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CHROMATIC EXCELLENCE IN FUZZY GRAPHS

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Abstract. Let G be a simple fuzzy graph. A family $\Gamma^f = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of fuzzy sets on a set V is called k-fuzzy colouring of $V = (V, \sigma, \mu)$ if
i) $\cup\Gamma^f = \sigma$,
ii) $\gamma_i \cap \gamma_j = \emptyset$,
iii) for every strong edge (x, y)(i.e., $\mu(xy) > 0$) of G

$$min\{\gamma_i(x), \gamma_i(y)\} = 0, (1 \leqslant i \leqslant k).$$

The minimum number of k for which there exists a k-fuzzy colouring is called the fuzzy chromatic number of G denoted as $\chi^f(G)$. Then Γ^f is the partition of independent sets of vertices of G in which each sets has the same colour is called the fuzzy chromatic partition. A graph G is called the χ^f-excellent if every vertex of G appears as a singleton in some χ^f-partitions of G. This paper aims at the study of the new concept namely Chromatic excellence in fuzzy graphs. Fuzzy corona and fuzzy independent sets is defined and studied. We explain these new concepts through examples.

1. Introduction

Many practical problems such as scheduling, allocation, network problems etc., can be modeled as coloring problems and hence coloring is one of the most studied areas in the research of graph theory. A large number of variations in coloring of graphs is available in literature. With the emergence of fuzzy set theory and fuzzy graph theory, most of the real situations are modeled with more precision and flexibility than their classical counterparts. A fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The concept of fuzzy sets and fuzzy relations was introduced by L.A.Zadeh in 1965 [**1**] and further studied [**2**]. It was Rosenfeld[**5**] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. The concepts of fuzzy trees, blocks, bridges and cut nodes in

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fuzzy graph has been studied [**3**]. Computing chromatic sum of an arbitrary graph introduced by Kubica [1989] is known as NP-complete problem. Graph coloring is the most studied problem of combinatorial optimization. As an advancement fuzzy coloring of a fuzzy graph was defined by authors Eslahchi and Onagh in 2004, and later developed by them as Fuzzy vertex coloring [**4**] in 2006. This fuzzy vertex coloring was extended to fuzzy total coloring in terms of family of fuzzy sets by Lavanya and Sattanathan [**6**]. In this paper we are introducing “Chromatic excellence in fuzzy graphs”.

2. Preliminaries

Definition 2.1. *A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ where for all $u, v \in V$, we have $\mu(u, v) \leqslant \sigma(u) \wedge \sigma(v)$.*

Definition 2.2. *The order p and size q of a fuzzy graph $G = (\sigma, \mu)$ are defined to be $p = \sum\limits_{x\in V} \sigma(x)$ and $q = \sum\limits_{xy\in E} \mu(xy)$.*

Definition 2.3. *The degree of vertex u is defined as the sum of the weights of the edges incident at u and is denoted by $d(u)$.*

Definition 2.4. *The union of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \cup G_2 : ((\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2))$ defined by*

$$\begin{aligned}
(\sigma_1 \cup \sigma_2)(u) &= \begin{cases} \sigma_1(u), \ \textit{if } u \in V_1 - V_2 \textit{ and} \\ \sigma_2(u) \ \textit{if } u \in V_2 - V_1 \end{cases} \\
(\mu_1 \cup \mu_2)(uv) &= \begin{cases} \mu_1(uv), \ \textit{if } uv \in E_1 - E_2 \textit{ and} \\ \mu_2(uv) \ \textit{if } uv \in E_2 - E_1 \end{cases}
\end{aligned}$$

Definition 2.5. *The join of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 + G_2 : ((\sigma_1 + \sigma_2, \mu_1 + \mu_2))$ defined by*

$$\begin{aligned}
(\sigma_1 + \sigma_2)(u) &= (\sigma_1 \cup \sigma_2)(u) \forall u \in V_1 \cup V_2 \\
(\mu_1 + \mu_2)(uv) &= \begin{cases} (\mu_1 \cup \mu_2)(uv) \ \textit{if } uv \in E_1 \cup E_2 \textit{ and} \\ \sigma_1(u) \wedge \sigma_2(v) \ \textit{if } uv \in E'. \end{cases}
\end{aligned}$$

where E' is the set of all edges joining the nodes of V_1 and V_2.

Definition 2.6. *The cartesian product of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ on $G* : (V, E)$ where $V = V_1 \times V_2$ and $E = \{((\sigma_1, \sigma_2), (\mu_1, \mu_2))/u_1 = v_1, u_2v_2 \in E_2 or u_2 = v_2, u_1v_1 \in E_1\}$ with*

$$\begin{aligned}
(\sigma_1 \times \sigma_2)(u_1, v_1) &= \sigma_1(u_1) \wedge \sigma_2(u_2) forall (u_1, u_2) \in V_1 \times V_2 \\
(\mu_1 \times \mu_2)((u_1, u_2)(v_1, v_2)) &= \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2, v_2), \ \textit{if } u_1 = v_1 \textit{ and } u_2v_2 \in E_2 \\ \sigma_2(u_2) \wedge \mu_1(u_1, v_1), \ \textit{if } u_2 = v_2 \textit{ and } u_1v_1 \in E_1 \end{cases}
\end{aligned}$$

Definition 2.7. *Let G be a fuzzy graph. A subset S of G is said to be fuzzy independent set of G, if there exists no $uv \in S$ such that $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$. The maximum cardinality of such fuzzy independent set is called fuzzy independence number and it is denoted by β_0^f.*

3. Main Definitions and Results

Definition 3.1. *Let G be a fuzzy graph. A family $\Gamma^f = \{\gamma_1.\gamma_2, \ldots, \gamma_k\}$ of fuzzy sets on a set V is called k-fuzzy coloring of $V = (V, \sigma, \mu)$ if*
(i) $\cup \Gamma^f = \sigma$,
(ii) $\gamma_i \cap \gamma_j = \emptyset$,
(iii) *for every strong edge (x, y)(i.e., $\mu(xy) > 0$) of G is*

$$min\{\gamma_i(x), \gamma_i(y)\} = 0, (1 \leqslant i \leqslant k).$$

The minimum number of k for which there exists a k-fuzzy colouring is called the fuzzy chromatic number of G denoted as $\chi^f(G)$.

Definition 3.2. *Γ^f is the partition of independent sets of vertices of G in which each sets has the same colour is called the fuzzy chromatic partition.*

Definition 3.3. *A vertex $v \in V(G)$ is called χ^f-good if $\{v\}$ belongs to a some Γ^f-partition. Otherwise v is said to be Γ^f-bad vertex.*

Definition 3.4. *A graph is called χ^f-excellent fuzzy graph if every vertex of G is χ^f-good.*

Definition 3.5. *A graph G is said to be Γ^f* ***- commendable*** *fuzzy graph if the number of Γ^f-good vertices is greater than the number of Γ^f-bad vertices.*

A graph G is said to be Γ^f ***- fair*** *fuzzy graph if the number of Γ^f-good vertices is equal to the number of Γ^f-bad vertices.*

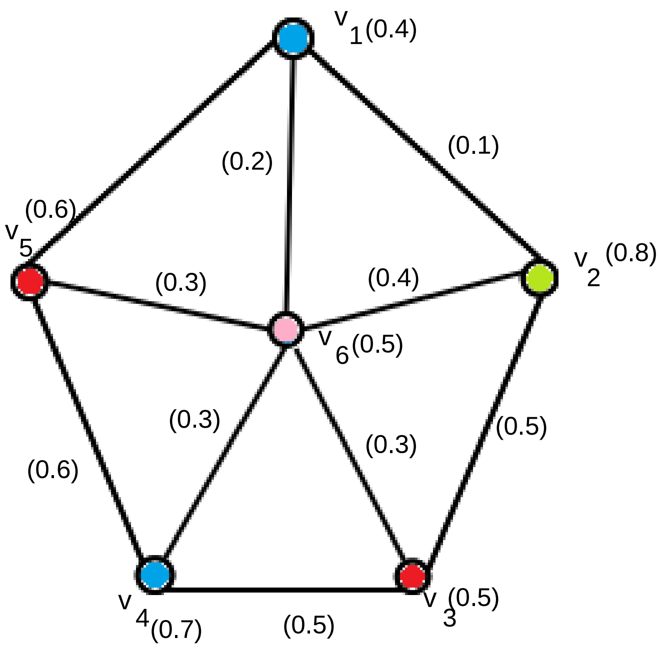
A graph G is said to be Γ^f ***- poor*** *fuzzy graph if the number of Γ^f-good vertices is less than the number of Γ^f-bad vertices.*

Definition 3.6. *Let $\{\Gamma_1^f, \Gamma_2^f, \ldots, \Gamma_k^f\}$ be a χ^f-partitions of G. Let $v \in V$, then*
(i) *v is χ^f* ***-fixed*** *if $\{v\} \in \Gamma_i^f$ for all i,$(1 \leqslant i \leqslant k)$*
(ii) *v is χ^f* ***-free*** *if for some $i, j, i \neq j$, $\{v\} \in \Gamma_i^f$ and $\{v\} \notin \Gamma_j^f$*
(iii) *v is χ^f* ***totally free*** *if $\{v\} \notin \Gamma_i^f$ for all i,$1 \leqslant i \leqslant k$.*

Definition 3.7. *G is fuzzy chromatic excellent if for every vertex of $v \in V(G)$, there exists a fuzzy chromatic partition Γ^f such that $\{v\} \in \Gamma^f$.*

Remark 3.1. (1) *K_n is χ^f-excellent*
(2) *C_{2n} is not χ^f-excellent but $C_{2n+1}(n \geqslant 1)$ is χ^f-excellent.*
(3) *$W_{2n}(n \geqslant 2)$ is χ^f-excellent.*

Example 3.1.



The fuzzy colouring $\Gamma^f = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$

$$\gamma_1(v_i) = \begin{cases} 0.4 & i = 1 \\ 0.7 & i = 4 \\ 0 & \text{otherwise} \end{cases}$$

$\gamma_2(v_i) = 0.8 \qquad i = 2$

$$\gamma_3(v_i) = \begin{cases} 0.5 & i = 3 \\ 0.6 & i = 5 \\ 0 & \text{otherwise} \end{cases}$$

$\gamma_4(v_i) = 0.5 \qquad i = 6$. Hence $\chi^f(G) = 4$. The fuzzy chromatic partitions are

$$\begin{aligned} \Gamma_1^f &= \{\{v_2\}, \{v_6\}, \{v_1, v_4\}, \{v_3, v_5\}\} \\ \Gamma_2^f &= \{\{v_1\}, \{v_6\}, \{v_2, v_4\}, \{v_3, v_5\}\} \\ \Gamma_3^f &= \{\{v_3\}, \{v_6\}, \{v_1, v_4\}, \{v_2, v_5\}\} \\ \Gamma_4^f &= \{\{v_4\}, \{v_6\}, \{v_3, v_4\}, \{v_2, v_5\}\} \\ \Gamma_5^f &= \{\{v_5\}, \{v_6\}, \{v_1, v_3\}, \{v_2, v_4\}\} \end{aligned}$$

Every vertex of G is appears in singleton in χ^f partition, hence above graph is χ^f-excellent.

EXAMPLE 3.2.

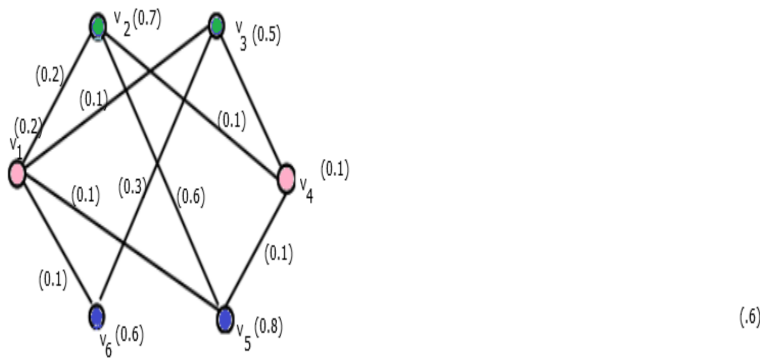


Fig 2: Non fuzzy chromatic excellent

The χ^f-partition is $\Gamma_1^f = \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_5, v_6\}\}$, no vertex in given graph is appears in singleton. Therefore the given graph is not χ^f-excellent.

DEFINITION 3.8. *A fuzzy graph G is β_0^f-excellent if every vertex belongs to a maximum fuzzy independent set.*

REMARK 3.2. *β_0^f-excellence and χ^f-excellence are unrelated proerties. That is, a graph may be β_0^f-excellent but not χ^f-excellent and conversely.*

REMARK 3.3. (1) *P_{2n} is not χ^f-excellent but not β_0^f-excellent.*

(2) *$K_{1,n}$ is neither χ^f-excellent and β_0^f-excellent.*

(3) *Graph in fig(i) is χ^f-excellent but not β_0^f-excellent.*

DEFINITION 3.9. *Let G_1 and G_2 be two fuzzy graphs. Then the fuzzy cororna $G_1 o G_2$ is denoted by G^{f+} as the fuzzy graph if taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining i^{th} vertex of G_1 to every vertex in i^{th} copy of G_2 such that $\mu(u_i, v_i) \leqslant \sigma(u_i) \wedge \sigma(v_i)$ for all $u_i \in i^{th}$ copy of G_2. If $G_2 = K_1$ then $G_1 o K_1$ is denoted by G_1^{f+}.*

DEFINITION 3.10. *Let G be a fuzzy graph. The Mycielskian of G is the fuzzy graph $\mu^f(G)$ with vertex set equal to $V \cup V' \cup \{u\}$ where $V' = \{x', x \in V\}$ and the edge set $E \cup \{xy', x'y : xy \in E\}$ and $\mu(xy') \leqslant \sigma(x) \wedge \sigma(y') \& \mu(x'y) \leqslant \sigma(x') \wedge \sigma(y)$. The vertex x' is called the twin of the vertex and the vertex u is called the root of $\mu^f(G)$.*

PROOSITION 3.1. *Let $\chi^f(G) = 2$. (Then G is a bipartite fuzzy graph). Let $|V(G)| \geqslant 3$. Then G is not χ^f - excellent.*

PROOF. Let us assume that G is χ^f excellent fuzzy graph. Since $\chi^f(G) = 2$, then there exists a chromatic partition $\Gamma^f = \{V_1, V_2\}$ of G such that $V_1 = \{v\}$.

Therefore $\langle V_2 \rangle = \langle G - \{v\} \rangle$ is totally disconnected graph. Clearly, $|V_2| \geqslant 2$ and $v \in V_1$ is adjacent to some vertex of V_2 such that $\mu(vu) \leqslant \sigma(v) \wedge \sigma(u)$ for some $u \in V_2$.

Let $w \in V_2$ be such that $wv \in E$ such that $\mu(vw) \leqslant \sigma(v) \wedge \sigma(w)$. let $w_1 \neq w \in V_2$. Since G is χ^f-excellent and $\chi^f(G) = 2$, then there exists a chromatic partition $\Gamma_1^f = \{\{w_1\}, V_3\}$. Therefore $v, w \in V_3$ which is a contradiction to $vw \in E(G)$. Hence G is not χ^f- excellent. □

PROOSITION 3.2. *χ^f- excellent graphs has no fuzzy isolates.*

PROOF. Suppose G is a χ^f-excellent graph, which has an isolate v. If $G = \bar{k_n}(n \geqslant 2)$, then $\chi^f(G) = 1$ and no vertex in G appears as a singleton in χ^f-partition of G. Therefore $G = \bar{k_n}$. Hence $\chi^f(G) \geqslant 2$, then there exists a chromatic partition

$$\Gamma^f = \{\{v\}, V_2, V_3, \ldots, V_\chi^f(G)\}$$

Therefore $deg(v) \geqslant \chi^f(G) - 1 \geqslant 1$. But v is an isolate, which is a contradiction. Hence any χ^f-excellent graph has no fuzzy isolates. □

REMARK 3.4. *If G is χ^f-excellent and $G \neq K_2$, then $\delta^f(G) \geqslant 2$*

PROOSITION 3.3. *Let G_1 and G_2 be two fuzzy graphs. Then $G_1 \cup G_2$ is not χ^f-excellent.*

PROOF. If $V(G_1)(orV(G_2))$ is a singleton, then clearly $G_1 \cup G_2$ is not χ^f-excellent. Let $|V(G_1)| \geqslant 2$, $|V(G_2)| \geqslant 2$.

Case (i). Let $\chi^f(G_1) = \chi^f(G_2) = k$

Suppose there exists a χ^f-partition of $G_1 \cup G_2$ such that $\{v\}$ is an element in the χ^f-partition for some $v \in V(G_1)$. Let $\Gamma^f = \{\{v\}, V_2, V_3, \ldots, V_k\}$ be a χ^f-partition of $G_1 \cup G_2$. Therefore $\{V_2 - V(G_1), V_3 - V(G_1), \ldots, V_k - V(G_1)$ be a proer color partition of G_2, and $\chi^f(G_2) \leqslant k - 1$, which is a contradiction that $\chi^f(G_2) = k$.

Similarly, we can show that there is no vertex $\{v\} \in V(G_2)$, can not appears as any χ^f-partition of $G_1 \cup G_2$. Hence $G_1 \cup G_2$ is not χ^f-excellent.

Case (ii). Let $\chi^f(G_1) \neq \chi^f(G_2)$. Let us assume that $\chi^f(G_2) = k$, then $\chi^f(G_1) \cup \chi^f(G_2) = k$. Suppose there exists a χ^f-partition of $G_1 \cup G_2$ such that $\{v\}$ is an element of the partition for some $v \in V(G_1)$. Let $\Gamma^f = \{\{v\}, V_2, V_3, \ldots, V_k\}$ be a χ^f-partition of $G_1 \cup G_2$. Then $\{V_2 - V(G_1), V_3 - V(G_1), \ldots, V_k - V(G_1)$ be a proer χ^f-partition of G_2 and hence $\chi^f(G_2) \leqslant k - 1$, which is a contradiction. Therefore $G_1 \cup G_2$ is not χ^f-excellent. □

COROLLARY 3.1. *If G is χ^f-excellent then G is a connected fuzzy graph.*

REMARK 3.5. *If G_1 and G_2 have same chromatic number, then no vertex of $G_1 \cup G_2$ can appear as a singleton in any χ^f-partition of $G_1 \cup G_2$.*

If $\chi^f(G_1) \leqslant \chi^f(G_2)$, then we know that no vertex in G_1 can appear as a singleton in χ^f-partition of $G_1 \cup G_2$. But a vertex of G_2 may appear as a singleton in any χ^f-partition of $G_1 \cup G_2$.

REMARK 3.6. *$P_n, (n \geqslant 3)$ is not χ^f-excellent but it is an induced subgraph of an odd cycle which is χ^f-excellent. P_n is an induced subgraph of C_{n+1} of n is even and C_{n+2} if n is odd).*

PROOSITION 3.4. *$K_{1,n}$ is not χ^f-excellent but it is an induced subgraph of a χ^f-excellent graph.*

PROOF. Let $V(K_{1,n}) = \{u, u_1, u_2, \ldots, u_n\}$ where $u_1, u_2, \ldots, u_n$ are independent (i.e., there is no edge $\mu(u_i, u_j) \leqslant \sigma(u_i) \wedge \sigma(u_j)$). Add vertices $v_1, v_2, \ldots, v_n$ to $K_{1,n}$ such that $\mu(v_i, v_j) \leqslant \sigma(v_i) \wedge \sigma(v_j)$ for all $i \neq j, 1 \leqslant j \leqslant n$, and also $\mu(v_i, u_k) \leqslant \sigma(v_i) \wedge \sigma(u_k)$ for all $i \neq k, 1 \leqslant k \leqslant n$). Let G be the resulting graph

$\delta^f(G) = n$. Since G contains K_n, then $\chi^f(G) \geqslant n$. Suppose let $\chi^f(G) = n$ and $\Gamma^f = \{V_1, V_2, \ldots, V_n\}$ be a χ^f-partition of G. Without loss of generality, $v_i \in V_i, 1 \leqslant i \leqslant n, |V_i| \leqslant 2$, Since V_i can contain u_i (or) u but not both.

Therefore $\sum_{i=1}^{n} |V_i| = |V(G)| = 2n + 1$,which is a contradiction. Therefore $\chi^f(G) = n + 1$.

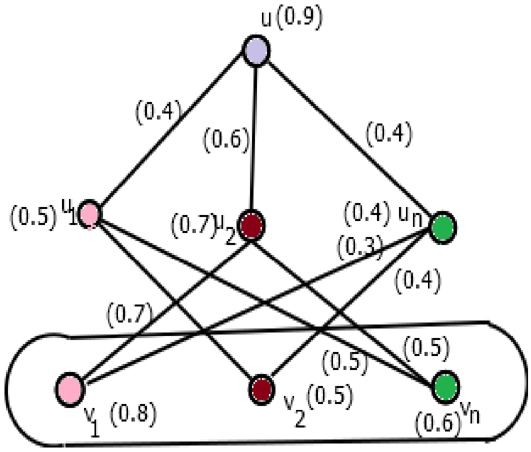


Fig 3 : Fuzzy Chromatic Excellent

Let

$$\begin{aligned}
\Gamma_1^f &= \{\{v_1, u_1\}, \ldots, \{v_n, u_n\}, \{u\}\}. \\
\Gamma_i^f &= \{\{v_i, u\}, \{v_1, u_1\}, \ldots, \{v_{i-1}, u_{i-1}\}, \{v_{i+1}, u_{i+1}\}, \ldots, \{v_n, u_n\}, \{u_i\}\}. \\
\Gamma_i^f &= \{\{u_1, u_2, \ldots, u_n\}, \{v_1, u\}, \{v_2\}, \ldots, \{v_i\}, \ldots, \{v_n\}\}.
\end{aligned}$$

From these we see that $K_{1,n}$ is an induced subgraph of G. □

PROOSITION 3.5. *$G = P_n \cup K_1$ can be embedded in a χ^f-excellent graph.*

PROOF. Let us add s vertices to $P_n \cup K_1$ such that the total number of vertices $s + n + 1$ is odd and we get the cycle.Since we know that the any odd cycle is χ^f-excellent. Hence we get the result. □

Proosition 3.6. *If G is χ^f-excellent then $\mu^f(G)$ is χ^f-excellent.*

Proof. Let us take $V(G) = \{u_1, u_2, \ldots, u_n\}$ and

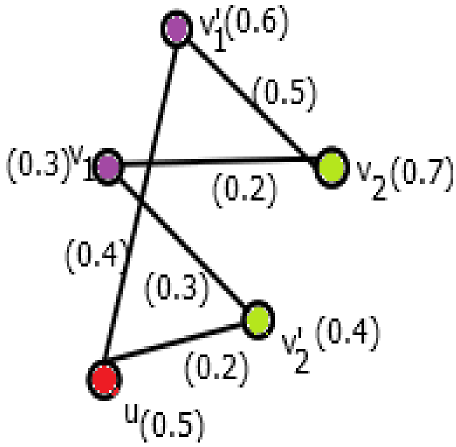
$$V(\mu^f(G)) = \{u_1, u_2, \ldots, u_n, u_1^{'}, u_2^{'}, \ldots, u_n^{'}, v\}.$$

Let G is χ^f-excellent, and $\chi^f(G) = k$ then $\Gamma^f = \{\{u_i\}, V_2, V_3, \ldots, V_k\}$ be a χ^f-partition of G. Then clearly $\chi^f(\mu^f(G)) = \chi^f(G) + 1$ and hence $\chi^f(\mu^f(G)) = k+1$. The following partitions are the χ^f-partitions of $\mu^f(G)$

$$\begin{aligned}
\Gamma_i^f &= \{\{u_i\}, V_2 \cup \{v\}, V_3, V_4, \ldots, V_k, \{u_1^{'}, u_2^{'}\}, \ldots, u_n^{'}\}\} \\
\Gamma_i^{f'} &= \{\{u_i^{'}\}, V_2 \cup V_2^{'}, \ldots, V_k \cup V_k^{'}, \{u_i, v\}\} \\
\Gamma_v^f &= \{\{v\}, \{u_i, u_i^{'}\}, V_2 \cup V_2^{'}, \ldots, V_k \cup V_k^{'}\}.
\end{aligned}$$

Therefore every vertex in $\mu^f(G)$ is appears in a singleton in above χ^f-partitions of $\mu^f(G)$, hence $\mu^f(G)$ is χ^f-excellent. $\square$

Example 3.3.



χ^f-partitions are $\Gamma_1^f = \{\{u\}, \{v_1, v_1^{'}\}, \{v_2, v_2^{'}\}\}$, $\Gamma_2^f = \{\{v_1^{'}\}, \{v_2, v_2^{'}\}, \{v_1, u\}\}$, $\Gamma_3^f = \{\{v_2^{'}\}, \{v_1, v_1^{'}\}, \{v_2, u\}\}$, $\Gamma_4^f = \{\{v_2\}, \{v_1, v_1^{'}\}, \{u, v_2^{'}\}\}$, $\Gamma_5^f = \{\{v_1\}, \{v_2, v_2^{'}\}, \{u, v_1^{'}\}\}$. Hence $\mu^f(G)$ is χ^f-excellent.

Theorem 3.1. *Let G be a simple fuzzy graph. G is χ^f-excellent if and only if G is critical.*

Proof. Let us assume that G be a critical graph with chromatic number χ^f. Let $u \in V(G)$, then $\chi^f(G-u) > \chi^(G)$. Suppose let $\chi^f(G-u) = \chi^f(G) - k, (k \geqslant 1)$. Let $\{V_1, V_2, \ldots, V_{\chi^f(G)-k}\}$ be a χ^f-partition of $G-u$,then $\{\{u\}, V_1, V_2, \ldots, V_{\chi^f(G)-k}\}$ be a proer color partition of G. Therefore $\chi^f(G) \leqslant \chi^f(G) - k + 1$. Therefore $k \leqslant 1$ which implies that $k = 1$. Therefore $\{\{u\}, V_1, V_2, \ldots, V_{\chi^f(G)-1}\}$ is a χ^f-partition of G. Hence G is χ^f-excellent.

Conversely, assume that G is χ^f-excellent. Then for any $u \in V(G)$, u is either fixed or free and the end vertices of any edge in the graph are free. We know that for a point $v \in G$, v is critical iff v is either fixed or free, then $\chi^f(G-u) < \chi^f(G)$ for every $v \in V(G)$.Also know that for a line $e = uv$ such that $\mu(uv) \leqslant \sigma(u) \wedge \sigma(v)$, e is critical iff each of u and v is either fixed or free, then $\chi^f(G-e) < \chi^f(G)$ for every $e \in E(G)$. Therefore any proer subgraph $H(G)$, $\chi^f(H) < \chi^f(G)$. Hence G is critical. $\square$

Theorem 3.2. *Let G be a fuzzy vertex transitive graph with a χ^f-partition containing a singleton. Then G is χ^f-excellent*

Proof. Let Γ^f be a partition of G containing say $\{u\}$ where $u \in V(G)$. Let $\Gamma^f = \{\{u\}, S_2, \ldots, S_\chi\}$. Let $v \in V(G), v \neq u$. Since G is vertex transitive there exists an automorphism ϕ^f such that $\phi^f(u) = v$. Let

$$\Gamma^f = \{\{\phi^f(u)\}, \phi^f(S_2), \ldots, \phi^f(S_\chi)\}.$$

Since ϕ^f is an automorphism, $\phi^f(S_2), \ldots, \phi^f(S_\chi)$ are all independent. Therefore there exists a χ^f-partition containing $\{v\}$. Hence the result. $\square$

Observation 3.1. *There exists a fuzzy vertex transitive which is not complete in which there exists a χ^f-partition containing singleton*

Observation 3.2. *There exists a fuzzy vertex transitive graph which is not complete in which there exists no χ^f-partition containing singleton.*

Definition 3.11. *Let $u, v \in V(G)$. u and v are said to be **relatively fuzzy fixed** if u and v belongs to the same set in every χ^f-partition of G.**Relatively fuzzy free** if u and v belongs to the same set in some χ^f-partition of G (i.e) u and v do not belongs to the same set in some other χ^f-partition. **Relatively fuzzy totally free** if u and v do not belongs to the same set in any χ^f-partition.*

Remark 3.7. *Let G be a χ^f-excellent graph and $e \in E(\bar{G})$. Then $G+e, G-e$ need not be χ^f-excellent.*

Proosition 3.7. *Let G be a χ^f-excellent and let $e = uv \in E(G)$. Suppose $\chi^f(G-e) = \chi^f(G)$. Then $G-e$ is χ^f-excellent.*

Proof. Let G be χ^f-excellent graph. Suppose let $\Gamma^f = \{\{u\}, V_2, \ldots, V_k\}$ be a χ^f-partition of G, where $k = \chi^f$. Let $v \in V_i, 2 \leqslant k \leqslant k$. If $|V_i| = 1$ then $\Gamma_1^f = \{\{u, v\}, V_2, \ldots, V_{i-1}, V_{i+1}, \ldots, V_k\}$ is a proer colour partition of G and hence $\chi^f(G-e) \leqslant k-1 = \chi^f(G) = k$, a contradiction. Therefore $|V_i| \geqslant 2$.

Therefore $\{u\}$ belongs to a χ^f-partition of $G-e$. Similarly it can be proved that $\{v\}$ belongs to a χ^f-partition of $G-e$. The remaining vertices are χ^f-good in G and hence in $G-e$. Therefore $G-e$ is χ^f-excellent. $\square$

Observation 3.3.
(1) If $G+H$ is χ^f-excellent if and only if G and H are χ^f-excellent.
(2) If $G+H$ is χ^f-excellent, then $G+H$ need not be complete.

Remark 3.8. *Given a positive integer $k \geqslant 2$, there exists a fuzzy graph with $\chi^f = k$ which is not χ^f-excellent.*

Proof. Let us consider a complete k-partite fuzzy graph G with each partition containing atleast two vertices. Then $\chi^f(G) = k$ which is not χ^f-excellent. $\square$

Remark 3.9. *Given a positive integer $t, l, k \geqslant 2$, and $k-1$ divides $k+l$, there exists a graph G with $\chi^f(G) = k$, $\delta^f(G) - \chi^f(G) = l$ and G is not χ^f-excellent.*

Proof. Let us consider a complete k-partite fuzzy graph G with each partition containing exactly $\frac{k+l}{k-1}$ vertices. Then $\delta^f(G) = k + l$ and $\chi^f(G) = k$ but which is not χ^f-excellent. $\square$

Observation 3.4. *A uniquely colorable graph G is χ^f-excellent iff $G = K_n$.*

Observation 3.5. *If G is χ^f-excellent then $G \times K_2$ need not be excellent.*

Proosition 3.8. *Suppose G is χ^f-excellent and $|V(G) \geqslant 2|$, then G^f+ is not χ^f-excellent.*

Proof. By the definition of G^{f+}, clearly $\chi^f(G+) = \chi^f(G)$. Since G is χ^f-excellent, let $\Gamma^f = \{\{u\}, S_2, S_3, \ldots, S_{\chi^f}\}$ where $u \in V(G)$, be a χ^f-partition of G.Let u' be a pendent vertex adjacent to u. Suppose $\Gamma_1^f = \{\{u'\}, T_2, \ldots, T_\chi^f\}$ be χ^f-partition of G^+. Then $\chi_2^f = \{T_2 \cap V(G), T_3 \cap V(G), \ldots, T_{\chi^f} \cap V(G)\}$ is a χ^f-partition of G which is a contradiction.Hence G^f+ is not χ^f-excellent. $\square$

Remark 3.10. *The only χ^f-good vertices of G^{f+} are those of G.*

4. Application

Fuzzy graph coloring has extensive applications in the following fields and solving different problems as follows: In Human Resource management such as assignment, job allocation, scheduling, In telecommunication process,In Bioinformatics, In traffic light problem.

5. Conclusion

In this paper we determine the excellent fuzzy graph by the fuzzy chromatic partition and also verify the fuzzy chromatic excellency in Fuzzy corona, Mycielskian fuzzy graph and union of two fuzzy graphs,addition of two fuzzy graphs. We can extend this concept to new type of fuzzy chromatic excellence and study the characteristics of this new parameter.

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