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PRODUCT VERSION OF RECIPROCAL DEGREE DISTANCE OF GRAPHS

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Abstract. In this paper, we present the various upper and lower bounds for the product version of reciprocal degree distance in terms of other graph inavriants. Finally, we obtain the upper bounds for the product version of reciprocal degree distance of the composition, Cartesian product and double of a graph in terms of other graph invariants including the Harary index and Zagreb indices. .

1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the degree of a vertex $v \in V(G)$. A *topological index* of a graph is a real number related to the graph; it does not depend on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [**17**].

Let G be a connected graph. Then *Wiener index* of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} d_G(u, v)$ with the summation going over all pairs of distinct vertices of G. Similarly, the *Harary index* of G is defined as $H(G) = \frac{1}{2} \sum\limits_{u,\, v \,\in\, V(G)} \frac{1}{d_G(u,v)}$.

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Gutman et al. [7,8] were introduced the *product version of Wiener index* which is defined as $W^*(G) = \prod_{\{u,v\}\subseteq V(G)} d_G(u,v)$.

Dobrynin and Kochetova [3] and Gutman [6] independently proposed a vertex-degree-weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2}\sum_{u,v\in V(G)} (d_G(u) + d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. Hua and Zhang [10] introduced a new graph invariant named reciprocal degree distance, which can be seen as a degree-weight version of Harary index, that is, $H_A(G) = \frac{1}{2}\sum_{u,v\in V(G)} \frac{(d_G(u)+d_G(v))}{d_G(u,v)}$. Hua and Zhang [10] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex and edge-connectivity. In this sequence, the product version of *reciprocal degree distance* is defined as $H_A^*(G) = \prod_{\{u,v\}\subseteq V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}$.

The *first Zagreb index* and *second Zagerb index* are defined as $M_1(G) = \sum_{u\in V(G)} d_G(u)^2 = \sum_{uv\in E(G)} (d_G(u)+d_G(v))$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$. Similarly, the *first Zagreb coindex* and *second Zagerb coindex* are defined as

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v))$$

and

$$\overline{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v).$$

The Zagreb indices are found to have appilications in QSPR and QSAR studies as well, see [4]. Various topological indices on tensor product, strong product have been studied various authors, see [1,2,9,11–13,15,16]. In this paper, we present the upper bounds for the product version of reciprocal degree distance of the tensor product, join and strong product of two graphs in terms of other graph invariants including the Harary index and Zagreb indices.

2. Bounds for H_A^*

In this section, we obtain the lower and upper bounds for H_A^* for a connected graph.

Theorem 2.1. *For any graph G, $H_A^*(G) \leqslant DD^*(G)$ with equality if and only if $G \cong K_n$.*

Proof. Let $u, v \in V(G)$. Clearly, $\frac{1}{d_G(u,v)} \leqslant d_G(u,v)$ with equality if and only if $d_G(u,v) = 1$. Therefore

$$H_A^*(G) = \prod_{\{u,v\}\subseteq V(G)} \frac{d_G(u) + d_G(v)}{d_G(u,v)} \leqslant \prod_{\{u,v\}\subseteq V(G)} (d_G(u) + d_G(v))d_G(u,v) = DD^*(G).$$

equality holds if and only if $d_G(u,v)=1$, for any two vertices $u,v\in V(G)$. Hence $G\cong K_n$. $\square$

THEOREM 2.2. *For any graph* G, $H_A^*(G)\leqslant M_1^*(G)\overline{M}_1^*(G)$ *with equality if and only if* $G\cong K_n$.

PROOF. One can see that $\frac{1}{d_G(u,v)}\leqslant 1$ with equality if and only if $d_G(u,v)=1$, for any two vertices $u,v\in V(G)$. Therefore

$$\begin{aligned}
H_A^*(G) &= \prod_{\{u,v\}\subseteq V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)} \\
&\leqslant \prod_{\{u,v\}\subseteq V(G)} (d_G(u)+d_G(v)) \\
&= \prod_{uv\in E(G)} (d_G(u)+d_G(v)) \prod_{uv\notin E(G)} (d_G(u)+d_G(v)) \\
&= M_1^*(G)\overline{M}_1^*(G).
\end{aligned}$$

equality holds if and only if $d_G(u,v)=1$, for any two vertices $u,v\in V(G)$. Hence $G\cong K_n$. $\square$

THEOREM 2.3. *For any connected graph* G, $H_A^*(G)\leqslant M_1^*(G)\overline{M}_1^*(G)$ *with either equality if and only if* G *is regular.*

PROOF. One can observe that $2\delta\leqslant d_G(u)+d_G(v)\leqslant 2\Delta$ for two vertices u and v in G. So

$$\begin{aligned}
H_A^*(G) &= \prod_{\{u,v\}\subseteq V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)} \\
&\leqslant 2\Delta \prod_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)} \\
&= 2\Delta\, H^*(G).
\end{aligned}$$

Hence $2\delta\, H^*(G)\leqslant H_A^*(G)\leqslant 2\Delta\, H^*(G)$. This completes the proof. $\square$

THEOREM 2.4. *For any graph* G, $H_A^*(G)\geqslant \frac{\left(M_1^*(G)\overline{M}_1^*(G)\right)^2}{DD^*(G)}$ *with equality if and only if* $G\cong K_n$.

PROOF. By the definitions of H_A^* and DD^*,

$$\begin{aligned}
&H_A^*(G)DD^*(G) \\
&= \Big(\prod_{\{u,v\}\subseteq V(G)} \frac{d_G(u)+d_G(v)}{d_G(u,v)}\Big)\Big(\prod_{\{u,v\}\subseteq V(G)} (d_G(u)+d_G(v))d_G(u,v)\Big) \\
&\geqslant \Big(\prod_{\{u,v\}\subseteq V(G)} (d_G(u)+d_G(v))\Big)^2
\end{aligned}$$

$$= \Big(\prod_{uv \in E(G)} (d_G(u) + d_G(v)) \prod_{uv \notin E(G)} (d_G(u) + d_G(v)) \Big)^2$$
$$= \Big(M_1^*(G) \overline{M}_1^*(G) \Big)^2.$$

Thus $H_A^*(G) \geqslant \frac{\left(M_1^*(G)\overline{M}_1^*(G)\right)^2}{DD^*(G)}$ with equality if and only if $d_G(u, v)$ is a constant. Hence $G \cong K_n$. $\square$

THEOREM 2.5. *Let G be a connected graph. Then*
$2\delta(G)\Big(H^*(G) + H^*(\overline{G})\Big) \leqslant H_A^*(G) + H_A^*(\overline{G}) \geqslant 2\Delta(G)\Big(H^*(G) + H^*(\overline{G})\Big).$

PROOF. Consider

$$\begin{aligned} H_A^*(G) + H_A^*(\overline{G}) &= \prod_{\{u,v\} \subseteq V(G)} \frac{d_G(u) + d_G(v)}{d_G(u, v)} + \prod_{\{u,v\} \subseteq V(\overline{G})} \frac{d_G(u) + d_G(v)}{d_G(u, v)} \\ &\leqslant 2\Delta(G)H^*(G) + 2\Delta(\overline{G})H^*(\overline{G}) \\ &= 2\Delta(G)H^*(G) + 2\delta(G)H^*(\overline{G}) \\ &\leqslant 2\Delta(G)H^*(G) + 2\Delta(G)H^*(\overline{G}) \\ &= 2\Delta(G)\Big(H^*(G) + H^*(\overline{G})\Big). \end{aligned}$$

On the other hand,

$$\begin{aligned} H_A^*(G) + H_A^*(\overline{G}) &\geqslant 2\delta(G)H^*(G) + 2\delta(\overline{G})H^*(\overline{G}) \\ &= 2\delta(G)H^*(G) + 2\Delta(G)H^*(\overline{G}) \\ &= 2\delta(G)\Big(H^*(G) + H^*(\overline{G})\Big). \end{aligned}$$

$\square$

3. Product graphs

In this section, we obtain the upper bounds for H_A^* of composition, Cartesian product and double of a graphs.

REMARK 3.1. (Arithmetic Geometric Inequality) Let $a_1, a_2, \ldots, a_n$ be non negative n numbers. Then

$$\sqrt[n]{a_1 a_2 \ldots a_n} \leqslant \tfrac{a_1 + a_2 + \ldots + a_n}{n}.$$

3.1. Composition. The *composition* of G and H, denoted by $G[H]$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever $g_1 g_2$ is an edge in G or, $g_1 = g_2$ and $h_1 h_2$ is an edge in H. In this section, we obtain the product version of reciprocal degree distance of the composition of two graphs.

THEOREM 3.1. *Let G_i be the connected graphs with n_i vertices and m_i edges, $i = 1, 2$. Then* $H_A^*(G_1[G_2]) \leqslant \left(\frac{1}{n_1 n_2}\right)^{3n_1 n_2} \Big[n_2 m_1 (n_2^2 + 2m_2 - n_2) + \frac{n_1}{2}(2M_1(G_2) +$

$\overline{M}_1(G_2))\Big]^{n_1n_2}\Big[n_2^2\,H_A(G_1)+4m_2H(G_1)\Big]^{n_1n_2}\Big[n_2^2(n_2-1)H_A(G_1)+2\,H(G_1)(M_1(G_2)$
$+\,\overline{M}_1(G_2))\Big]^{n_1n_2}$.

Proof. Let $V(G_1)=\{u_1,u_2,\ldots,u_{n_1}\}$ and let $V(G_2)=\{v_1,v_2,\ldots,v_{n_2}\}$. Let x_{ij} denote the vertex (u_i,v_j) of $G_1[G_2]$. The degree of the vertex x_{ij} in $G_1[G_2]$ is $n_2d_{G_1}(u_i)+d_{G_2}(v_j)$. By the definition of H_A^*

$$H_A^*(G_1[G_2])=\prod_{x_{ij},x_{k\ell}\in V(G_1[G_2])}\frac{d_{G_1[G_2]}(x_{ij})+d_{G_1[G_2]}(x_{k\ell})}{d_{G_1[G_2]}(x_{ij},x_{k\ell})}$$

$$=\prod_{i=0}^{n_1-1}\prod_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{d_{G_1[G_2]}(x_{ij})+d_{G_1[G_2]}(x_{i\ell})}{d_{G_1[G_2]}(x_{ij},x_{i\ell})}\times\prod_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod_{j=0}^{n_2-1}\frac{d_{G_1[G_2]}(x_{ij})+d_{G_1[G_2]}(x_{kj})}{d_{G_1[G_2]}(x_{ij},x_{kj})}$$

$$\times\prod_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{d_{G_1[G_2]}(x_{ij})+d_{G_1[G_2]}(x_{k\ell})}{d_{G_1[G_2]}(x_{ij},x_{k\ell})}.$$

We shall calculate the above sums are separately. First we compute $\prod_{i=0}^{n_1-1}\prod_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{d_{G_1[G_2]}(x_{ij})+d_{G_1[G_2]}(x_{i\ell})}{d_{G_1[G_2]}(x_{ij},x_{i\ell})}$.

$$\begin{aligned}
&\prod_{i=0}^{n_1-1}\prod_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{d_{G_1[G_2]}(x_{ij})+d_{G_1[G_2]}(x_{i\ell})}{d_{G_1[G_2]}(x_{ij},x_{i\ell})}\\
=\;&\prod_{i=0}^{n_1-1}\prod_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{2n_2d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G'}(v_\ell)}{d_{G_2}(v_j,v_\ell)}\\
\leqslant\;&\left[\frac{\frac{1}{2}\sum_{i=0}^{n_1-1}\sum_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{2n_2d_{G_1}(u_i)+d_{G_2}(v_j)+d_{G_2}(v_\ell)}{d_{G_2}(v_j,v_\ell)}}{n_1n_2}\right]^{n_1n_2}\quad\text{by Remark 3.1}\\
=\;&\left[\frac{S_1}{n_1n_2}\right]^{n_1n_2},
\end{aligned}$$

where

$$\begin{aligned}
S_1\;&=\;\frac{1}{2}\sum_{i=0}^{n_1-1}\sum_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{2n_2d_{G_1}(u_i)}{d_{G_2}(v_j,v_\ell)}+\frac{1}{2}\sum_{i=0}^{n_1-1}\sum_{\substack{j,\ell=0\\ j\neq\ell}}^{n_2-1}\frac{d_{G_2}(v_j)+d_{G_2}(v_\ell)}{d_{G_2}(v_j,v_\ell)}\\
&=\;n_2\sum_{i=0}^{n_1-1}d_{G_1}(u_i)\left(\sum_{v_jv_\ell\in E(G_2)}\frac{1}{d_{G_2}(v_j,v_\ell)}+\sum_{v_jv_\ell\notin E(G_2)}\frac{1}{d_{G_2}(v_j,v_\ell)}\right)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{2}\sum_{i=0}^{n_1-1}\left(\sum_{v_jv_\ell\in E(G_2)}\frac{d_{G_2}(v_j)+d_{G_2}(v_\ell)}{d_{G_2}(v_j,v_\ell)}+\sum_{v_jv_\ell\notin E(G_2)}\frac{d_{G_2}(v_j)+d_{G_2}(v_\ell)}{d_{G_2}(v_j,v_\ell)}\right)\\
= \; & 2n_2m_1\left(\sum_{v_j\in V(G_2)}d_{G_2}(v_j)+\sum_{v_j\in V(G_2)}\frac{1}{2}(m-d_{G_2}(v_j)-1)\right)\\
& +\frac{1}{2}\sum_{i=0}^{n_1-1}\left(\sum_{v_jv_\ell\in E(G_2)}(d_{G_2}(v_j)+d_{G_2}(v_\ell))+\sum_{v_jv_\ell\notin E(G_2)}\frac{d_{G_2}(v_j)+d_{G_2}(v_\ell)}{2}\right),\\
& \text{since each row}\\
& \text{induces a copy of } G_2 \text{ and } d_{G_1[G_2]}(x_{ij},x_{i\ell})=\begin{cases}1, & \text{if } v_jv_\ell\in E(G_2)\\ 2, & \text{if } v_jv_\ell\notin E(G_2).\end{cases}\\
= \; & n_2m_1(n_2^2+2m_2-n_2)+\frac{n_1}{2}(2M_1(G_2)+\overline{M}_1(G_2)).
\end{aligned}
$$

Next we obtain $\prod\limits_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod\limits_{j=0}^{n_2-1}\frac{d(x_{ij})+d(x_{kj})}{d_{G_1[G_2]}(x_{ij},x_{kj})}$.

$$
\begin{aligned}
\prod_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod_{j=0}^{n_2-1}\frac{d(x_{ij})+d(x_{kj})}{d_{G_1[G_2]}(x_{ij},x_{kj})} &= \prod_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod_{j=0}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))+2d(v_j)}{d_{G_1}(u_i,u_k)},\\
& \quad \text{since the distance between a pair of vertices in a column is}\\
& \quad \text{same as the distance between the corresponding vertices of}\\
& \quad \text{other column}\\
&\leqslant \left[\frac{S_2}{n_1n_2}\right]^{n_1n_2}, \text{ by Remark 3.1}
\end{aligned}
$$

where

$$
\begin{aligned}
S_2 &= \frac{1}{2}\sum_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))+2d(v_j)}{d_{G_1}(u_i,u_k)},\\
&= \frac{1}{2}\sum_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))}{d_{G_1}(u_i,u_k)}+\frac{1}{2}\sum_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\sum_{j=0}^{n_2-1}\frac{2d(v_j)}{d_{G_1}(u_i,u_k)}\\
&= n_2^2H_A(G_1)+4m_2H(G_1).
\end{aligned}
$$

Finally, we compute $\prod\limits_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod\limits_{\substack{j,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{d(x_{ij})+d(x_{k\ell})}{d_{G_1[G_2]}(x_{ij},x_{k\ell})}$. Since $d_{G_1[G_2]}(x_{ij},x_{k\ell})=d_{G_1}(u_i,u_k)$ for all j and k and further the distance between the corresponding vertices of the layers is counted in previous sum. Hence

$$\prod_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\prod_{\substack{j,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{d(x_{ij})+d(x_{k\ell})}{d_{G_1[G_2]}(x_{ij},x_{k\ell})} \leqslant \left[\frac{\frac{1}{2}\sum\limits_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\sum\limits_{\substack{j,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{n_2d(u_i)+d(v_j)+n_2d(u_k)+d(v_\ell)}{d_{G_1}(u_i,u_k)}}{n_1n_2}\right]^{n_1n_2}$$

$$= \left[\frac{S_3}{n_1n_2}\right]^{n_1n_2}, \text{ by Remark 3.1}$$

where

$$\begin{aligned} S_3 &= \frac{1}{2}\sum_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{n_2(d(u_i)+d(u_k))}{d_{G_1}(u_i,u_k)}+\frac{1}{2}\sum_{\substack{i,k=0\\ i\neq k}}^{n_1-1}\sum_{\substack{j,\ell=0\\ j\neq \ell}}^{n_2-1}\frac{d(v_j)+d(v_\ell)}{d_{G_1}(u_i,u_k)}, \\ &= n_2^2(n_2-1)H_A(G_1)+2\,H(G_1)(M_1(G_2)+\overline{M}_1(G_2)). \end{aligned}$$

Combine the aboves we get the desired result. □

3.2. Cartesian product. The *Cartesian product,* $G\,\square\,H$, of graphs G and H has the vertex set $V(G\,\square\,H)=V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G\,\square\,H$ if $u=v$ and $xy\in E(H)$ or, $uv\in E(G)$ and $x=y$. To each vertex $u\in V(G)$, there is an isomorphic copy of H in $G\,\square\,H$ and to each vertex $v\in V(H)$, there is an isomorphic copy of G in $G\,\square\,H$. The following lemma follows from the structure of $G\square H$.

LEMMA 3.1. *Let G and H be two connected graphs with n_1 and n_2 vertices, respectively. Then*

(*i*) *The distance between two vertices of $G\square H$ is given by*

$$d_{G\square H}((u_i,v_j),(u_p,v_q))=d_G(u_i,u_p)+d_H(v_j,v_q).$$

(*ii*) *The degree of a vertex (u_i,v_j) of $G\square H$ is $d_G(u_i)+d_H(v_j)$.*

Now we obtain the upper bound for product version of reciprocal degree distance of Cartesian product of two connected graphs.

THEOREM 3.2. *Let G_i be the connected graphs with n_i vertices and m_i edges, $i=1,2$. Then*

$$H_A^*(G_1\square G_2)\leqslant\left[\tfrac{n_2H_A(G_1)+n_1H_A(G_2)+4m_1H(G_2)+4m_2H(G_1)}{n_1n_2}\right]^{n_1n_2}.$$

PROOF. By the definition of H_A^*,

$$H_A^*(G_1\square G_2)=\prod_{(u,x),(v,y)\in V(G_1\square G_2)}\frac{d_{G_1\square G_2}((u,x))+d_{G_1\square G_2}((v,x))}{d_{G_1\square G_2}((u,x),(v,y))}.$$

By Lemma 3.1, we have

$$H_A^*(G_1\square G_2)=\prod_{(u,x),(v,y)\in V(G_1\square G_2)}\frac{d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(v)+d_{G_2}(y)}{d_{G_1}(u,v)+d_{G_2}(x,y)}$$

$$\leqslant \left[\frac{\sum\limits_{(u,x),(v,y)\in V(G_1\Box G_2)} \frac{d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(v)+d_{G_2}(y)}{d_{G_1}(u,v)+d_{G_2}(x,y)}}{n_1 n_2}\right]^{n_1 n_2} \quad \text{by Remark 3.1}$$
$$= \left[\frac{A}{n_1 n_2}\right]^{n_1 n_2},$$

where

$$\begin{aligned} A &= \sum_{(u,x),(v,y)\in V(G_1\Box G_2)} \frac{d_{G_1}(u)+d_{G_2}(x)+d_{G_1}(v)+d_{G_2}(y)}{d_{G_1}(u,v)+d_{G_2}(x,y)} \\ &\leqslant \sum_{r\in V(G_1)} \sum_{x,y\in V(G_2)} \Big(\frac{d_{G_1}(u)+d_{G_1}(v)}{d_{G_2}(x,y)+t} + \frac{d_{G_2}(x)+d_{G_2}(y)}{d_{G_2}(x,y)+t}\Big) \\ &\quad + \sum_{u,v\in V(G_1)} \sum_{z\in V(G_2)} \Big(\frac{d_{G_1}(u)+d_{G_1}(v)}{d_{G_1}(u,v)+t} + \frac{d_{G_2}(x)+d_{G_2}(y)}{d_{G_1}(u,v)+t}\Big) \\ &= n_2 H_A(G_1) + n_1 H_A(G_2) + 4m_1 H(G_2) + 4m_2 H(G_1). \end{aligned}$$

Hence

$$H_A^*(G_1\Box G_2) \leqslant \left[\frac{n_2 H_A(G_1) + n_1 H_A(G_2) + 4m_1 H(G_2) + 4m_2 H(G_1)}{n_1 n_2}\right]^{n_1 n_2}.$$

□

3.3. Double graph. Let us denote the double graph of a graphG by G^*, which is constructed from two copies of G in the following manner. Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots, v_n\}$, and the vertices of G^* are given by the two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G^*)$. The *double graph* G^* includes the initial edge set of each copies of G, and for any edge $v_iv_j \in E(G)$, two more edges x_iy_j and x_jy_i are added. Now we obtain the H_A^* of double graph.

THEOREM 3.3. *Let G be a connected graph. Then*

$$H_A^*(G^*) \leqslant \frac{\Big(H_A(G)\Big)^{6n}\Big(M_1(G)\Big)^{2n}}{n^{8n}}.$$

PROOF. From the structure of the double graph, the distance between two vertices of G^* are given as follows.
$d_{G^*}(x_i, x_j) = d_G(x_i, x_j),\ i, j \in \{1, 2, \ldots, n\}$.
$d_{G^*}(x_i, y_j) = d_G(x_i, x_j),\ i, j \in \{1, 2, \ldots, n\}$.
$d_{G^*}(x_i, y_i) = 2,\ i \in \{1, 2, \ldots, n\}$.
Similarly, the degree of the vertex of G^* is

$$d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(x_i), i \in \{1, 2, \ldots, n\}.$$

$$H_A^*(G^*) = \prod_{1\leqslant i<j\leqslant n} \frac{d_{G^*}(v_i) + d_{G^*}(v_j)}{d_{G^*}(v_i, v_j)}$$

$$
\begin{aligned}
&= \prod_{1\leqslant i<j\leqslant n} \frac{d_{G^*}(x_i)+d_{G^*}(x_j)}{d_{G^*}(x_i,x_j)} \times \prod_{1\leqslant i<j\leqslant n} \frac{d_{G^*}(y_i)+d_{G^*}(y_j)}{d_{G^*}(y_i,y_j)} \\
&\quad \times \prod_{i,j=1,i\neq j}^{n} \frac{d_{G^*}(x_i)+d_{G^*}(y_j)}{d_{G^*}(x_i,y_j)} \times \prod_{i=1}^{n} \frac{d_{G^*}(x_i)+d_{G^*}(y_i)}{d_{G^*}(x_i,y_i)} \\
&\leqslant \left[\frac{\sum\limits_{1\leqslant i<j\leqslant n} \frac{2d_G(x_i)+2d_G(x_j)}{d_G(x_i,x_j)}}{2n}\right]^{2n} \left[\frac{\sum\limits_{1\leqslant i<j\leqslant n} \frac{2d_G(x_i)+2d_G(x_j)}{d_G(x_i,x_j)}}{2n}\right]^{2n} \\
&\quad \left[\frac{\sum\limits_{i,j=1,i\neq j}^{n} \frac{2d_G(x_i)+2d_G(x_j)}{d_G(x_i,x_j)}}{2n}\right]^{2n} \left[\frac{\sum\limits_{x_i\in V(G)} \frac{2d_G(x_i)+2d_G(x_i)}{2}}{2n}\right]^{2n} \text{ by Remark 3.1} \\
&= \left(\frac{H_A(G)}{n}\right)^{2n} \left(\frac{H_A(G)}{n}\right)^{2n} \left(\frac{2H_A(G)}{n}\right)^{2n} \left(\frac{M_1(G)}{2n}\right)^{2n} \\
&= \frac{\Big(H_A(G)\Big)^{6n}\Big(M_1(G)\Big)^{2n}}{n^{8n}}.
\end{aligned}
$$

□

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