BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(2016), 173-179

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

MULTIPLICATIVE ZAGREB INDICES OF GENERALIZED TRANSFORMATION GRAPHS

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Abstract. The first, second and modified first multiplicative Zagreb indices of a graph G are defined, respectively, as

$$\prod_1(G) = \prod_{u \in V(G)} d_G(u)^2, \prod_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v)$$

and

$$\prod_1^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)]$$

where $d_G(w)$ is the degree of vertex w in G. In the present study, we obtain the expressions for $\prod_1$, $\prod_2$ and $\prod_1^*$ of generalized transformation graphs G^{ab}.

1. Introduction

In this paper we are concerned with finite, simple, nontrivial and undirected graphs. Let G be such a graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$, $|E(G)| = m$. As usual, n is order and m is size of G. The degree of a vertex $w \in V(G)$ is the number of vertices adjacent to w and is denoted by $d_G(w)$. We use [**7**] for terminology and notations not defined here.

A graphical invariant is a number related to a graph, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also called the topological indices. In 1984, Narumi and Katayama [**9**] considered the product index as

$$NK(G) = \prod_{u \in V(G)} d_G(u)$$

for representing the carbon skeleton of a saturated hydrocarbon, and named it as simple topological index. Tomović and Gutman, this molecular structure descriptor was renamed as Narumi-Katayama index [**15**]. In 2010, Todeshine et al. [**13, 14**]

2010 *Mathematics Subject Classification.* 05C07.

Key words and phrases. degree, multiplicative Zagreb indices, generalized transformation graphs G^{ab}.

have proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows:

$$\prod_1(G) = \prod_{u\in V(G)} d_G(u)^2 = [NK(G)]^2 \text{ and } \prod_2(G) = \prod_{uv\in E(G)} d_G(u)d_G(v).$$

These two graph invariants are called first and second multiplicative Zagreb indices by Gutman [**5**]. And recently, Eliasi et al. [**4**] introduced further multiplicative version of the first Zagreb index as

$$\prod_1^*(G) = \prod_{uv\in E(G)} [d_G(u) + d_G(v)]$$

and in [**6**], Gutman called it as modified first multiplicative Zagreb index. The main properties of multiplicative Zagreb indices are summarized in [**2, 3, 8, 10, 12**].

2. Generalized transformation graphs G^{ab}

The semitotal-point graph $T_2(G)$ of a graph G is a graph whose vertex set is $V(T_2(G)) = V(G) \cup E(G)$ and two vertices are adjacent in $T_2(G)$ if and only if (i) they are adjacent vertices of G or (ii) one is a vertex of G and other is an edge of G incident with it. It was introduced by Sampathkumar and Chikkodimath [**11**]. Recently some new graphical transformations were defined by Basavanagoud et al. [**1**], which generalizes the concept of semitotal-point graph.

The generalized transformation graph G^{ab} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{ab})$. The vertices α and β are adjacent in G^{ab} if and only if $(*)$ and $(**)$ holds:
$(*)$ $\alpha, \beta \in V(G)$, α, β are adjacent in G if $a = +$ and α, β are not adjacent in G if $a = -$. $(**)$ $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $b = +$ and α, β are not incident in G if $b = -$.

One can obtain the four graphical transformations of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. The vertex v_i of G^{ab} corresponding to a vertex v_i of G is referred to as *point vertex* and vertex e_i of G^{ab} corresponding to an edge e_i of G is referred to as *line vertex*.
The following propositions will be useful in proof of our results.

Proposition 2.1. [**1**] *Let G be a graph of order n and size m. Then the degree of point vertex u_i and line vertex e_i in G^{ab} are*
(i) $d_{G^{++}}(u_i) = 2d_G(u_i)$ *and* $d_{G^{++}}(e_i) = 2$
(ii) $d_{G^{+-}}(u_i) = m$ *and* $d_{G^{+-}}(e_i) = n-2$
(iii) $d_{G^{-+}}(u_i) = n-1$ *and* $d_{G^{-+}}(e_i) = 2$
(iv) $d_{G^{--}}(u_i) = n+m-1-2d_G(u_i)$ *and* $d_{G^{--}}(e_i) = n-2$.

Proposition 2.2. [**1**] *Let G be a graph of order n and size m. Then order of G^{ab} is $n+m$ and*
(i) Size of $G^{++} = 3m$
(ii) Size of $G^{+-} = m(n-1)$
(iii) Size of $G^{-+} = \binom{n}{2} + m$
(iv) Size of $G^{--} = \frac{n(n-1)}{2} + m(n-3)$.

In this paper, we obtain expressions for $\prod_1$, $\prod_2$ and $\prod_1^*$ of generalized transformation graphs.

3. Results

THEOREM 3.1. *Let G be a graph of order $n > 2$ and size m. Then*

$$\prod\nolimits_1(G^{+-}) = m^{2n}(n-2)^{2m}.$$

PROOF. Since G^{+-} has $m+n$ vertices.

$$\begin{aligned}\prod\nolimits_1(G^{+-}) &= \prod_{u\in V(G^{+-})} d_{G^{+-}}(u)^2 \\ &= \prod_{u\in V(G^{+-})\cap V(G)} d_{G^{+-}}(u)^2 \prod_{e_i\in V(G^{+-})\cap E(G)} d_{G^{+-}}(e_i)^2.\end{aligned}$$

From Proposition 2.1, we have

$$\begin{aligned}\prod\nolimits_1(G^{+-}) &= \prod_{u\in V(G)} m^2 \prod_{e_i\in E(G)} (n-2)^2 \\ &= m^{2n}(n-2)^{2m}.\end{aligned}$$

□

THEOREM 3.2. *Let G be a graph of order $n > 2$ and size $m \geqslant 1$. Then*

$$\prod\nolimits_2(G^{+-}) = m^{mn}(n-2)^{m(n-2)}.$$

PROOF. Since G^{+-} has $m+n$ vertices and $m(n-1)$ edges.

$$\begin{aligned}\prod\nolimits_2(G^{+-}) &= \prod_{uv\in E(G^{+-})} [d_{G^{+-}}(u)d_{G^{+-}}(v)] \\ &= \prod_{uv\in E(G^{+-})\cap E(G)} [d_{G^{+-}}(u)d_{G^{+-}}(v)] \prod_{uv\in E(G^{+-})-E(G)} [d_{G^{+-}}(u)d_{G^{+-}}(v)].\end{aligned}$$

From Proposition 2.1, we have

$$\begin{aligned}\prod\nolimits_2(G^{+-}) &= \prod_{uv\in E(G)} mm \prod_{uv\in E(G^{+-})-E(G)} m(n-2) \\ &= m^{2m}[m(n-2)]^{m(n-1)-m} \\ &= m^{mn}(n-2)^{m(n-2)}.\end{aligned}$$

□

THEOREM 3.3. *Let G be a graph of order n and size $m \geqslant 1$. Then*

$$\prod\nolimits_1^*(G^{+-}) = (2m)^m(m+n-2)^{m(n-2)}.$$

PROOF. Since G^{+-} has $n+m$ vertices and $m(n-1)$ edges.

$$\begin{aligned}\prod\nolimits_1^*(G^{+-}) &= \prod_{uv\in E(G^{+-})} [d_{G^{+-}}(u)+d_{G^{+-}}(v)] \\ &= \prod_{uv\in E(G^{+-})\cap E(G)} [d_{G^{+-}}(u)+d_{G^{+-}}(v)] \prod_{uv\in E(G^{+-})-E(G)} [d_{G^{+-}}(u)+d_{G^{+-}}(v)]\end{aligned}$$

From Proposition 2.1, we have

$$\begin{aligned}\prod\nolimits_1^*(G^{+-}) &= \prod_{uv\in E(G)} (m+m) \prod_{uv\in E(G^{+-})-E(G)} (m+n-2) \\ &= (2m)^m(m+n-2)^{m(n-2)}.\end{aligned}$$

□

THEOREM 3.4. *Let G be a graph of order n and size m. Then*

$$\prod_1(G^{-+}) = 4^m(n-1)^{2n}.$$

PROOF. Since G^{-+} has $m+n$ vertices.

$$\prod_1(G^{-+}) = \prod_{u\in V(G^{-+})} d_{G^{-+}}(u)^2$$
$$= \prod_{u\in V(G^{-+})\cap V(G)} d_{G^{-+}}(u)^2 \prod_{e_i\in V(G^{-+})\cap E(G)} d_{G^{-+}}(e_i)^2.$$

From Proposition 2.1, we have

$$\prod_1(G^{-+}) = \prod_{u\in V(G)} (n-1)^2 \prod_{e_i\in E(G)} 2^2$$
$$= 4^m(n-1)^{2n}.$$

□

THEOREM 3.5. *Let G be a graph of order n and size m. Then*

$$\prod_2(G^{-+}) = 4^m(n-1)^{n(n-1)}.$$

PROOF. Since G^{-+} has $m+n$ vertices and $\binom{n}{2}+m$ edges.

$$\prod_2(G^{-+}) = \prod_{uv\in E(G^{-+})} [d_{G^{-+}}(u)d_{G^{-+}}(v)]$$
$$= \prod_{uv\in E(G^{-+})\cap E(\overline{G})} [d_{G^{-+}}(u)d_{G^{-+}}(v)] \prod_{uv\in E(G^{-+})-E(\overline{G})} [d_{G^{-+}}(u)d_{G^{-+}}(v)].$$

From Proposition 2.1, we have

$$\prod_2(G^{-+}) = \prod_{uv\in E(\overline{G})} (n-1)(n-1) \prod_{uv\in E(G^{-+})-E(\overline{G})} 2(n-1)$$
$$= [n-1]^{[n(n-1)-2m]}2^{2m}(n-1)^{2m}$$
$$= 4^m(n-1)^{n(n-1)}.$$

□

THEOREM 3.6. *Let G be a graph of order n and size m. Then*

$$\prod_1^*(G^{-+}) = [2(n-1)]^{[\binom{n}{2}-m]}(n+1)^{2m}.$$

PROOF. Since G^{-+} has $n+m$ vertices and $[\binom{n}{2}+m]$ edges.

$$\prod_1^*(G^{-+}) = \prod_{uv\in E(G^{-+})} [d_{G^{-+}}(u)+d_{G^{-+}}(v)]$$
$$= \prod_{uv\in E(G^{-+})\cap E(\overline{G})} [d_{G^{-+}}(u)+d_{G^{-+}}(v)] \prod_{uv\in E(G^{-+})-E(\overline{G})} [d_{G^{-+}}(u)+d_{G^{-+}}(v)].$$

From Proposition 2.1, we have

$$\prod_1^*(G^{-+}) = \prod_{uv\in E(\overline{G})} (n-1+n-1) \prod_{uv\in E(G^{-+})-E(\overline{G})} (n-1+2)$$
$$= [2(n-1)]^{[\binom{n}{2}-m]}(n+1)^{2m}.$$

□

THEOREM 3.7. *Let G be a graph of order n and size m. Then*

$$\prod_1(G^{--}) = (n-2)^{2m} \prod_{u\in V(G)\ and\ d_G(u)\neq n-1} (n+m-1-2d_G(u))^2.$$

PROOF. Since G^{--} has $m+n$ vertices.

$$\textstyle\prod_1(G^{--}) = \prod_{u\in V(G^{--})} d_{G^{--}}(u)^2$$

$$= \prod_{u\in V(G^{--})\cap V(G)} d_{G^{--}}(u)^2 \prod_{e_i\in V(G^{--})\cap E(G)} d_{G^{--}}(e_i)^2.$$

From Proposition 2.1, we have

$$\textstyle\prod_1(G^{--}) = \prod_{u\in V(G)} (n+m-2d_G(u)-1)^2 \prod_{e_i\in E(G)} (n-2)^2$$

$$\textstyle\prod_1(G^{--}) = (n-2)^{2m} \prod_{u\in V(G)\ and\ d_G(u)\neq n-1} [n+m-1-2d_G(u)]^2. \quad \square$$

THEOREM 3.8. *Let G be a graph of order n and size m. Then*

$$\textstyle\prod_2(G^{--}) = \Big[\prod_{uv\notin E(G)} [n+m-1-2d_G(u)][n+m-1-2d_G(v)]\Big]$$

$$\Big[(n-2)^{2m} \prod_{v\in V(G)\ and\ d_G(v)\neq n-1} [n+m-1-2d_G(v)]^{m-d_G(v)}\Big].$$

PROOF. Since G^{--} has $m+n$ vertices and $\frac{n(n-1)}{2}+m(n-3)$ edges.

$$\textstyle\prod_2(G^{--}) = \prod_{uv\in E(G^{--})} [d_{G^{--}}(u)d_{G^{--}}(v)]$$

$$= \prod_{uv\in E(G^{--})\cap E(\overline{G})} [d_{G^{--}}(u)d_{G^{--}}(v)] \prod_{uv\in E(G^{--})-E(\overline{G})} [d_{G^{--}}(u)d_{G^{--}}(v)]$$

From Proposition 2.1, we have

$$\textstyle\prod_2(G^{--}) = \prod_{uv\in E(\overline{G})} [n+m-1-2d_G(u)][n+m-1-2d_G(v)] \prod_{uv\in E(G^{--})-E(\overline{G})} (n-2)[n+m-1-2d_G(v)]$$

$$\textstyle\prod_2(G^{--}) = \Big[\prod_{uv\notin E(G)} [n+m-1-2d_G(u)][n+m-1-2d_G(v)]\Big]$$

$$\Big[(n-2)^{2m} \prod_{v\in V(G)\ and\ d_G(v)\neq n-1} [n+m-1-2d_G(v)]^{m-d_G(v)}\Big]. \quad \square$$

THEOREM 3.9. *Let G be a graph of order n and size m. Then*

$$\textstyle\prod_1^*(G^{--}) = \prod_{uv\notin E(G)} 2[n+m-1-d_G(u)-d_G(v)] \prod_{v\in V(G)} [2n+m-3-2d_G(v)]^{m-d_G(v)}.$$

PROOF. Since G^{--} has $n+m$ vertices and $\frac{n(n-1)}{2}+m(n-3)$ edges. Then

$$\textstyle\prod_1^*(G^{--}) = \prod_{uv\in E(G^{--})} [d_{G^{--}}(u)+d_{G^{--}}(v)]$$

$$= \prod_{uv\in E(G^{--})\cap E(\overline{G})} [d_{G^{--}}(u)+d_{G^{--}}(v)] \prod_{uv\in E(G^{--})-E(\overline{G})} [d_{G^{--}}(u)+d_{G^{--}}(v)]$$

From Proposition 2.1, we have

$$= \prod_{uv\in E(\overline{G})} [n+m-1-2d_G(u)+n+m-1-2d_G(v)] \prod_{uv\in E(G^{--})-E(\overline{G})} [n-2+n+m-1-2d_G(v)]$$

and

$$\prod_1^*(G^{--}) = \prod_{uv\notin E(G)} 2[n+m-1-d_G(u)-d_G(v)] \prod_{v\in V(G)} [2n+m-3-2d_G(v)]^{m-d_G(v)}.$$

□

The expressions for $\prod_1$, $\prod_2$ and $\prod_1^*$ of semitotal point graph G^{++} was obtained in [**2**]. We nevertheless state it for the sake of completeness:

THEOREM 3.10. [**2**]*Let G be a graph of order n and size m. Then*

(1) $\prod_1(G^{++}) = 4^{n+m}\prod_1(G)$

(2) $\prod_2(G^{++}) = 64^m\prod_1(G)\prod_2(G)$

(3) $\prod_1^*(G^{++}) = 8^m\prod_1^*(G)\prod_{u\in V(G)}[1+d_G(u)]^{d_G(u)}$.

4. Acknowledgement

This research is supported by UGC-SAP DRS-III, New Delhi, India for 2016-2021: F.510/3/DRS-III/2016(SAP-I) Dated: 29^{th} Feb. 2016. (B. Basavanagoud)

This research is supported by UGC- National Fellowship (NF) New Delhi. No. F./2014-15/NFO-2014-15-OBC-KAR-25873/(SA-III/Website) Dated: March-2015. (Veena R. Desai)

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Received by editors 26.10.2016; Avalanche online 14.11.2016.

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