h}, & i = 1 \\ \frac{u_{i+\frac{1}{2}} - u_{i-\frac{3}{2}}}{2h}, & i = 2, 3 < i < N \\ \frac{3u_{i-\frac{1}{2}} - 4u_{i-\frac{3}{2}} + u_{i-\frac{5}{2}}}{2h}, & i = 3 \\ \frac{u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}} + 2hu'_i}{2h}, & i = N \end{cases} \tag{2.2}$$

$$\overline{u}''_{i-\frac{1}{2}} = \begin{cases} \frac{h(u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) - u'_{i-1}}{h^2}, & i = 1 \\ \frac{u_{i+\frac{1}{2}} - 2u_{i-\frac{1}{2}} + u_{i-\frac{3}{2}}}{h^2}, & i = 2, 3 < i < N \\ \frac{-88u_{i-1} + 5u_{i-\frac{1}{2}} - 6u_{i-\frac{3}{2}} + 40u_{i-\frac{5}{2}}}{8h^2}, & i = 3 \\ \frac{26u_{i-\frac{3}{2}} - 25u_{i-\frac{1}{2}} - u_{i-\frac{5}{2}} + 24hu'_i}{23h^2}, & i = N \end{cases} \tag{2.3}$$

and

$$\begin{aligned} \overline{f}_{i+\frac{1}{2}} &= f(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}, \overline{u}'_{i+\frac{1}{2}}, \overline{u}''_{i+\frac{1}{2}}), \quad i = 1, 2, ..., N-1, \\ \overline{f}_{i-\frac{1}{2}} &= f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}, \overline{u}'_{i-\frac{1}{2}}, \overline{u}''_{i-\frac{1}{2}}), \quad i = 1, 2, ..., N \end{aligned} \tag{2.4}$$

Following the idea in [**15**] and using approximations (2.2), (2.3) and (2.4), we discretize problem (2.1) in $[a, b]$ at nodes $x_{i-\frac{1}{2}}$ as,

$$\begin{aligned} 9u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= 8u_{i-1} + 3hu'_{i-1} - \frac{3h^3}{8}\overline{f}_{i-\frac{1}{2}} + t_i, \qquad i = 1 \\ -15u_{i-\frac{3}{2}} + 10u_{i-\frac{1}{2}} - 3u_{i+\frac{1}{2}} &= -8u_{i-2} - \frac{h^3}{48}(165\overline{f}_{i-\frac{1}{2}} - 45\overline{f}_{i+\frac{1}{2}}) + t_i, \quad i = 2 \\ u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= -\frac{h^3}{2}(\overline{f}_{i-\frac{3}{2}} + \overline{f}_{i-\frac{1}{2}}) + t_i, \quad 3 \leqslant i \leqslant N-1 \\ u_{i-\frac{5}{2}} - 3u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} &= hu'_i + \frac{h^3}{48}(-25\overline{f}_{i-\frac{3}{2}} + 21\overline{f}_{i-\frac{1}{2}}) + t_i, \quad i = N \end{aligned} \tag{2.5}$$

where $t_i, i = 1, 2, ..., N$ is truncation error. We have not used any special discretization technique for boundary conditions and we have used boundary conditions in our method in a natural way.

After neglecting the t_i in (2.5), at nodal points $x_{i-\frac{1}{2}}, i = 1, 2, ..., N$, we will obtain the $N \times N$ linear or nonlinear system of equations in unknown $u_{i-\frac{1}{2}}$ depends on the source function $f(x, u, u', u'')$. We have to solve a system of equations by an appropriate method. We have applied either Gauss Seidel or Newton-Raphson iterative method to solve above system of equations (2.5) respectively for linear and nonlinear system of equations.

We compute numerical value of $u_i, i = 1, 2, ..., N$ by using following second order approximation,

$$u_i = \begin{cases} \frac{1}{2}(u_{i-\frac{1}{2}} + u_{i+\frac{1}{2}}), & i = 1, 2, ..., N-1 \\ u_{i-\frac{1}{2}} + \frac{1}{2}hu'_i, & i = N \end{cases} \tag{2.6}$$

3. The Difference Method 2

In this section we derive proposed finite difference method (2.5). From approximations (2.2), we find that $\overline{u}'_{i-\frac{1}{2}}$ will provide $O(h^2)$ approximation for $u'_{i-\frac{1}{2}}$ i.e.

$$\overline{u}'_{i-\frac{1}{2}} = u'_{i-\frac{1}{2}} + O(h^2). \tag{3.1}$$

From approximations (2.3),we find that $\overline{u}''_{i-\frac{1}{2}}$ will provide $O(h^2)$ approximation for $u''_{i-\frac{1}{2}}$ i.e.

$$\overline{u}''_{i-\frac{1}{2}} = u''_{i-\frac{1}{2}} + O(h^2). \tag{3.2}$$

Using (3.1) and (3.2) in (2.4), after linearization of $f_{i+\frac{1}{2}}$, we find that $\overline{f}_{i+\frac{1}{2}}$ will provide $O(h^2)$ approximation for $f_{i+\frac{1}{2}}$ i.e.

$$\overline{f}_{i-\frac{1}{2}} = f_{i-\frac{1}{2}} + O(h^2). \tag{3.3}$$

Thus from (3.3), we can conclude that method (2.5) is of $O(h^2)$ discretization for the problem (1.1) at nodes $x_{i-\frac{1}{2}}, \quad i = 1, 2, ..., N$.

4. Convergence Analysis

We will consider following linear test equation for convergence analysis of the proposed method (2.5).

$$u'''(x) = f(x, u(x), u'(x), u''(x)), \qquad a \leqslant x \leqslant b. \tag{4.1}$$

subject to the boundary conditions $u_0 = \alpha, \quad u'_0 = \beta$ and $u'_N = \gamma$. We can write the proposed method (2.5) for exact solution $\mathbf{U}$ in the matrix form as,

$$\mathbf{DU} = \mathbf{a(U)} + \mathbf{t} \tag{4.2}$$

where $\mathbf{t}$ is truncation error matrix of which each element is of $O(h^2)$. The terms in (4.2) are respectively

$$\mathbf{D} = \begin{pmatrix} 9 & -1 & & & & & 0 \\ -15 & 10 & -3 & & & & \\ 1 & -3 & 3 & -1 & & & \\ & 1 & -3 & 3 & -1 & & \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ & & & 1 & -3 & 3 & -1 \\ 0 & & & & 1 & -3 & 2 \end{pmatrix}_{N \times N},$$

$\mathbf{U} = (U_{i-\frac{1}{2}}), \quad \mathbf{a} = (a_i),$

$$a_i = \begin{cases} 8\alpha + 3h\beta - \frac{3h^3}{8}\overline{f}_{i-\frac{1}{2}}, & i = 1 \\ -8\alpha - \frac{5h^3}{16}(11\overline{f}_{i-\frac{1}{2}} - 3\overline{f}_{i+\frac{1}{2}}), & i = 2 \\ -\frac{h^3}{2}(\overline{f}_{i-\frac{3}{2}} + \overline{f}_{i-\frac{1}{2}}), & 3 \leqslant i \leqslant N-1 \\ h\gamma + \frac{h^3}{48}(-25\overline{f}_{i-\frac{3}{2}} + 21\overline{f}_{i-\frac{1}{2}}), & i = N \end{cases}$$

and $\mathbf{t} = (t_i)$,

$$t_i = \begin{cases} -\frac{27h^5}{1920}u^{(5)}_{i-\frac{1}{2}}, & i = 1 \\ -\frac{7h^5}{8}u^{(5)}_{i-\frac{1}{2}}, & i = 2 \\ o(h^6), & 3 \leqslant i \leqslant N-1 \\ \frac{31h^5}{1920}u^{(5)}_{i-\frac{1}{2}}, & i = N \end{cases}$$

are N-dimensional column vectors. After neglecting the terms t_i in (4.2), we will obtain a system of equations in $u_{i-\frac{1}{2}}$. We can write system of equations in the matrix form ,

$$\mathbf{Du} = \mathbf{a(u)} \tag{4.3}$$

where $\mathbf{u} = (u_{i-\frac{1}{2}})$ is N-dimensional column vector of approximate solution of system of equations obtained from (4.2). Let us define

$$\overline{F}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, U_{i-\frac{1}{2}}, \overline{U}_{xi-\frac{1}{2}}, \overline{U}_{xxi-\frac{1}{2}})$$

and

$$\overline{f}_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}, \overline{u}_{xi-\frac{1}{2}}, \overline{u}_{xxi-\frac{1}{2}})$$

After linearization of $\overline{f}_{i-\frac{1}{2}}$ we have,

$$\overline{f}_{i-\frac{1}{2}} - \overline{F}_{i-\frac{1}{2}} = (u_{i-\frac{1}{2}} - U_{i-\frac{1}{2}})G_{i-\frac{1}{2}} + (\overline{u}_{xi-\frac{1}{2}} - \overline{U}_{xi-\frac{1}{2}})H_{i-\frac{1}{2}} + (\overline{u}_{xxi-\frac{1}{2}} - \overline{U}_{xxi-\frac{1}{2}})I_{i-\frac{1}{2}}. \tag{4.4}$$

where $G = \frac{\partial f}{\partial U}$, $H = \frac{\partial f}{\partial \overline{U}_x}$ and $I = \frac{\partial f}{\partial \overline{U}_{xx}}$.
Let us define

$$e_{i-\frac{1}{2}} = u_{i-\frac{1}{2}} - U_{i-\frac{1}{2}} \tag{4.5}$$

Application of approximations (2.2),(2.3) in (4.4), a Taylor series expansions of $G_{i-\frac{1}{2}}, H_{i-\frac{1}{2}}$ and $I_{i-\frac{1}{2}}$ at mesh point $x_{i-\frac{1}{2}}$ and (4.5), we have get the matrix equation from (4.2) and (4.3),

$$\mathbf{a(U)} - \mathbf{a(u)} = \mathbf{Pe} \tag{4.6}$$

where $\mathbf{P} = (-P_{l,m})_{N\times N}$ is the Toeplitz matrix defined as,

$$(P_{l,l-2}) = \begin{cases} \frac{h^3}{2}(-\frac{1}{2h}H_{i-\frac{1}{2}} + \frac{1}{h^2}I_{i-\frac{1}{2}} + \frac{1}{2}H'_{i-\frac{1}{2}} - \frac{1}{h}I'_{i-\frac{1}{2}}), & 3 \leqslant l = i \leqslant N-1 \\ \frac{h^3}{48}(-\frac{25}{2h}H_{i-\frac{1}{2}} + \frac{596}{23h^2}I_{i-\frac{1}{2}} + \frac{25}{2}H'_{i-\frac{1}{2}} - \frac{25}{h}I'_{i-\frac{1}{2}}), & l = i = N \end{cases}$$

$$(P_{l,l-1}) = \begin{cases} \frac{h^3}{48}(-\frac{105}{h}H_{i-\frac{1}{2}} - \frac{60}{h^2}I_{i-\frac{1}{2}} - \frac{45}{2}H'_{i-\frac{1}{2}} - \frac{225}{h}I'_{i-\frac{1}{2}}), & l = i = 2 \\ \frac{h^3}{2}(G_{i-\frac{1}{2}} - \frac{1}{2h}H_{i-\frac{1}{2}} - \frac{1}{h^2}I_{i-\frac{1}{2}} - hG'_{i-\frac{1}{2}} + \frac{2}{h}I'_{i-\frac{1}{2}}), & 3 \leqslant l = i \leqslant N-1 \\ \frac{h^3}{48}(25G_{i-\frac{1}{2}} + \frac{21}{2h}H_{i-\frac{1}{2}} - \frac{2116}{23h^2}I_{i-\frac{1}{2}} - 25hG'_{i-\frac{1}{2}} + \frac{50}{h}I'_{i-\frac{1}{2}}), & l = i = N \end{cases}$$

$$(P_{l,l}) = \begin{cases} \frac{3h^3}{8}(1 - \frac{1}{h^2}I_{i-\frac{1}{2}}), & l = i = 1 \\ \frac{h^3}{48}(165G_{i-\frac{1}{2}} + \frac{90}{h}H_{i-\frac{1}{2}} - \frac{60}{h^2}I_{i-\frac{1}{2}} + 90H'_{i-\frac{1}{2}} + \frac{270}{h}I'_{i-\frac{1}{2}}), & l = i = 2 \\ \frac{h^3}{2}(G_{i-\frac{1}{2}} + \frac{1}{2h}H_{i-\frac{1}{2}} - \frac{1}{h^2}I_{i-\frac{1}{2}} - \frac{1}{2}H'_{i-\frac{1}{2}} - \frac{1}{h}I'_{i-\frac{1}{2}}), & 3 \leqslant l = i \leqslant N-1 \\ \frac{h^3}{48}(-21G_{i-\frac{1}{2}} + \frac{2}{h}H_{i-\frac{1}{2}} + \frac{600}{23h^2}I_{i-\frac{1}{2}} - \frac{25}{2}H'_{i-\frac{1}{2}} - \frac{25}{h}I'_{i-\frac{1}{2}}), & l = i = N \end{cases}$$

and

$$(P_{l,l+1}) = \begin{cases} \frac{3h^3}{8}(-\frac{4}{h}H_{i-\frac{1}{2}} + \frac{1}{h^2}I_{i-\frac{1}{2}}), & l = i = 1 \\ \frac{h^3}{48}(-45G_{i-\frac{1}{2}} + \frac{15}{h}H_{i-\frac{1}{2}} + \frac{1095}{8h^2}I_{i-\frac{1}{2}} & \\ -45hG'_{i-\frac{1}{2}} - \frac{135}{2}H'_{i-\frac{1}{2}} - \frac{225}{8h}I'_{i-\frac{1}{2}}), & l = i = 2 \\ \frac{h^2}{4}H_{i-\frac{1}{2}}, & 3 \leqslant l = i \leqslant N-1 \end{cases}$$

where $u_{i-\frac{1}{2}}$ is an approximate value of $U_{i-\frac{1}{2}}$, $i = 1, 2, ..., N$. Let there are no roundoff errors in solution of difference equation (2.5), so from (4.2),(4.3) and (4.6) we can write an error equation

$$(\mathbf{D} + \mathbf{P})\mathbf{e} = \mathbf{t} \tag{4.7}$$

where $\mathbf{e} = (e_{i-\frac{1}{2}})$, $i = 1, 2, ..., N$ is N-dimnsional column vector. Define a sets

$$G_0 = \{G_{i-\frac{1}{2}}, \quad i = 1, 2, ..., N\}$$

$$H_0 = \{H_{i-\frac{1}{2}}, \quad i = 1, 2, ..., N\}$$

$$I_0 = \{I_{i-\frac{1}{2}}, \quad i = 1, 2, ..., N\}$$

$$D_x = \{H_{xi-\frac{1}{2}}, \quad I_{xi-\frac{1}{2}}, \quad i = 1, 2, ..., N\}$$

Let

$$G_* = \min_{x \in (a,b)} \frac{\partial f}{\partial U}, \quad G^* = \max_{x \in (a,b)} \frac{\partial f}{\partial U}$$

Then

$$0 < G_* \leqslant t \leqslant G^* \quad ,\forall \quad t \in G_0.$$

Let us assume that

$$0 < |\theta| < q_k, \quad q_k > 0, \quad k = 1(1)3 \quad ,\forall \quad \theta \in H_0, I_0, D_x.$$

So, we can assume that $(\mathbf{D} + \mathbf{P}) \geqslant \mathbf{D}$ for small h. Let $\mathbf{K} = (k_{ij})$ be the explicit inverse as defined in [**16, 17, 18**] of nonsymmetric Toeplitz matrix $\mathbf{D}$,

$$k_{ij} = \begin{cases} \frac{(2i-1)(4N-2i+1)}{24N}, & 1 \leqslant i \leqslant N, \quad j = 1 \\ (2i-1)^2 c_1, & i \leqslant j \leqslant N, \quad 2 \leqslant j \\ \frac{(N-i)(N-i+1)}{2}c_2 - \frac{(N-i+2)(N-i-1)}{2}c_3, & j+1 \leqslant i \leqslant N-1 \end{cases} \tag{4.8}$$

where

$$c_1 = \begin{cases} \frac{(N+1-j)(2j+1)^2}{40N(8j-(2j-5)^2)}, & j=2 \\ \frac{N+1-j}{8N}, & 2<j\leqslant N \end{cases} \tag{4.9}$$

$$c_2 = \begin{cases} \frac{(N+1-j)(4N(j-1)+1)-8(j-1)}{24N}, & j=2 \\ \frac{(N+1-j)(4N(j-1)+1)-8(j-1)}{8N}, & j+1\leqslant i<N \end{cases} \tag{4.10}$$

$$c_3 = \begin{cases} \frac{(N+1-j)(4N(j-1)+1)}{24N}, & j=2 \\ \frac{(N+1-j)(4N(j-1)+1)}{8N}, & j+1\leqslant i<N \end{cases} \tag{4.11}$$

From (4.9), (4.10) and (4.11), we can prove that $\mathbf{K}$ is nonsymmetric and positive matrix. So $(\mathbf{D}+\mathbf{P})^{-1} \leqslant \mathbf{K}$. Let matrix $\mathbf{R} = (R_{i1})_{N\times 1}$, denotes the matrix of the row sum of the matrix $\mathbf{K} = (k_{ij})_{N\times N}$ where,

$$R_{i1} = \sum_{j=1}^{N} k_{ij} \tag{4.12}$$

Hence we have obtained

$$\|\mathbf{K}\| = \max_{1\leqslant i\leqslant N} |R_{i1}| = \frac{4N^4-(9N^2-9N-2)}{48N} \tag{4.13}$$

Thus for large N, from (4.13) we conclude that

$$\|\mathbf{K}\| \leqslant \frac{(b-a)^3}{12h^3} \tag{4.14}$$

Let

$$M = \max_{x\in[a,b]} \left| u^{(5)}(x) \right|, \tag{4.15}$$

Then from (4.7), (4.14) and (4.15) we have

$$\|\mathbf{e}\| \leqslant \frac{7h^2(b-a)^3}{96} M \tag{4.16}$$

From equation (4.16) it follows that $\|\mathbf{e}\| \to 0$ as $h \to 0$. This establishes the convergence of the method (2.5) and the order of convergence of method (2.5) is at least $O(h^2)$.

5. Numerical Results

To illustrate our method and demonstrate its computational efficiency, we have considered four model problems. In each model problem, we took uniform step size h. In Table 1 - Table 4, we have shown MAU the maximum absolute error in the solution u of the problems (1.1) for different values of N. We have used the following formula in computation of MAU ,

$$MAU = \max_{1\leqslant i\leqslant N} |u(x_i) - u_i|.$$

We have used Gauss Seidel and Newton-Raphson iteration method to solve respectively for linear and nonlinear system of equations arised from equation (2.5). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. The solutions are computed on N nodes and iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iteration reached 10^3.

Problem 1. The nonlinear model problem given by

$$u'''(x) = x^2(u''(x) - u'(x)) + y^2(x) + f(x), \quad 0 \leqslant x \leqslant 1$$

subject to the boundary conditions

$$u(0) = 0 \quad , \quad u'(0) = -1 \qquad \text{and} \quad u'(1) = \sin(1)$$

where $f(x)$ is calculated so that the analytical solution of the problem is $u(x) = (x-1)\sin(x)$. The MAU computed by method (2.5) and no. of iterations $Iter.$ required for different values of N are presented in Table 1.

Problem 2. The linear model problem [**19**] given by,

$$u'''(x) = -xu''(x) - 6x^2 + 3x - 6, \quad 0 \leqslant x \leqslant 1$$

subject to the boundary conditions

$$u(0) = 0 \quad , \quad u'(0) = 0 \qquad \text{and} \quad u'(1) = 0$$

The analytical solution of the problem is $u(x) = x^2(\frac{3}{2} - x)$. The MAU computed by method (2.5) and no. of iterations $Iter.$ required for different values of N are presented in Table 2.

Problem 3. The linear model problem [**14**] given by,

$$u'''(x) = K^2u'(x) - r, \quad 0 \leqslant x \leqslant 1$$

subject to the boundary conditions

$$u(0) = \frac{r(-K + 2\tanh(\frac{K}{2}))}{2K^3} \quad , \quad u'(0) = 0 \qquad \text{and} \quad u'(1) = 0$$

The analytical solution of the problem is

$$u(x) = \frac{r(K(2x-1) - 2\sinh(Kx) + 2\cosh(Kx)\tanh(\frac{K}{2}))}{2K^3}$$

and MAU computed by method (2.5) and no. of iterations $Iter.$ required for different values of N, $K = 5, 10$ and $r = 1$ are presented in Table 3 and Table 4.

Table 1. Maximum absolute error (Problem 1).

	N				
	32	64	128	256	512
MAU	.79360791(-3)	.64820051(-4)	.16197562(-4)	.40680170(-5)	.11026859(-5)
$Iter.$	4533	1666	352	118	33

TABLE 2. Maximum absolute error (Problem 2).

	N				
	128	256	512	1024	2048
MAU	.30696392(-4)	.61094761(-5)	.14379621(-5)	.41723251(-6)	.16298145(-6)
$Iter.$	53	5	3	4	5

TABLE 3. Maximum absolute error (Problem 3).

	r=1, K=5.0, N				
	64	128	256	512	1024
MAU	.36522746(-4)	.43436885(-5)	.51409006(-6)	.99651515(-7)	.25145710(-7)
$Iter.$	2475	441	38	3	5

TABLE 4. Maximum absolute error (Problem 3).

	r=1, K=10.0, N				
	64	128	256	512	1024
MAU	.17987564(-4)	.40354207(-5)	.40931627(-6)	.55879354(-7)	.13038516(-7)
$Iter.$	2222	1107	127	6	6

We have described a numerical method for numerical solution of third order boundary value problem and three model problems considered to test the performance of the proposed method. Numerical results for example 1 for different values of N which is presented in table 1, maximum absolute errors in solution decreases with decrease in step size h. The order of accuracy in the result is appreciable. Same observation can be drawn from the results for the other model problems. It is evident that method (2.5) is convergent and the rate of convergence is at least quadratic.

6. Conclusion

A finite difference method to find the numerical solution of third order boundary value problems has been developed. At nodal point $x = x_{i-\frac{1}{2}}, i = 1, 2...N$ we have obtained a system of algebraic equations given by (2.5). Thus we have a system of linear equations if source function $f(x, u, u', u'')$ is linear otherwise system of nonlinear equations. The propose method produces good approximate numerical value of the solution for model problems and it is computationally efficient and accurate method. The idea presented in this article leads to the possibility to develop finite difference methods for the numerical solution of higher odd order boundary value problems. Works in these directions are in progress.

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