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ω-TOPOLOGY AND $\star$-TOPOLOGY

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Abstract. In this paper, we introduce some generalizations of ω-open sets in ideal topological spaces and investigate some properties of the sets. Moreover, we use them to obtain decompositions of continuity via idealization.

1. Introduction

By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If $H \subset X$, $cl(H)$ and $int(H)$ will, respectively, denote the closure and interior of H in (X, τ).

An ideal I ([**19**]) on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

(1) $H \in I$ and $G \subset H$ imply $G \in I$ and
(2) $H \in I$ and $G \in I$ imply $H \cup G \in I$.

Given a space (X, τ) with an ideal I on X if $\mathbb{P}(X)$ is the set of all subsets of X, a set operator $(.)^\star : \mathbb{P}(X) \to \mathbb{P}(X)$, called a local function of H with respect to τ and I is defined as follows: for $H \subset X$,

$$H^\star(I, \tau) = \{x \in X : U \cap H \notin I \text{ for every } U \in \tau(x)\}$$

where $\tau(x) = \{U \in \tau : x \in U\}$ ([**14**]). A Kuratowski closure operator $cl^\star(.)$ for a topology $\tau^\star(I, \tau)$, called the $\star$-topology, finer than τ is defined by $cl^\star(H) = H \cup H^\star(I, \tau)$ ([**18**]). We will simply write $H^\star$ for $H^\star(I, \tau)$ and $\tau^\star$ for $\tau^\star(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space or an ideal space.

In this paper, we introduce and investigate the new notions called b-I_ω-open sets, α-I_ω-open sets and pre-I_ω-open sets which are weaker than ω-open sets. Moreover, we use these notions to obtain decompositions of continuity via idealization.

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2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{Q}$, $\mathbb{Q}^\star$, $\mathbb{N}$) denotes the set of all real numbers (resp. the set of all rational numbers, the set of all irrational numbers, the set of all natural numbers).

Definition 2.1. *A subset H of a space (X, τ) is said to be*

1. *α-open* [**16**] *if $H \subset int(cl(int(H)))$,*
2. *pre-open* [**15**] *if $H \subset int(cl(H))$,*
3. *β-open* [**1**] *if $H \subset cl(int(cl(H)))$,*
4. *b-open* [**5**] *if $H \subset int(cl(H)) \cup cl(int(H))$.*

Definition 2.2. [**20**] *Let H be a subset of a space (X, τ), a point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 2.3. [**12**] *A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points.*

The complement of an ω-closed set is called ω-open.

It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 2.1. [**12**] *Let H be a subset of a space (X, τ). Then*

1. *H is ω-closed in X if and only if $H = cl_\omega(H)$.*
2. *$cl_\omega(X \backslash H) = X \backslash int_\omega(H)$.*
3. *$cl_\omega(H)$ is ω-closed in X.*
4. *$x \in cl_\omega(H)$ if and only if $H \cap G \neq \emptyset$ for each ω-open set G containing x.*
5. *$cl_\omega(H) \subset cl(H)$.*
6. *$int(H) \subset int_\omega(H)$.*

Remark 2.1. [**3, 12**] *In a space (X, τ), every closed set is ω-closed but not conversely.*

Definition 2.4. [**17**] *A subset H of a space (X, τ) is said to be*

1. *α-ω-open if $H \subset int_\omega(cl(int_\omega(H)))$,*
2. *pre-ω-open if $H \subset int_\omega(cl(H))$,*
3. *β-ω-open if $H \subset cl(int_\omega(cl(H)))$,*
4. *b-ω-open if $H \subset int_\omega(cl(H)) \cup cl(int_\omega(H))$.*

Definition 2.5. [**4**] *A space (X, τ) is said to be anti-locally countable if every non-empty open set is uncountable.*

Lemma 2.2. [**4**] *If (X, τ) is an anti-locally countable space, then $int_\omega(H) = int(H)$ for every ω-closed set H of X and $cl_\omega(H) = cl(H)$ for every ω-open set H of X.*

Definition 2.6. *A subset H of an ideal topological space (X,τ,I) is said to be*

(1) *α-I-open* [**10**] *if $H \subset int(cl^\star(int(H)))$,*
(2) *semi-I-open* [**10**] *if $H \subset cl^\star(int(H))$,*
(3) *pre-I-open* [**7**] *if $H \subset int(cl^\star(H))$,*
(4) *strongly β-I-open* [**11**] *if $H \subset cl^\star(int(cl^\star(H)))$,*
(5) *b-I-open* [**9**] *if $H \subset int(cl^\star(H)) \cup cl^\star(int(H))$.*

Remark 2.2. [**8**] *The following diagram holds for a subset of an ideal topological space:*

Diagram-1

$$\textbf{open} \longrightarrow \alpha\text{-}I\textbf{-open} \longrightarrow \textbf{pre-}I\textbf{-open} \longrightarrow \textbf{strongly } \beta\text{-}I\textbf{-open}$$

The converses of the implications in this diagram are not true, in general.

Lemma 2.3. [**2**] *Let (X,τ,I) be an ideal topological space and H a subset of X. Then the following properties hold:*

(1) *If O is open in (X,τ,I), then $O \cap cl^\star(H) \subset cl^\star(O \cap H)$.*
(2) *If $H \subset X_0 \subset X$, then $cl^\star_{X_0}(H) = cl^\star(H) \cap X_0$.*

Proposition 2.1. [**2**] *Let (X,τ,I) be an ideal topological space and H a subset of X. If $I = \{\phi\}$ (resp. $\mathbb{P}(X)$, $\mathcal{N}$), then $H^\star = cl(H)$ (resp. ϕ, $cl(int(cl(H)))$) and $cl^\star(H) = cl(H)$ (resp. H, $H \cup cl(int(cl(H)))$) where $\mathcal{N}$ is the ideal of all nowhere dense sets of (X,τ).*

Lemma 2.4. [**14**] *Let (X,τ) be a space with an arbitrary index set $\triangle$, I an ideal of subsets of X and $\mathbb{P}(X)$ the power set of X. If $\{H_\alpha : \alpha \in \triangle\} \subset \mathbb{P}(X)$, then the following property holds:*

$$(\cup_{\alpha\in\triangle} H^\star_\alpha) \subset (\cup_{\alpha\in\triangle} H_\alpha)^\star.$$

3. Weaker forms of ω-open sets

In this section we introduce the following notions.

Definition 3.1. *A subset H of an ideal topological space (X,τ,I) is said to be*

(1) *α-I_ω-open if $H \subset int_\omega(cl^\star(int_\omega(H)))$,*
(2) *pre-I_ω-open if $H \subset int_\omega(cl^\star(H))$,*
(3) *β-I_ω-open if $H \subset cl^\star(int_\omega(cl^\star(H)))$,*
(4) *b-I_ω-open if $H \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H))$.*

Theorem 3.1. *For a subset of an ideal topological space (X,τ,I), the following properties hold:*

(1) *Every ω-open set is α-I_ω-open.*
(2) *Every α-I_ω-open set is pre-I_ω-open.*
(3) *Every pre-I_ω-open set is b-I_ω-open.*
(4) *Every b-I_ω-open set is β-I_ω-open.*

Proof. (1). If H is an ω-open set, then $H = int_\omega(H)$. Since

$$H \subset cl^\star(H),\ H \subset cl^\star(int_\omega(H)) \text{ and } int_\omega(H) \subset int_\omega(cl^\star(int_\omega(H))).$$

Therefore $H \subset int_\omega(cl^\star(int_\omega(H)))$ and H is α-I_ω-open.

(2). If H is an α-I_ω-open set, then $H \subset int_\omega(cl^\star(int_\omega(H))) \subset int_\omega(cl^\star(H))$. Therefore H is pre-I_ω-open.

(3). If H is a pre-I_ω-open set, then

$$H \subset int_\omega(cl^\star(H)) \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)).$$

Therefore H is b-I_ω-open.

(4). If H is a b-I_ω-open set, then

$$H \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)) \subset cl^\star(int_\omega(cl^\star(H))) \cup cl^\star(int_\omega(H)) \subset cl^\star(int_\omega(cl^\star(H))).$$

Therefore H is β-I_ω-open. □

EXAMPLE 3.1. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$ and ideal $I = \{\phi\}$,*

(1) $H = \mathbb{Q} \cup \{\sqrt{2}\}$ *is α-I_ω-open, since* $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(\mathbb{Q})) = int_\omega(cl(\mathbb{Q})) = int_\omega(\mathbb{R}) = \mathbb{R} \supset H$. *But* $H = \mathbb{Q} \cup \{\sqrt{2}\}$ *is not ω-open, since* $int_\omega(H) = \mathbb{Q} \neq H$.

(2) $H = \mathbb{Q}$ *is pre-I_ω-open, since* $int_\omega(cl^\star(H)) = int_\omega(cl(\mathbb{Q})) = int_\omega(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q} = H$. *But* $H = \mathbb{Q}$ *is not α-I_ω-open, since* $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(\phi)) = int_\omega(\phi) = \phi \not\supseteq \mathbb{Q} = H$.

(3) $H = (0,1]$ *is b-I_ω-open, since* $int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)) = int_\omega(cl(H)) \cup cl^\star((0,1)) = int_\omega([0,1]) \cup cl((0,1)) = (0,1) \cup [0,1] = [0,1] \supset H$. *But* $H = (0,1]$ *is not pre-I_ω-open, since* $int_\omega(cl^\star(H)) = int_\omega(cl(H)) = int_\omega([0,1]) = (0,1) \not\supseteq (0,1] = H$.

(4) $H = [0,1) \cap \mathbb{Q}$ *is β-I_ω-open, since* $cl^\star(int_\omega(cl^\star(H))) = cl^\star(int_\omega(cl(H))) = cl^\star(int_\omega([0,1])) = cl^\star((0,1)) = cl((0,1)) = [0,1] \supset H$. *But* $H = [0,1) \cap \mathbb{Q}$ *is not b-I_ω-open, since* $int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)) = int_\omega(cl(H)) \cup cl^\star(\phi) = int_\omega([0,1]) \cup \phi = (0,1) \cup \phi = (0,1) \not\supseteq H$.

THEOREM 3.2. *For a subset of an ideal topological space (X, τ, I), the following properties hold:*

(1) *Every α-I-open set is α-I_ω-open.*
(2) *Every pre-I-open set is pre-I_ω-open.*
(3) *Every b-I-open set is b-I_ω-open.*
(4) *Every strongly β-I-open set is β-I_ω-open.*

PROOF. (1). If H is an α-I-open set, then

$$H \subset int(cl^\star(int(H))) \subset int_\omega(cl^\star(int_\omega(H))).$$

Hence H is α-I_ω-open.

(2). If H is a pre-I-open set, then $H \subset int(cl^\star(H)) \subset int_\omega(cl^\star(H))$. Therefore H is pre-I_ω-open.

(3). If H is a b-I-open set, then

$$H \subset int(cl^\star(H)) \cup cl^\star(int(H)) \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)).$$

Therefore H is b-I_ω-open.

(4). If H is a strongly β-I-open set, then

$$H \subset cl^\star(int(cl^\star(H))) \subset cl^\star(int_\omega(cl^\star(H))).$$

Therefore H is β-I_ω-open. □

EXAMPLE 3.2. *Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}\}$ and ideal $I = \mathbb{P}(X)$. Then*

(1) *$H = \{b\}$ is α-I_ω-open, since*
$$int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(H)) = int_\omega(H) = H \supset H.$$
But $H = \{b\}$ is not α-I-open, since
$$int(cl^\star(int(H))) = int(cl^\star(\phi)) = \phi \not\supseteq H.$$

(2) *$H = \{b\}$ is pre-I_ω-open, since $int_\omega(cl^\star(H)) = int_\omega(H) = H \supset H$. But $H = \{b\}$ is not pre-I-open, since $int(cl^\star(H)) = int(H) = \phi \not\supseteq H$.*

(3) *$H = \{b\}$ is b-I_ω-open, since $int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)) = int_\omega(H) \cup cl^\star(H) = H \cup H = H \supset H$. But $H = \{b\}$ is not b-I-open, since $int(cl^\star(H)) \cup cl^\star(int(H)) = int(H) \cup cl^\star(\phi) = \phi \cup \phi = \phi \not\supseteq H$.*

(4) *$H = \{b\}$ is β-I_ω-open, since $cl^\star(int_\omega(cl^\star(H))) = cl^\star(int_\omega(H)) = cl^\star(H) = H \supset H$. But $H = \{b\}$ is not strongly β-I-open, since $cl^\star(int(cl^\star(H))) = cl^\star(int(H)) = cl^\star(\phi) = \phi \not\supseteq H$.*

PROPOSITION 3.1. *For a subset of an ideal topological space (X, τ, I), the following properties hold:*

(1) *Every α-I-open set is α-ω-open.*
(2) *Every pre-I-open set is pre-ω-open.*
(3) *Every b-I-open set is b-ω-open.*
(4) *Every strongly β-I-open set is β-ω-open.*

PROOF. (1). Let H be an α-I-open set. Then

$$H \subset int(cl^\star(int(H))) \subset int_\omega(cl^\star(int_\omega(H))) \subset int_\omega(cl(int_\omega(H))).$$

This shows that H is α-ω-open.

(2). Let H be a pre-I-open set. Then

$$H \subset int(cl^\star(H)) \subset int_\omega(cl(H)).$$

This shows that H is pre-ω-open.

(3). Let H be a b-I-open set. Then

$$H \subset int(cl^\star(H)) \cup cl^\star(int(H)) \subset int_\omega(cl(H)) \cup cl(int_\omega(H)).$$

This shows that H is b-ω-open.

(4) Let H be a strongly β-I-open set. Then

$$H \subset cl^\star(int(cl^\star(H))) \subset cl(int_\omega(cl(H))).$$

This shows that H is β-ω-open. □

EXAMPLE 3.3. *In Example 3.2,*

(1) $H = \{b\}$ *is* α-ω-*open, since*
$$int_\omega(cl(int_\omega(H))) = int_\omega(cl(H)) = int_\omega(\{b,c\}) = \{b,c\} \supset H.$$
But $H = \{b\}$ *is not* α-I-*open by (1) of Example 3.2.*

(2) $H = \{b\}$ *is pre-*ω-*open, since*
$$int_\omega(cl(H)) = int_\omega(\{b,c\}) = \{b,c\} \supset H.$$
But $H = \{b\}$ *is not pre-*I-*open by (2) of Example 3.2.*

(3) $H = \{b\}$ *is b-*ω-*open, since*
$$int_\omega(cl(H)) \cup cl(int_\omega(H)) = int_\omega(\{b,c\}) \cup cl(H) = \{b,c\} \cup \{b,c\} = \{b,c\} \supset H.$$
But $H = \{b\}$ *is not b-*I-*open by (3) of Example 3.2.*

(4) $H = \{b\}$ *is* β-ω-*open, since*
$$cl(int_\omega(cl(H))) = cl(int_\omega(\{b,c\})) = cl(\{b,c\}) = \{b,c\} \supset H.$$
But $H = \{b\}$ *is not strongly* β-I-*open by (4) of Example 3.2.*

PROPOSITION 3.2. *For a subset of an ideal topological space* (X, τ, I), *the following properties hold:*

(1) *Every pre-*I-*open set is b-*I-*open.*
(2) *Every b-*I-*open is strongly* β-I-*open.*

PROOF. (1). Let H be a pre-I-open set. Then
$$H \subset int(cl^\star(H)) \subset int(cl^\star(H)) \cup cl^\star(int(H)).$$
This shows that H is b-I-open.
(2). Let H be a b-I-open set. Then
$$H \subset int(cl^\star(H)) \cup cl^\star(int(H)) \subset cl^\star(int(cl^\star(H))) \cup cl^\star(int(H)) = cl^\star(int(cl^\star(H))).$$
This shows that H is strongly β-I-open. □

EXAMPLE 3.4. *Let* $X = \{a,b,c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ *and* $I = \{\phi, \{b\}\}$. *Then* $H = \{a,c\}$ *is b-*I-*open, since* $int(cl^\star(H)) \cup cl^\star(int(H)) = int(H \cup H^\star) \cup cl^\star(\{a\}) = int(H \cup H) \cup (\{a\} \cup \{a\}^\star) = int(H) \cup (\{a\} \cup H) = \{a\} \cup H = H \supset H$. *But* $H = \{a,c\}$ *is not pre-*I-*open, since* $int(cl^\star(H)) = \{a\} \not\supseteq H$.

EXAMPLE 3.5. *Let* $X = \{a,b,c,d\}$, $\tau = \{\phi, X, \{d\}, \{a,c\}, \{a,c,d\}\}$ *and* $I = \{\phi, \{b\}\}$. *Then* $H = \{a,b\}$ *is strongly* β-I-*open, since* $cl^\star(int(cl^\star(H))) = cl^\star(int(H \cup H^\star)) = cl^\star(int(H \cup \{a,b,c\})) = cl^\star(int(\{a,b,c\})) = cl^\star(\{a,c\}) = \{a,c\} \cup \{a,c\}^\star = \{a,c\} \cup \{a,b,c\} = \{a,b,c\} \supset H$. *But* $H = \{a,b\}$ *is not b-*I-*open, since* $int(cl^\star(H)) \cup cl^\star(int(H)) = int(\{a,b,c\}) \cup cl^\star(\phi) = \{a,c\} \cup \phi = \{a,c\} \not\supseteq H$.

REMARK 3.1. *Since every open set is* ω-*open* [**3, 12**], *we have the following diagram for properties of subsets.*

Diagram-2

$$\begin{array}{ccccccccc}
\textbf{open} & \longrightarrow & \alpha\text{-}I\textbf{-open} & \longrightarrow & \textbf{pre-}I\textbf{-open} & \longrightarrow & \textbf{b-}I\textbf{-open} & \longrightarrow & \textbf{strongly } \beta\text{-}I\textbf{-open} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\omega\textbf{-open} & \longrightarrow & \alpha\text{-}I_\omega\textbf{-open} & \longrightarrow & \textbf{pre-}I_\omega\textbf{-open} & \longrightarrow & \textbf{b-}I_\omega\textbf{-open} & \longrightarrow & \beta\text{-}I_\omega\textbf{-open}
\end{array}$$

The converses of the above implications are not true in general as can be seen from the above Examples.

THEOREM 3.3. *If H is a pre-I_ω-open subset of an ideal topological space (X, τ, I) such that $U \subset H \subset cl^\star(U)$ for a subset U of X, then U is a pre-I_ω-open set.*

PROOF. Since $H \subset int_\omega(cl^\star(H))$, $U \subset int_\omega(cl^\star(H)) \subset int_\omega(cl^\star(U))$ since $cl^\star(H) \subset cl^\star(U)$. Thus U is a pre-I_ω-open set. □

THEOREM 3.4. *A subset H of an ideal topological space (X, τ, I) is semi-I-open if and only if H is β-I_ω-open and $int_\omega(cl^\star(H)) \subset cl^\star(int(H))$.*

PROOF. Let H be semi-I-open. Then $H \subset cl^\star(int(H)) \subset cl^\star(int_\omega(cl^\star(H)))$ and hence H is β-I_ω-open. In addition $cl^\star(H) \subset cl^\star(int(H))$ and hence

$$int_\omega(cl^\star(H)) \subset cl^\star(int(H)).$$

Conversely let H be β-I_ω-open and $int_\omega(cl^\star(H)) \subset cl^\star(int(H))$. Then

$$H \subset cl^\star(int_\omega(cl^\star(H))) \subset cl^\star(cl^\star(int(H))) = cl^\star(int(H))$$

and hence H is semi-I-open. □

PROPOSITION 3.3. *The intersection of a pre-I_ω-open set and an open set is pre-I_ω-open.*

PROOF. Let H be a pre-I_ω-open set and U an open set. Then $H \subset int_\omega(cl^\star(H))$. Since every open set is ω-open,

$$U \cap H \subset int_\omega(U) \cap int_\omega(cl^\star(H)) = int_\omega(U \cap cl^\star(H)) \subset int_\omega(cl^\star(U \cap H))$$

by Lemma 2.3(1). This shows that $U \cap H$ is pre-I_ω-open. □

PROPOSITION 3.4. *The intersection of a β-I_ω-open set and an open set is β-I_ω-open.*

PROOF. Let H be a β-I_ω-open set and U an open set. Then

$$H \subset cl^\star(int_\omega(cl^\star(H))).$$

Since every open set is

$$\omega\text{-open, } U \cap H \subset U \cap cl^\star(int_\omega(cl^\star(H))) \subset cl^\star(U \cap int_\omega(cl^\star(H))) \subset$$
$$cl^\star(int_\omega(U) \cap int_\omega(cl^\star(H))) = cl^\star(int_\omega(U \cap cl^\star(H))) \subset cl^\star(int_\omega(cl^\star(U \cap H)))$$

by Lemma 2.3(1). This shows that $U \cap H$ is β-I_ω-open. □

PROPOSITION 3.5. *The intersection of a b-I_ω-open set and an open set is b-I_ω-open.*

PROOF. Let H be a b-I_ω-open and U an open set. Then

$$H \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)).$$

Since every open set is ω-open,

$$U \cap H \subset U \cap [int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H))] = [U \cap int_\omega(cl^\star(H))] \cup [U \cap cl^\star(int_\omega(H))] = [int_\omega(U) \cap int_\omega(cl^\star(H))] \cup [U \cap cl^\star(int_\omega(H))] \subset [int_\omega(U \cap cl^\star(H))] \cup [cl^\star(U \cap int_\omega(H))]$$

by Lemma 2.3(1). Thus

$$U \cap H \subset [int_\omega(cl^\star(U \cap H))] \cup [cl^\star(int_\omega(U \cap H))].$$

This shows that $U \cap H$ is b-I_ω-open. □

REMARK 3.2. *The intersection of two pre-I_ω-open (resp. b-I_ω-open, β-I_ω-open) sets need not be pre-I_ω-open (resp. b-I_ω-open, β-I_ω-open) as can be seen from the following Example.*

EXAMPLE 3.6. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \{\phi\}$,*

(1) *$A = \mathbb{Q}$ is pre-I_ω-open, since $int_\omega(cl^\star(A)) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset A$. Also $B = \mathbb{Q}^\star \cup \{1\}$ is pre-I_ω-open, since $int_\omega(cl^\star(B)) = int_\omega(cl(B)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset B$. But $A \cap B = \{1\}$ is not pre-I_ω-open, since $int_\omega(cl^\star(A \cap B)) = int_\omega(cl^\star(\{1\})) = int_\omega(cl(\{1\})) = int_\omega(\{1\}) = \phi \not\supseteq A \cap B$.*
(2) *$A = \mathbb{Q}$ and $B = \mathbb{Q}^\star \cup \{1\}$ are b-I_ω-open, by (1) of Example 3.6 and Theorem 3.1 (3). But $A \cap B = \{1\}$ is not b-I_ω-open, since $int_\omega(cl^\star(\{1\})) \cup cl^\star(int_\omega(\{1\})) = \phi \cup cl^\star(\phi) = \phi \cup \phi = \phi \not\supseteq \{1\} = A \cap B$.*
(3) *$A = \mathbb{Q}$ and $B = \mathbb{Q}^\star \cup \{1\}$ are β-I_ω-open by (2) of Example 3.6 and Theorem 3.1 (4). But $A \cap B = \{1\}$ is not β-I_ω-open, since $cl^\star(int_\omega(cl^\star(\{1\}))) = cl^\star(int_\omega(cl(\{1\}))) = cl^\star(int_\omega(\{1\})) = cl^\star(\phi) = \phi \not\supseteq \{1\} = A \cap B$.*

PROPOSITION 3.6. *The intersection of an α-I_ω-open set and an open set is α-I_ω-open.*

PROOF. Let H be α-I_ω-open and U be open. Then

$$U = int_\omega(U) \text{ and } H \subset int_\omega(cl^\star(int_\omega(H))).$$
$$U \cap H \subset int_\omega(U) \cap [int_\omega(cl^\star(int_\omega(H))] = int_\omega[U \cap cl^\star(int_\omega(H))] \subset int_\omega[cl^\star(U \cap int_\omega(H))]$$

by Lemma 2.3(1). Thus

$$U \cap H \subset int_\omega[cl^\star(int_\omega(U) \cap int_\omega(H))] = int_\omega[cl^\star(int_\omega(U \cap H))]$$

which implies $U \cap H$ is α-I_ω-open. □

PROPOSITION 3.7. *If $\{H_\alpha : \alpha \in \triangle\}$ is a collection of pre-I_ω-open sets of an ideal topological space (X, τ, I), then $\cup_{\alpha\in\triangle} H_\alpha$ is pre-I_ω-open.*

PROOF. Since $H_\alpha \subset int_\omega(cl^\star(H_\alpha))$ for every $\alpha \in \triangle$,

$$\cup_{\alpha\in\triangle} H_\alpha \subset \cup_{\alpha\in\triangle} int_\omega(cl^\star(H_\alpha)) \subset int_\omega(\cup_{\alpha\in\triangle} cl^\star(H_\alpha)) = int_\omega(\cup_{\alpha\in\triangle}({H_\alpha}^\star \cup H_\alpha))$$
$$= int_\omega((\cup_{\alpha\in\triangle}{H_\alpha}^\star) \cup (\cup_{\alpha\in\triangle} H_\alpha)) \subset int_\omega((\cup_{\alpha\in\triangle} H_\alpha)^\star \cup (\cup_{\alpha\in\triangle} H_\alpha))$$

by Lemma 2.4. Thus

$$\cup_{\alpha\in\triangle}H_\alpha \subset int_\omega[(\cup_{\alpha\in\triangle}H_\alpha)^\star \cup (\cup_{\alpha\in\triangle}H_\alpha)] = int_\omega(cl^\star(\cup_{\alpha\in\triangle}H_\alpha)).$$

Therefore, $\cup_{\alpha\in\triangle}H_\alpha$ is pre-I_ω-open. □

THEOREM 3.5. *If $\{H_\alpha : \alpha \in \triangle\}$ is a collection of b-I_ω-open (resp. β-I_ω-open) sets of an ideal topological space (X, τ, I), then $\cup_{\alpha\in\triangle}H_\alpha$ is b-I_ω-open (resp. β-I_ω-open).*

PROOF. We prove only the first result since the other result follows similarly. Since H_α is b-I_ω-open for every $\alpha \in \triangle$, $H_\alpha \subset int_\omega(cl^\star(H_\alpha)) \cup cl^\star(int_\omega(H_\alpha))$ for every $\alpha \in \triangle$.

$$\begin{aligned} Then \cup_{\alpha\in\triangle} H_\alpha &\subset \cup_{\alpha\in\triangle}[int_\omega(cl^\star(H_\alpha)) \cup cl^\star(int_\omega(H_\alpha))] \\ &= [\cup_{\alpha\in\triangle}int_\omega(cl^\star(H_\alpha))] \cup [\cup_{\alpha\in\triangle}cl^\star(int_\omega(H_\alpha))] \\ &\subset [int_\omega(\cup_{\alpha\in\triangle}cl^\star(H_\alpha))] \cup [cl^\star(\cup_{\alpha\in\triangle}int_\omega(H_\alpha))] \\ &\subset [int_\omega(cl^\star(\cup_{\alpha\in\triangle}H_\alpha))] \cup [cl^\star(int_\omega(\cup_{\alpha\in\triangle}H_\alpha))]. \end{aligned}$$

Therefore, $\cup_{\alpha\in\triangle}H_\alpha$ is b-I_ω-open. □

PROPOSITION 3.8. *Let H be a b-I_ω-open set such that $int_\omega(H) = \phi$. Then H is pre-I_ω-open.*

Recall that a space (X, τ) is called a door space if every subset of X is open or closed.

PROPOSITION 3.9. *If (X, τ) is a door space, then every pre-I_ω-open set in (X, τ, I) is ω-open.*

PROOF. Let H be a pre-I_ω-open set. If H is open, then H is ω-open. Otherwise, H is closed and hence $H \subset int_\omega(cl^\star(H)) \subset int_\omega(cl(H)) = int_\omega(H) \subset H$. Therefore, $H = int_\omega(H)$ and thus H is an ω-open set. □

THEOREM 3.6. *Let (X, τ) be an anti-locally countable space and H a subset of (X, τ, I). Then the following properties hold:*

(1) *if H is pre-I_ω-open, then it is pre-open.*
(2) *if H is b-I_ω-open and ω-closed, then it is b-open.*
(3) *if H is β-I_ω-open, then it is β-open.*

PROOF. (1) Let H be a pre-I_ω-open set. Then

$$H \subset int_\omega(cl^\star(H)) \subset int_\omega(cl(H)) = int(cl(H))$$

by Lemma 2.2, since every closed set is ω-closed. This shows that H is pre-open.

(2) Let H be a b-I_ω-open and ω-closed set. Since H and $cl(H)$ are ω-closed, $int_\omega(cl^\star(H)) \subset int_\omega(cl(H)) = int(cl(H))$ and $cl^\star(int_\omega(H)) \subset cl(int(H))$ by Lemma 2.2. Since H is b-I_ω-open, $H \subset int_\omega(cl^\star(H)) \cup cl^\star(int_\omega(H)) \subset int(cl(H)) \cup cl(int(H))$. This shows that H is b-open.

(3) Let H be a β-I_ω-open set. Then

$$H \subset cl^\star(int_\omega(cl^\star(H))) \subset cl(int_\omega(cl(H))) = cl(int(cl(H)))$$

by Lemma 2.2. This shows that H is β-open.

□

PROPOSITION 3.10. *For an ideal topological space (X, τ, I) and $H \subset X$,*

(1) *If $I = \{\phi\}$, then H is pre-I_ω-open (resp. b-I_ω-open) if and only if H is pre-ω-open (resp. b-ω-open).*
(2) *If $I = \mathbb{P}(X)$, then H is pre-I_ω-open (resp. b-I_ω-open) if and only if H is ω-open.*

PROOF. (1) It is obvious from Proposition 2.1.
(2) It is obvious from Proposition 2.1.

□

4. Decompositions of continuity via idealization

DEFINITION 4.1. *A subset H of an ideal topological space (X, τ, I) is called*

(1) *a t-I_ω-set if $int(H) = int_\omega(cl^\star(H))$;*
(2) *a B-I_ω-set if $H = U \cap V$, where $U \in \tau$ and V is an t-I_ω-set.*

EXAMPLE 4.1. (1) *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \{\phi\}$, $H = \mathbb{Q}$ is not a t-I_ω-set, since $int_\omega(cl^\star(H)) = int_\omega(cl(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \phi = int(H)$.*

(2) *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and ideal $I = \mathbb{P}(\mathbb{R})$, $H = \mathbb{Q}$ is a t-I_ω-set, since $int_\omega(cl^\star(H)) = int_\omega(H) = \phi = int(H)$.*

REMARK 4.1. *In an ideal topological space (X, τ, I),*

(1) *Every open set is a B-I_ω-set.*
(2) *Every t-I_ω-set is a B-I_ω-set.*

The converses of (1) and (2) in Remark 4.1 are not true in general as illustrated in the following Examples.

EXAMPLE 4.2. *In Example 4.1 (2), $H = \mathbb{Q}$ is a t-I_ω-set and hence by (2) of Remark 4.1, $H = \mathbb{Q}$ is a B-I_ω-set. But $H = \mathbb{Q}$ is not open, since $\mathbb{Q} \notin \tau$.*

EXAMPLE 4.3. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and ideal $I = \{\phi\}$, $H = \mathbb{Q}^\star$ is open in $\mathbb{R}$ and hence by (1) of Remark 4.1, H is a B-I_ω-set. But $int_\omega(cl^\star(H)) = int_\omega(cl(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \mathbb{Q}^\star = int(H)$. Thus $H = \mathbb{Q}^\star$ is not a t-I_ω-set.*

EXAMPLE 4.4. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \{\phi\}$, $H = \mathbb{Q}$ is not a B-I_ω-set. If $H = U \cap V$, where $U \in \tau$ and V is t-I_ω-set, then $H \subset U$. But $\mathbb{R}$ is the only open set containing H. Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$ which is a contradiction, since $H = V$ is not a t-I_ω-set by Example 4.1 (1). This proves that $H = \mathbb{Q}$ is not a B-I_ω-set.*

PROPOSITION 4.1. *Let A and B be subsets of an ideal topological space (X, τ, I). If A and B are t-I_ω-sets, then $A \cap B$ is a t-I_ω-set.*

PROOF. Let A and B be t-I_ω-sets. Then we have $int(A \cap B) \subset int_\omega(cl^\star(A \cap B)) \subset int_\omega(cl^\star(A) \cap cl^\star(B)) = int_\omega(cl^\star(A)) \cap int_\omega(cl^\star(B)) = int(A) \cap int(B) = int(A \cap B)$. Then $int(A \cap B) = int_\omega(cl^\star(A \cap B))$ and hence $A \cap B$ is a t-I_ω-set. □

PROPOSITION 4.2. *For a subset H of an ideal topological space (X, τ, I), the following properties are equivalent:*

(1) *H is open;*
(2) *H is pre-I_ω-open and a B-I_ω-set.*

PROOF. (1) $\Rightarrow$ (2): Let H be open. Then $H = int(H) \subset int_\omega(cl^\star(H))$ and H is pre-I_ω-open. Also by Remark 4.1 H is a B-I_ω-set.
(2) $\Rightarrow$ (1): Given H is a B-I_ω-set. So $H = U \cap V$ where $U \in \tau$ and $int(V) = int_\omega(cl^\star(V))$. Then $H \subset U = int(U)$. Also, H is pre-I_ω-open implies $H \subset int_\omega(cl^\star(H)) \subset int_\omega(cl^\star(V)) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. $\square$

REMARK 4.2. *The following Examples show that the concepts of pre-I_ω-openness and being a B-I_ω-set are independent.*

EXAMPLE 4.5. *In Example 4.4, $H = \mathbb{Q}$ is pre-I_ω-open, since $int_\omega(cl^\star(H)) = int_\omega(cl(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \supset \mathbb{Q} = H$. But $H = \mathbb{Q}$ is not a B-I_ω-set.*

EXAMPLE 4.6. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and ideal $I = \mathbb{P}(\mathbb{R})$, $H = \mathbb{Q}$ is a t-I_ω-set, by (2) of Example 4.1. Hence $H = \mathbb{Q}$ is a B-I_ω-set by (2) of Remark 4.1. But $H = \mathbb{Q}$ is not pre-I_ω-open, since $int_\omega(cl^\star(H)) = int_\omega(H) = \phi \not\supset \mathbb{Q} = H$.*

DEFINITION 4.2. *A subset H of an ideal topological space (X, τ, I) is called*

(1) *a t_α-I_ω-set if $int(H) = int_\omega(cl^\star(int_\omega(H)))$;*
(2) *a B_α-I_ω-set if $H = U \cap V$, where $U \in \tau$ and V is a t_α-I_ω-set.*

EXAMPLE 4.7. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \mathbb{P}(\mathbb{R})$, $H = \mathbb{R}\backslash\{0\}$ is a t_α-I_ω-set, since $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(H)) = int_\omega(H) = H = int(H)$.*

EXAMPLE 4.8. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \mathbb{P}(\mathbb{R})$, $H = \mathbb{Q}^\star$ is not a t_α-I_ω-set, since $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(H)) = int_\omega(H) = H \neq \phi = int(H)$.*

REMARK 4.3. *In an ideal topological space (X, τ, I),*

(1) *Every open set is a B_α-I_ω-set.*
(2) *Every t_α-I_ω-set is a B_α-I_ω-set.*

EXAMPLE 4.9. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^\star, \mathbb{Q}^\star \cup \mathbb{N}\}$ and ideal $I = \{\phi\}$, $H = \mathbb{Q}$ is a t_α-I_ω-set, since $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(\mathbb{N})) = int_\omega(cl(\mathbb{N})) = int_\omega(\mathbb{Q}) = \mathbb{N} = int(H)$. Hence by (2) of Remark 4.3, $H = \mathbb{Q}$ is a B_α-I_ω-set. But $H = \mathbb{Q}$ is not open, since $\mathbb{Q} \notin \tau$.*

EXAMPLE 4.10. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$ and ideal $I = \{\phi\}$, $H = \mathbb{Q}^\star$ is open, since $H \in \tau$ and hence $H = \mathbb{Q}^\star$ is a B_α-I_ω-set by (1) of Remark 4.3. But $H = \mathbb{Q}^\star$ is not a t_α-I_ω-set, since $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(H)) = int_\omega(cl(H)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \mathbb{Q}^\star = H = int(H)$.*

EXAMPLE 4.11. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \mathbb{P}(\mathbb{R})$, $H = \mathbb{Q}^\star$ is not a B_α-I_ω-set. If $H = U \cap V$ where $U \in \tau$ and V is t_α-I_ω-set, then $H \subset U$. But $\mathbb{R}$ is the only open set containing H. Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$ which is* *a contradiction, since $H = V$ is not a t_α-I_ω-set by Example 4.8. This proves that $H = \mathbb{Q}^\star$ is not a B_α-I_ω-set.*

PROPOSITION 4.3. *If A and B are t_α-I_ω-sets of an ideal topological space (X, τ, I), then $A \cap B$ is a t_α-I_ω-set.*

PROOF. Let A and B be t_α-I_ω-sets. Then we have

$$int(A \cap B) \subset int_\omega(cl^\star(int_\omega(A \cap B))) \subset int_\omega[cl^\star(int_\omega(A)) \cap cl^\star(int_\omega(B))] =$$
$$int_\omega(cl^\star(int_\omega(A))) \cap int_\omega(cl^\star(int_\omega(B))) = int(A) \cap int(B) = int(A \cap B).$$

Then $int(A \cap B) = int_\omega(cl^\star(int_\omega(A \cap B)))$ and hence $A \cap B$ is a t_α-I_ω-set. □

PROPOSITION 4.4. *For a subset H of an ideal topological space (X, τ, I), the following properties are equivalent:*

(1) *H is open;*
(2) *H is α-I_ω-open and a B_α-I_ω-set.*

PROOF. (1) ⇒ (2): Let H be open. Then $H = int_\omega(H) \subset cl^\star(int_\omega(H))$ and $H = int_\omega(H) \subset int_\omega(cl^\star(int_\omega(H)))$. Therefore H is α-I_ω-open. Also by (1) of Remark 4.3, H is a B_α-I_ω-set.

(2) ⇒ (1): Given H is a B_α-I_ω-set. So $H = U \cap V$ where $U \in \tau$ and $int(V) = int_\omega(cl^\star(int_\omega(V)))$. Then $H \subset U = int(U)$. Also H is α-I_ω-open implies $H \subset int_\omega(cl^\star(int_\omega(H))) \subset int_\omega(cl^\star(int_\omega(V))) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and H is open. □

REMARK 4.4. *The following Examples show that the concepts of α-I_ω-openness and being a B_α-I_ω-set are independent.*

EXAMPLE 4.12. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \mathbb{P}(\mathbb{R})$, $H = \mathbb{Q}^\star$ is α-I_ω-open, since $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star(H)) = int_\omega(H) = H \supset H$. But $H = \mathbb{Q}^\star$ is not a B_α-I_ω-set by Example 4.11.*

EXAMPLE 4.13. *In $\mathbb{R}$ with usual topology τ_u and ideal $I = \mathbb{P}(\mathbb{R})$, $H = (0, 1]$ is a t_α-I_ω-set, since $int_\omega(cl^\star(int_\omega(H))) = int_\omega(cl^\star((0, 1))) = int_\omega((0, 1)) = (0, 1) = int(H)$. Hence $H = (0, 1]$ is a B_α-I_ω-set by (2) of Remark 4.3. But $H = (0, 1]$ is not α-I_ω-open, since $int_\omega(cl^\star(int_\omega(H))) = (0, 1) \not\supseteq (0, 1] = H$.*

DEFINITION 4.3. *A function $f : X \to Y$ is said to be ω-continuous* [**13**] *(resp. pre-I_ω-continuous, B-I_ω-continuous, α-I_ω-continuous, B_α-I_ω-continuous) if $f^{-1}(V)$ is ω-open (resp. pre-I_ω-open, a B-I_ω-set, α-I_ω-open, a B_α-I_ω-set) for each open set V in Y.*

By Propositions 4.2 and 4.4 we have the immediate result.

THEOREM 4.1. *For a function $f : X \to Y$, the following properties are equivalent:*

(1) *f is continuous.*
(2) *f is pre-I_ω-continuous and B-I_ω-continuous.*
(3) *f is α-I_ω-continuous and B_α-I_ω-continuous.*

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