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STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS TO THE SOLUTION OF TOTAL ASYMPTOTICALLY NON-EXPANSIVE NONLINEAR SYSTEM

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Abstract. We prove strong convergence of an implicit iteration procedure equipped with error terms to the solution of a system involving total asymptotically nonexpansive mappings, in uniformly convex Banach spaces.

1. Introduction

Let K be a nonempty subset of a real normed linear space E. A self mapping $T : K \to K$ is called

- nonexpansive if $\| Tx - Ty \| \leqslant \| x - y \|$ for every $x, y \in K$.
- asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \geqslant 1$, $\| T^n x - T^n y \| \leqslant k_n \| x - y \|$ for all $x, y \in K$.
- asymptotically quasi-nonexpansive if $F(T) = \{x \in K : Tx = x\} \neq \phi$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\| T^n x - y \| \leqslant k_n \| x - y \|$ for all $x \in K$, $y \in F(T)$ and every $n \geqslant 1$.
- uniformly L-Lipschitzian if there exists a real number $L > 0$ such that $\| T^n x - T^n y \| \leqslant L \| x - y \|$ for all $x, y \in K$ and all $n \geqslant 1$.

The class of asymptotically nonexpansive mappings introdued by Goebel and Kirk [**11**] represents a significant generalization of nonexpansive mappings. It was proved in [**11**] that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self mapping on K, then T does have a fixed point.

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A mapping $T : K \to K$ is called asymptotically nonexpansive in the intermediate sense (see for example [**4**]) if it is continuous and satisfies the following inequality:

$$\limsup_{n\to\infty} \sup_{x,\, y\in K} \{\| T^n x - T^n y \| - \| x - y \|\} \leqslant 0. \tag{1.1}$$

If $F(T) \neq \emptyset$ and (1.1) holds for all $x \in K$, $y \in F(T)$, then T is called asymptotically quasi-nonexpansive in the intermediate sense. It is obvious that if

$$\sigma_n = \max\{ \sup_{x,\, y\in K} \{\| T^n x - T^n y \| - \| x - y \|\}, 0\},$$

then σ_n as $n \to \infty$ and (1.1) reduces to

$$\| T^n x - T^n y \| \leqslant \| x - y \| + \sigma_n.$$

The class of mappings that are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [**4**]. It is known (refer to [**14**]) that if K be a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is a self mapping of K which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings. The main tool for approximation of fixed points of generalizations of nonexpansive mappings remains iterative technique.

Iterative methods for approximating fixed points of nonexpansive mappings have been studied by many authors (see for example [**1**], [**5**], [**7**], [**8**], [**9**], [**13**], [**16**], [**19**–**21**], [**22**], [**24**] and the references therein). In Most of these papers, the well-known Mann iteration process [**15**],

$$x_1 \in K, \qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n \geqslant 1, \qquad (*)$$

has been studied and the operator T has been assumed to map K into itself. The convexity of K then ensures that the sequence $\{x_n\}$ generated by (*) is well defined. In 2001, Xu and Ori [**28**] introduced the following implicit iteration process for a finite family of nonexpansive self mappings $\{T_i, i \in I\}$, where $I = \{1, 2, ..., N\}$. For any initial point $x_0 \in K$,

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n \qquad n \geqslant 1,$$

where $\{\alpha_n\}$ is a real sequence in (0,1) and $T_n = T_{n(modN)}$, the mod N function takes values in I. They proved weak convergence of the above process to a common fixed point of $\{T_i, i \in I\}$. Later on, this implicit iteration method has been used to study the common fixed point of a finite family of strictly pseudocontractive self mappings [**18**], asymptotically nonexpansive self mappings [**26**], and asymptotically quasi-nonexpansive self mappings [**29**]. In 1991, Schu [**25**] introduced a modified iteration process to approximate fixed points of asymptotically nonexpansive self mappings in Hilbert spaces. More precisely, he proved the following theorem.

THEOREM 1.1. ([**25**]) *Let H be a Hilbert space, K a nonempty closed convex and bounded subset of H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geqslant 1$, $\lim k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) <$*

∞. *Let* $\{\alpha_n\}$ *be a real sequence in* $[0,1]$ *satisfying the condition* $0 < a \leqslant \alpha_n \leqslant b < 1$, $n \geqslant 1$, *for some constants* a *and* b. *Then the sequence* $\{x_n\}$ *generated from* $x_1 \in K$ *by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \qquad n \geqslant 1,$$

converges strongly to some fixed point of T.

Since then, Schu's iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self mappings in a Hilbert or Banach space (see for example [**17**], [**22**], [**23**], [**27**]).

Recently, Alber et al. [**2**] introduced a more general class of asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings, and studied methods of approximation of their fixed points.

Definition 1.1. ([**2**]) *A self mapping* $T : K \to K$ *is said to be total asymptotically nonexpansive if there exist nonnegative real sequences* $\{\mu_n\}_{n=1}^{\infty}$ *and* $\{l_n\}_{n=1}^{\infty}$ *with* μ_n, $l_n \to 0$ *as* $n \to \infty$ *and a strictly increasing continuous function* $\phi : \Re^+ \to \Re^+$ *with* $\phi(0) = 0$ *such that for all* x, $y \in K$,

$$\| T^n x - T^n y \| \quad \leqslant \quad \| x - y \| + \mu_n \phi(\| x - y \|) + l_n, \qquad n \geqslant 1. \tag{1.2}$$

Remark 1.1. *If* $\phi(\lambda) = \lambda$, *then (1.2) reduces to*

$$\| T^n x - T^n y \| \quad \leqslant \quad (1+\mu_n) \| x - y \| + l_n, \qquad n \geqslant 1.$$

If, in addition, $l_n = 0$, $n \geqslant 1$, *then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If* $\mu_n = 0$ *and* $l_n = 0$ *for all* $n \geqslant 1$, *we obtain from (1.2) the class of mappings that includes the class of nonexpansive mappings. If* $\mu_n = 0$ *and* $l_n = \max\{\sup_{x,\, y \in K}\{\| T^n x - T^n y \| - \| x - y \|\}, 0\}$ *for all* $n \geqslant 1$, *then (1.2) reduces to (1.1) which has been studied as mappings asymptotically nonexpansive in the intermediate sense. This goes to show how Definition 1.1 unifies various generalizations of asymptotically nonexpansive mappings.*

Very recently, Chang et al. [**6**] constructed an implicit iterative sequence with errors to approximate a common fixed point of a finite family of asymptotically nonexpansive mappings $T_1, T_2, ..., T_N$.

Definition 1.2. ([**6**]) *Let* K *be a nonempty closed convex subset of a real normed linear space* E *satisfying* $K + K \subset K$ *and* $\{T_1, T_2, ..., T_N\} : K \to K$ *be* N *asymptotically nonexpansive mappings. Let* α_n *be a sequence in [0,1] and* $\{u_n\}$ *be a bounded sequence in* K. *Then for any given point* $x_0 \in K$, *the sequence* $\{x_n\}$ *is* *defined as follows*

$$
\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 + u_1, \qquad m \geqslant 1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 + u_2,\\
&\dots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N + u_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1} + u_{N+1},\\
x_{N+2} &= \alpha_{N+2} x_{N+1} + (1-\alpha_{N+2})T_2^2 x_{N+2} + u_{N+2},\\
&\dots\\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N} + u_{2N},\\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1} + u_{2N+1},\\
&\dots
\end{aligned}
$$

which can be rewritten in a compact form as follows

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n + u_n, \qquad n \geqslant 1, \tag{1.3}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in I = \{1, 2, ..., N\}$, $k(n) \geqslant 1$ *is a positive integer such that* $k(n) \to \infty$ *as* $n \to \infty$.

Motivated and inspired by the previous facts, we extend the results obtained by Chang et al. [**6**] to the class of total asymptotically nonexpansive self mappings.

2. Preliminaries

Let E be a real normed linear space. *The modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2} \parallel x+y \parallel : \parallel x \parallel = \parallel y \parallel = 1, \parallel x-y \parallel = \epsilon\}.$$

E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$.

A mapping $T : K \to K$ is said to be *semicompact* if, for any bounded sequence $\{x_n\}$ in K such that $\parallel x_n - Tx_n \parallel \to 0$ as $n \to 0$, there exists a subsequence $\{x_{n_j}\}$, say, of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some x^* in K. T is said to be *completely continuous* if, for any bounded sequence $\{x_n\}$, there exists a subsequence $\{Tx_{n_j}\}$, say, of $\{Tx_n\}$ such that $\{Tx_{n_j}\}$ converges strongly to some element of the range of the range of T.

We need the following lemmas in order to prove the main results of this paper.

Lemma 2.1. ([**27**]) *Let* $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be sequences of non-negative real numbers such that* $a_{n+1} \leqslant (1+b_n)a_n + c_n$, $n \geqslant 1$. *If* $\sum_{n=1}^{\infty} b_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$ *then* $\lim_{n\to\infty} a_n$ *exists.*

Lemma 2.2. ([**25**]) *Let* E *be a real uniformly convex Banach space and* $0 < \alpha \leqslant t_n \leqslant \beta < 1$ *for all positive integers* $n \geqslant 1$. *Suppose that* $\{x_n\}$ *and* $\{y_n\}$ *are* *two sequences of E such that*

$$\limsup_{n\to\infty} \| x_n \| \leqslant r, \qquad \limsup_{n\to\infty} \| y_n \| \leqslant r \quad \text{and} \qquad \limsup_{n\to\infty} \| t_n x_n + (1-t_n) y_n \| = r$$

hold for some $r \geqslant 0$, then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.

3. Main Results

THEOREM 3.1. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E with $K + K \subset K$. Let $\{T_i : K \to K, i \in I\}$ be a finite family of N uniformly continuous total asymptotically nonexpansive mappings with a nonempty common fixed point set $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ such that*

$$\| T_i^n x - T_i^n y \| \quad \leqslant \quad \| x - y \| + \mu_{in} \phi_i(\| x - y \|) + l_{in}, \qquad n \geqslant 1, \quad i \in I,$$

where $\{\mu_{in}\}_{n=1}^\infty$ and $\{l_{in}\}_{n=1}^\infty$, $i \in I$ are nonnegative real sequences with $\sum_{n=1}^\infty \mu_{in} < \infty$, $\sum_{n=1}^\infty l_{in} < \infty$, $i \in I$ and $\phi_i : \Re^+ \to \Re^+$, $i \in I$ are strictly increasing continuous functions with $\phi_i(0) = 0$, $i \in I$. Suppose that there exist constants M_i, $M_i^* > 0$ such that $\phi_i(\lambda) \leqslant M_i^* \lambda$ for all $\lambda \geqslant M_i$, $i \in I$. Let $\{u_n\}_{n=1}^\infty$ is a bounded sequence in K such that $\sum_{n=1}^\infty u_n < \infty$. Let $\{x_n\}$ be the implicit iterative sequence defined by (1.3) such that $\{\alpha_n\}$ is a sequence in [0,1] satisfying that $\tau_1 \leqslant (1-\alpha_n) \leqslant \tau_2$, $n \geqslant 1$ for some constants $\tau_1, \tau_2 \in (0,1)$. Then, for each $i \in I$, $\lim_{n\to\infty} \| x_n - T_i x_n \| = 0$.

REMARK 3.1. *Theorem 3.1 generalizes and extends the results obtained by Chang et al.* [**6**] *which in their turn extend the corresponding results in* [**3**], [**12**], [**26**], [**28**], [**29**] *and many others.*

PROOF. Assume that $\phi : \Re^+ \to \Re^+$ is defined by $\phi(\nu) = \max_{i\in I} \phi_i(\nu)$, $\nu \geqslant 0$. Then $\phi : \Re^+ \to \Re^+$ is continuous and strictly increasing with $\phi(0) = 0$. Let $\mu_n = \max\{\mu_{in}, i \in I\}$, $n \geqslant 1$, then we have that $\{\mu_n\} \subset [0, \infty)$ with $\sum_{n=1}^\infty \mu_n < \infty$. Similarly, let $l_n = \max\{l_{in}, i \in I\}$, $n \geqslant 1$, so that $\{l_n\} \subset [0,\infty)$ and $\sum_{n=1}^\infty l_n < \infty$.

Hence, we have

$$\| T_i^n x - T_i^n y \| \quad \leqslant \quad \| x - y \| + \mu_n \phi(\| x - y \|) + l_n, \qquad n \geqslant 1. \tag{3.1}$$

Given any $q \in F = \cap_{i=1}^N F(T_i) \neq \emptyset$, it follows from (1.3) and (3.1) that for any $n \geqslant 1$,

$$\begin{aligned} \| x_n - q \| &= \| \alpha_n x_{n-1} + (1-\alpha_n) T_{i(n)}^{k(n)} x_n + u_n - q \| \\ &\leqslant \alpha_n \| x_{n-1} - q \| + (1-\alpha_n) \| T_{i(n)}^{k(n)} x_n - q \| + \| u_n \| \\ &\leqslant \alpha_n \| x_{n-1} - q \| + (1-\alpha_n)[\| x_n - q \| + \mu_n \phi(\| x_n - q \|) + l_n] + \| u_n \|. \end{aligned}$$

By assumptions of Theorem 3.1, there exist constants $M = \max_{i\in I} M_i$, $M^* = \max_{i\in I} M_i^* > 0$ such that $\phi(\lambda) \leqslant M^* \lambda$ for all $\lambda \geqslant M$. Since $\phi(.)$ is an increasing function, then

$$\begin{cases} \phi(\| x_n - q \|) \leqslant \phi(M), & \text{whenever } \| x_n - q \| \leqslant M; \\ \phi(\| x_n - q \|) \leqslant \| x_n - q \| M^*, & \text{whenever } \| x_n - q \| \geqslant M. \end{cases}$$

In either case, we have

$$\phi(\| x_n - q \|) \quad \leqslant \quad \| x_n - q \| M^* + \phi(M) \qquad for\ some\ M, M^* > 0.$$

Therefore

$$\| x_n - q \| \leqslant$$
$$\alpha_n \| x_{n-1} - q \| + (1-\alpha_n)[\| x_n - q \| + \mu_n(\| x_n - q \| M^* + \phi(M)) + l_n] + \| u_n \| .$$

Hence

$$(\alpha_n - (1-\alpha_n)\mu_n M^*) \| x_n - q \|$$
$$\leqslant \alpha_n \| x_{n-1} - q \| + (1-\alpha_n)[\mu_n \phi(M) + l_n] + \| u_n \|,$$

that is

$$(1 - \tfrac{(1-\alpha_n)\mu_n M^*}{\alpha_n}) \| x_n - q \| \leqslant$$
$$\| x_{n-1} - q \| + \tfrac{(1-\alpha_n)}{\alpha_n}[\mu_n \phi(M) + l_n] + \tfrac{1}{\alpha_n} \| u_n \|,$$

simplifying, we have

$$\| x_n - q \| \leqslant$$
$$\| x_{n-1} - q \| + \tfrac{(1-\alpha_n)\mu_n M^*}{\alpha_n} \| x_n - q \| + \tfrac{1-\alpha_n}{\alpha_n}[\mu_n \phi(M) + l_n] + \tfrac{1}{\alpha_n} \| u_n \| .$$

Since

$$\tau_1 \leqslant (1-\alpha_n) \leqslant \tau_2,\ n \geqslant 1,$$

then

$$\| x_n - q \| \leqslant$$
$$\| x_{n-1} - q \| + \tfrac{\tau_2 M^* \mu_n}{1-\tau_2} \| x_n - q \| + \tfrac{\tau_2}{1-\tau_2}[\mu_n \phi(M) + l_n] + \tfrac{1}{1-\tau_2} \| u_n \| .$$

Further simplification implies

$$\| \tfrac{1-\tau_2-\tau_2 M^* \mu_n}{1-\tau_2} \| x_n - q \| \leqslant$$
$$\| x_{n-1} - q \| + \tfrac{\tau_2}{1-\tau_2}[\mu_n \phi(M) + l_n] + \tfrac{1}{1-\tau_2} \| u_n \|,$$

i. e.,

$$\| x_n - q \| \leqslant$$
$$(1 + \tfrac{\tau_2 M^* \mu_n}{1-\tau_2-\tau_2 M^* \mu_n}) \| x_{n-1} - q \| + \tfrac{\tau_2}{1-\tau_2-\tau_2 M^* \mu_n}[\mu_n \phi(M) + l_n] + \tfrac{1}{1-\tau_2-\tau_2 M^* \mu_n} \| u_n \| .$$

By virtue of $\sum_{n=1}^{\infty} \mu_n$, $\mu_n \to \infty$ as $n \to \infty$, therefore there exists a positive integer n_0 such that $\mu_n \leqslant \frac{(1-\tau_2)^2}{\tau_2 M^*}$ for all $n \geqslant n_0$. Hence

$$\| x_n - q \| \leqslant$$
$$(1 + \tfrac{M^*}{1-\tau_2}\mu_n) \| x_{n-1} - q \| + \tfrac{1}{1-\tau_2}[\mu_n \phi(M) + l_n] + \tfrac{1}{\tau_2(1-\tau_2)} \| u_n \|,$$
$$n \geqslant n_0.$$

Taking

$$\begin{aligned} a_n &= \| x_{n-1} - q \|, \\ b_n &= \frac{M^*}{1-\tau_2}\mu_n, \\ c_n &= \frac{1}{1-\tau_2}[\mu_n \phi(M) + l_n] + \frac{1}{\tau_2(1-\tau_2)} \| u_n \| \end{aligned}$$

in Lemma 2.1 and using that $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$ and $\sum_{n=1}^{\infty} u_n < \infty$, it is easy to see that $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Hence, it follows from Lemma 2.1 that $\lim_{n\to\infty} \| x_n - q \|$ for any $q \in F$. Without loss of generality, we can assume that $\lim_{n\to\infty} \| x_n - q \| = d$, where $d \geqslant 0$ is some number. Since $\{\| x_n - q \|\}$ is a convergent real sequence, then $\{x_n\}$ is a bounded sequence in K. Now, we have

$$\lim_{n\to\infty} \| \alpha_n[x_{n-1} - q + u_n] + (1-\alpha_n)[T_{i(n)}^{k(n)} x_n - q + u_n] \| = \lim_{n\to\infty} \| x_n - q \| = d. \tag{3.2}$$

Since

$$\limsup_{n\to\infty} \| x_{n-1} - q + u_n \| \leqslant \limsup_{n\to\infty} \| x_{n-1} - q \| + \limsup_{n\to\infty} \| u_n \| .$$

Then

$$\limsup_{n\to\infty} \| x_{n-1} - q + u_n \| \leqslant d. \tag{3.3}$$

Also, since $T_{i(n)}$ is total asymptotically nonexpansive, $i \in I$, then

$$\begin{aligned} \| T_{i(n)}^{k(n)} x_n - q + u_n \| &\leqslant \| T_{i(n)}^{k(n)} x_n - q \| + \| u_n \| \\ &\leqslant \| x_n - q \| + \mu_n \phi(\| x_n - q \|) + l_n + \| u_n \| \\ &\leqslant \| x_n - q \| + M^* \| x_n - q \| \mu_n + \phi(M)\mu_n + l_n + \| u_n \|, \end{aligned}$$

for some constants $M, M^* > 0$. By boundedness of the sequence $\{\| x_n - q \|\}$ and the fact that $\mu_n, l_n, u_n \to 0$, $n \to \infty$, it follows that

$$\limsup_{n\to\infty} \| T_{i(n)}^{k(n)} x_n - q + u_n \| \leqslant d. \tag{3.4}$$

Applying Lemma 2.2 in regard of (3.2)-(3.4) implies

$$\lim_{n\to\infty} \| T_{i(n)}^{k(n)} x_n - x_{n-1} \| = 0. \tag{3.5}$$

Moreover, since

$$\begin{aligned} \| x_n - x_{n-1} \| &\leqslant \| (1-\alpha_n) T_{i(n)}^{k(n)} x_n - (1-\alpha_n) x_{n-1} + u_n \| \\ &\leqslant (1-\alpha_n) \| T_{i(n)}^{k(n)} x_n - x_{n-1} \| + \| u_n \| . \end{aligned}$$

It follows from (3.5) that

$$\lim_{n\to\infty} \| x_n - x_{n-1} \| = 0. \tag{3.6}$$

Generally

$$\lim_{n\to\infty} \| x_n - x_{n+j} \| = 0, \qquad j = 1, 2, ...N. \tag{3.7}$$

For any positive integer $n > N$ $(k(n) \geqslant 2)$, we have

$$\begin{aligned} \| T_{i(n)}^{k(n)-1} x_n - x_n \| &\leqslant \| T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N} \| + \\ &\quad \| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N-1} \| + \| x_{n-N-1} - x_n \| . \end{aligned} \tag{3.8}$$

Since for each $n > N$, $n = (k(n) - 1)N + i(n)$, $i(n) \in I$, then $n - N = (k(n) - 1)N + i(n) - N = [(k(n) - 1) - 1]N + i(n) = (k(n - N) - 1)N + i(n - N)$, thus $k(n - N) = k(n) - 1$ and $i(n - N) = i(n)$. Hence, uniform continuity of $T_{i(n)}$, $i(n) \in I$ and (3.7) imply

$$\begin{aligned} \| T_{i(n)}^{k(n)-1} x_n - T_{i(n-N)}^{k(n)-1} x_{n-N} \| &= \| T_{i(n)}^{k(n)-1} x_n - T_{i(n)}^{k(n)-1} x_{n-N} \| \longrightarrow 0 \\ as \quad n &\longrightarrow \infty. \end{aligned} \tag{3.9}$$

Also, (3.5) yields

$$\begin{aligned} \| T_{i(n-N)}^{k(n)-1} x_{n-N} - x_{n-N-1} \| = \| T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{n-N-1} \| \longrightarrow 0 \\ as \quad n \longrightarrow \infty. \end{aligned} \tag{3.10}$$

Hence, using (3.7), (3.9) and (3.10), it follows from (3.8) that

$$\lim_{n\to\infty} \| T_{i(n)}^{k(n)-1} x_n - x_n \| = 0.$$

Again $T_{i(n)}$ is uniformly continuous for any $i \in I$, thus

$$\lim_{n\to\infty} \| T_{i(n)}^{k(n)} x_n - T_{i(n)} x_n \| = 0, \tag{3.11}$$

since

$$\| x_{n-1} - T_{i(n)} x_n \| \leqslant \| x_{n-1} - T_{i(n)}^{k(n)} x_n \| + \| T_{i(n)}^{k(n)} x_n - T_{i(n)} x_n \|,$$

then, using (3.5) and (3.11), we obtain

$$\lim_{n\to\infty} \| x_{n-1} - T_{i(n)} x_n \| = 0. \tag{3.12}$$

It follows from (3.6) and (3.12) that

$$\begin{aligned} &\lim_{n\to\infty} \| x_n - T_{i(n)} x_n \| \leqslant \\ &\lim_{n\to\infty} \| x_n - x_{n-1} \| + \lim_{n\to\infty} \| x_{n-1} - T_{i(n)} x_n \| = 0. \end{aligned} \tag{3.13}$$

Consequently, for any $j \in \{1, 2, ..., N\}$, we have

$$\begin{gathered} \| x_n - T_{n+j} x_n \| \leqslant \\ \| x_n - x_{n+j} \| + \| x_{n+j} - T_{n+j} x_{n+j} \| + \| T_{n+j} x_{n+j} - T_{n+j} x_n \| . \end{gathered}$$

It follows from (3.7), (3.13) and uniform continuity of T_{n+j}, that

$$\lim_{n\to\infty} \| x_n - T_{n+j} x_n \| = 0, \qquad j = 1, 2, ..., N.$$

This implies that the sequence $\bigcup_{j=1}^{N}\{\| x_n - T_{n+j}x_n \|\}_{n=1}^{\infty} \to 0$ as $n \to \infty$. Since, $\{\| x_n - T_l x_n \|\}_{n=1}^{\infty}$ is a subsequence of $\bigcup_{j=1}^{N}\{\| x_n - T_{n+j}x_n \|\}_{n=1}^{\infty}$, $l = 1, 2, ..., N$. Then, we have

$$\lim_{n\to\infty} \| x_n - T_l x_n \| \quad = \quad 0, \qquad l = 1, 2, ..., N.$$

This completes the proof. □

THEOREM 3.2. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E with $K + K \subset K$. Let $\{T_i : K \to K, i \in I\}$ be a finite family of N uniformly continuous total asymptotically nonexpansive mappings with a nonempty common fixed point set $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ such that*

$$\| T_i^n x - T_i^n y \| \quad \leqslant \quad \| x - y \| + \mu_{in}\phi_i(\| x - y \|) + l_{in}, \qquad n \geqslant 1, \quad i \in I,$$

where $\{\mu_{in}\}_{n=1}^{\infty}$ and $\{l_{in}\}_{n=1}^{\infty}$, $i \in I$ are nonnegative real sequences with $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i \in I$ and $\phi_i : \Re^+ \to \Re^+$, $i \in I$ are strictly increasing continuous functions with $\phi_i(0) = 0$, $i \in I$. Suppose that there exist constants M_i, $M_i^* > 0$ such that $\phi_i(\lambda) \leqslant M_i^*\lambda$ for all $\lambda \geqslant M_i$, $i \in I$. Let $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in K such that $\sum_{n=1}^{\infty} u_n < \infty$. Let $\{x_n\}$ be the implicit iterative sequence defined by (1.3) such that $\{\alpha_n\}$ is a sequence in [0,1] satisfying that $\tau_1 \leqslant (1 - \alpha_n) \leqslant \tau_2$, $n \geqslant 1$ for some constants $\tau_1, \tau_2 \in (0, 1)$. If at least one of the mappings T_i, $i \in I$ is semi-compact, then $\{x_n\}$ converges strongly to a common fixed point of T_i, $i \in I$ in K.

PROOF. The proof follows from the proof of Theorem 3 in [**6**]. □

4. Some Remarks on the Main Result

(i) The main results of this paper are obviously valid for the case when $\{T_i : K \to K, i \in I\}$ is a finite family of N uniformly continuous asymptotically nonexpansive and uniformly continuous asymptotically nonexpansive in the intermediate sense mappings with a nonempty common fixed point set, since in these cases we no longer need to assume that there exist constants M_i, $M_i^* > 0$ such that $\phi_i(\lambda) \leqslant M_i^*\lambda$ for all $\lambda \geqslant M_i$, $i \in I$.

(ii) Our main results surely includes the iterative sequence $\{x_n\}$ defined by

$$x_n \quad = \quad \alpha_n x_{n-1} + (1 - \alpha_n)T_{i(n)}^{k(n)} x_n, \qquad n \geqslant 1, \tag{4.1}$$

where x_0 is any point in K, $n = (k(n) - 1)N + i(n)$, $i(n) \in I = \{1, 2, ..., N\}$ and $k(n) \geqslant 1$ is a positive integer such that $k(n) \to \infty$ as $n \to \infty$. In this case, there will be no need for the assumption that $K + K \subset K$.

(iii) Theorems 3.1 and 3.2 of this paper trivially carry over to the class of total asymptotically quasi-nonexpansive mappings defined below with little or no modification:

DEFINITION 4.1. ([**10**]) *A self mapping $T : K \to K$ is said to be total asymptotically quasi-nonexpansive if there exist nonnegative real sequences $\{\mu_n\}_{n=1}^{\infty}$ and*

$\{l_n\}_{n=1}^{\infty}$ *with* μ_n, $l_n \to 0$ *as* $n \to \infty$ *and a strictly increasing continuous function* $\phi : \Re^+ \to \Re^+$ *with* $\phi(0) = 0$ *such that for all* $x \in K$, $x^* \in F(T)$,

$$\| T^n x - x^* \| \quad \leqslant \quad \| x - x^* \| + \mu_n \phi(\| x - x^* \|) + l_n, \qquad n \geqslant 1. \tag{4.2}$$

If $\phi(\lambda) = \lambda$, then (3.15) reduces to

$$\| T^n x - x^* \| \quad \leqslant \quad (1 + \mu_n) \| x - x^* \| + l_n, \qquad n \geqslant 1.$$

If, in addition, $l_n = 0$, $n \geqslant 1$, then total asymptotically quasi-nonexpansive mappings coincide with asymptotically quasi- nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geqslant 1$, we obtain from (1.2) the class of mappings that includes the class of quasi-nonexpansive mappings. If $\mu_n = 0$ and $l_n = \max\{\sup_{x \in K,\, x^* \in F(T)} \{\| T^n x - x^* \| - \| x - x^* \|\}, 0\}$ for all $n \geqslant 1$, then the mappings which has been studied as mappings asymptotically quasi-nonexpansive in the intermediate sense.

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