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EDGE VERSION OF HARMONIC INDEX AND POLYNOMIAL OF SOME CLASSES OF BRIDGE GRAPHS

Rabia Nazir, Muhammad Shoaib Sardar, Sohail Zafar and Zohaib Zahid

Abstract. We report explicit formulas for the edge version of harmonic index and harmonic polynomial of several classes of bridge graphs.

1. Introduction and Preliminaries

A graph is a representation of a set of objects where some pairs of objects are connected by links. The interconnected objects are represented by mathematical abstractions called vertices (also called nodes or points), and the links that connect some pairs of vertices are called edges (also called arcs or lines) (see [**26**]) . In the mathematical discipline of graph theory, the line graph of an undirected graph G is another graph $L(G)$ that represents the adjacencies between edges of G. The name line graph comes from a paper by Harary and Norman (see [**11**]) although both Whitney (see [**27**]) and Krausz (see [**15**]) used the construction before. Given a graph G, its line graph $L(G)$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint ("are incident") in G. Let $\{G_i\}_{i=1}^{n}$ be a set of finite pairwise disjoint graphs with distinct vertices $v_i \in V(G_i)$. The bridge graph $B(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d)$ of G_i respect to the vertices v_i is the graph

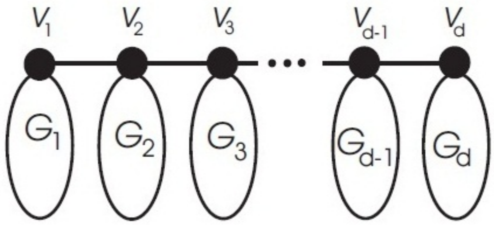

FIGURE 1. The Bridge graph $B(G_1, G_2, \dots, G_d; v_1, v_2, \dots, v_d)$

obtained from the graphs $G_1, G_2, \dots, G_d$ by connecting the vertices v_i and v_{i+1} by an edge for all $i \in 1, 2, \dots, d-1$, as shown in Figure 1.

In chemistry, molecular structure descriptors are used to model information of molecules, which are known as topological indices. A topological index, sometimes also known as a graph-theoretic index, is a numerical invariant of a chemical graph (see [**23**]). Particular topological indices include the Balaban index, Harary index, molecular topological index, and Wiener index. Unless otherwise stated, hydrogen atoms are usually ignored in the computation of such indices as organic chemists usually do when they write a benzene ring as a hexagon (see [**3**]) .

Line graphs are useful in chemical graph theory. The first topological index on the basis of the line graph was introduced by Bertz [**2**] in 1981 , when he was working on molecular branching. After that a number of topological indices based on line graphs were introduced (see [**6, 7, 13**]). For more information about the applications of line graphs in chemistry, we refer the articles (see [**8, 9, 10**]).

On the basis of the vertex-degrees of graphs many topological indices are defined. (see [**16, 17, 19, 20, 21, 22, 24, 28**]). One of the vertex-degree based index namely harmonic index $H(G)$ is first time introduced in [**4**]:

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v}.$$

For more results on harmonic index we refer to the articles [**5, 29, 30, 31**]. The harmonic polynomial is defined in [**14**] as follows

$$H(G, x) = \sum_{uv \in E(G)} 2x^{d_u + d_v - 1}.$$

Note that $\int_0^1 H(G, x)dx = H(G)$.

The edge version of harmonic polynomial and index is introduced by Rabia et al. in (see [**18**]). They also computed the edge version of harmonic polynomial and index of wheel graph, helm graph, ladder graph and linear $[n]$−pentacene. The

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edge version of harmonic polynomial is defined as

$$H_e(G,x) = \sum_{ef \in E(L(G))} 2x^{d_e+d_f-1}.$$

Similarly the edge version of harmonic index is defined as

$$H_e(G) = \sum_{ef \in E(L(G))} \frac{2}{d_e+d_f}.$$

Clearly $\int_0^1 H_e(G,x)dx = H_e(G)$.

The following lemma is helpful for computing the degree of a vertex of line graph.

LEMMA 1.1. *Let G be a graph with $u, v \in V(G)$ and $e = uv \in E(G)$. Then:*

$$d_e = d_u + d_v - 2.$$

In order to calculate the number of edges of an arbitrary graph, the following Lemma is significant for us.

LEMMA 1.2. *Let G be a graph. Then*

$$\sum_{u \in V(G)} d_u = 2|E(G)|.$$

This is also known as handshaking Lemma.

In this paper we computed edge version of harmonic index and harmonic polynomial for some classes of bridge graph.

2. Main results

2.1. Square comb lattice graph. Let P_i be the path of length i and let $BP_n = B(P_n, P_n, \ldots, P_n, v_1, v_2, \ldots, v_n)$ be a square comb lattice graph as shown in Figure 2. It is easy to see that BP_n has n^2 vertices and n^2-1 edges. For $n = 1$, BP_1 consist of only one vertex and for $n = 2$, BP_2 consist of a path of length 3. We are interested in finding edge version of Harmonic index and Harmonic polynomial of BP_n.

EXAMPLE 2.1. *Consider square comb lattice graph for $n = 3$ then the edge partition of $E\big(L(BP_3)\big)$ based on the degree of the vertices is shown in Table 1.*

(d_e, d_f)	$(1,2)$	$(2,3)$	$(1,3)$	$(3,3)$
Number of edges	2	2	1	3

TABLE 1. The edge partition of $L(BP_3)$

We get $H_e(BP_3, x) = 6x^5 + 4x^4 + 2x^3 + 4x^2$ and $H_e(BP_3) = \frac{109}{30}$.

PROPOSITION 2.2. *Let BP_n be a square comb lattice graph for $n > 3$, then*

(1) $H_e(BP_n, x) = 2(n-4)x^7 + 2(2n-4)x^6 + 4x^5 + 2nx^4 + (n^2-4n+2)2x^3 + 2nx^2$.

(2) $H_e(BP_n) = \frac{n^2-4n+2}{2} + \frac{2n}{5} + \frac{2}{3} + \frac{2(2n-4)}{7} + \frac{n-4}{7} + \frac{2n}{3}$.

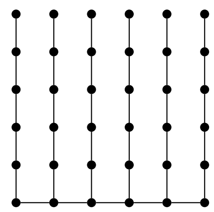

FIGURE 2. The square comb lattice graph BP_6

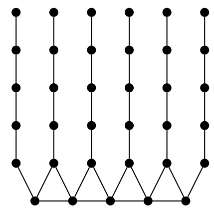

FIGURE 3. The line graph of square comb lattice graph BP_6

PROOF. The line graph $L(BP_n)$ for $n = 6$ is shown in Figure 3. It is easy to see that in line graph $L(BP_n)$, the total number of vertices are $n^2 - 1$ out of which n vertices are of degree 1, n vertices are of degree 3 , $n - 3$ vertices of degree 4 and $n^2 - 3n + 2$ vertices are of degree 2 . It is easily seen from Lemma 1.2 that the total number of edges in $L(BP_n)$ are $n^2 + n - 4$. The edge partition of $E\big(L(BP_n)\big)$ based on the degree of the vertices is shown in Table 2.

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$	$(3,4)$	$(4,4)$	$(1,2)$
Number of edges	$n^2 - 4n + 2$	n	2	$2(n-2)$	$n-4$	n

TABLE 2. The edge partition of $L(BP_n)$

Consequently, we get $H_e(BP_n, x) = 2(n-4)x^7 + 2(2n-4)x^6 + 4x^5 + 2nx^4 + (n^2 - 4n + 2)2x^3 + 2nx^2$ and $H_e(BP_n) = \frac{n^2-4n+2}{2} + \frac{2n}{5} + \frac{2}{3} + \frac{2(2n-4)}{7} + \frac{n-4}{7} + \frac{2n}{3}$. □

2.2. Van hove comb lattice graph. Let P_i be the path of length i and let $VP_n = B(P_1, P_2, \ldots, P_{n-1}, P_n, P_{n-1}, \ldots, P_2, P_1; v_{1,1}, v_{1,2}, \ldots, v_{1,n-1}, v_{1,n}, v)$ be a van hove lattice graph as shown in Figure 4. Note that VP_n has n^2 vertices and $n^2 - 1$ edges. For $n = 1$, VP_1 consist of only one vertex and for $n = 2$, VP_2 consist of a path of length 3. We are interested in finding edge version of Harmonic index and Harmonic polynomial of VP_n.

EXAMPLE 2.3. *Consider Van hove comb lattice graph for $n = 3$ then the edge partition of $E\big(L(VP_3)\big)$ based on the degree of the vertices is shown in Table 3.*

(d_e, d_f)	$(2,2)$	$(2,4)$	$(1,3)$	$(4,4)$	$(3,4)$
Number of edges	2	4	1	1	2

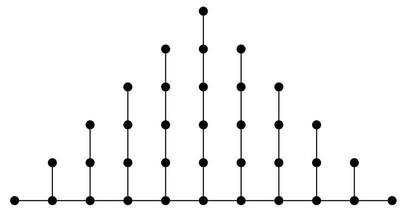

FIGURE 4. The Van Hove Lattice graph VP_6

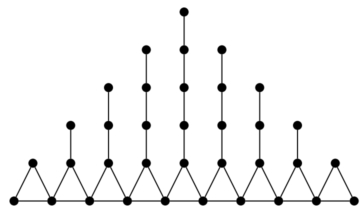

FIGURE 5. The line graph of Van Hove lattice graph VP_6

TABLE 3. The edge partition of $L(VP_3)$

We get $H_e(VP_3, x) = 2x^7 + 4x^6 + 8x^5 + 6x^3$ *and* $H_e(VP_3) = \frac{307}{84}$.

PROPOSITION 2.4. *Let* VP_n *be a square comb lattice graph for* $n > 3$ *, then*

(1) $H_e(VP_n, x) = (2n-5)2x^7 + (4n-10)2x^6 + 8x^5 + (2n-7)2x^4 + 4x^3 + (2n-7)2x^2$.

(2) $H_e(VP_n) = \frac{2n-5}{4} + \frac{16(2n-7)}{75} + \frac{2(4n-10)}{7}$.

PROOF. The line graph $L(VP_n)$ for $n = 6$ is shown in Figure 5. It is easy to see that in line graph $L(VP_n)$, the total number of vertices are $n^2 - 1$ out of which $2n-5$ vertices are of degree 1, $2n-5$ vertices of degree 3 , $2n-4$ vertices of degree 4 and $n^2 - 6n + 13$ vertices are of degree 2 as shown in figure 5. It is easily seen from Lemma 1.2 that the total number of edges in $L(VP_n)$ are $n^2 + 2n - 5$. The edge partition of $E\big(L(VP_n)\big)$ based on the degree of the vertices is shown in Table 4.

(d_e, d_f)	$(4,4)$	$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$	$(3,4)$	(2,2)
Number of edges	$2n-5$	$2n-7$	2	$2n-7$	4	4n-10	$n^2 - 8n + 18$

TABLE 4. The edge partition of $L(VP_n)$

consequently we get $H_e(VP_n, x) = (2n-5)2x^7 + (4n-10)2x^6 + 8x^5 + (2n-7)2x^4 + 4x^3 + (2n-7)2x^2$ and $H_e(VP_n) = \frac{2n-5}{4} + \frac{16(2n-7)}{75} + \frac{2(4n-10)}{7}$.

□

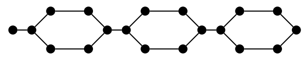

FIGURE 6. Polyphenylene chain R_3

2.3. Bridge of $[k]$-polyphenylene. Let R_k be a $[k]$-polyphenylene chain as shown in Figure 6. Let $B_nR_k = B(R_k, R_k, \ldots, R_k, v_1, v_2, \ldots, v_n)$ be a bridge graph of $[k]$-polyphenylene as shown in Figure 7. It is to easy to see that B_nR_k has $n(1+6k)$ vertices and $7nk+n-1$ edges .

EXAMPLE 2.5. *The edge partition of* $E\big(L(B_1R_1)\big)$ *based on the degree of the vertices is shown in Table 5.*

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$
Number of edges	3	4	1

TABLE 5. The edge partition of $L(B_1R_1)$

We get $H_e(B_1R_1, x) = 2x^5 + 8x^4 + 6x^3$ *and* $H_e(B_1R_1) = \frac{103}{30}$.

EXAMPLE 2.6. *The edge partition of* $E\big(L(B_1R_k)\big)$ *based on the degree of the vertices is shown in Table 6.*

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$	$(3,4)$
Number of edges	3	$4k$	$2k-1$	$4(k-1)$

TABLE 6. The edge partition of $L(B_1R_k)$

We get $H_e(B_1R_k, x) = 8(k-1)x^6 + 2(2k-1)x^5 + 8kx^4 + 6x^3$ *and* $H_e(B_1R_k) = \frac{47k-3}{21}$.

EXAMPLE 2.7. *The edge partition of* $E\big(L(B_2R_1)\big)$ *based on the degree of the vertices is shown in Table 7.*

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$
Number of edges	6	6	6

TABLE 7. The edge partition of $L(B_2R_1)$

We get $H_e(B_2R_1, x) = 12(x^5 + x^4 + x^3)$ *and* $H_e(B_2R_1) = \frac{111}{15}$.

EXAMPLE 2.8. *The edge partition of* $E\big(L(B_2R_k)\big)$ *based on the degree of the vertices is shown in Table 8.*

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$	$(3,4)$
Number of edges	6	$8k-2$	$4k+2$	$8(k-1)$

TABLE 8. The edge partition of $L(B_2R_k)$

We get $H_e(B_2R_k, x) = 16(k-1)x^6 + 4(2k+1)x^5 + 4(4k-1)x^4 + 12x^3$ *and* $H_e(B_2R_k) = \frac{716k+61}{105}$.

EXAMPLE 2.9. *The edge partition of* $E\big(L(B_3R_1)\big)$ *based on the degree of the vertices is shown in Table 9.*

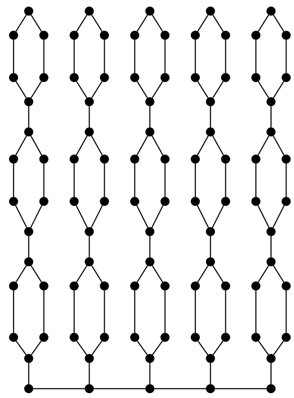

FIGURE 7. The graph B_5R_3

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$	$(3,4)$
Number of edges	9	6	10	4

TABLE 9. The edge partition of $L(B_3R_1)$

We get $H_e(B_3R_1, x) = 8x^6 + 10x^5 + 12x^4 + 18x^3$ *and* $H_e(B_3R_1) = \frac{19}{10}$.

EXAMPLE 2.10. *The edge partition of* $E\big(L(B_3R_k)\big)$ *based on the degree of the vertices is shown in Table 10.*

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$	$(3,4)$
Number of edges	9	$12k-6$	$6k+4$	$4(3k-2)$

TABLE 10. The edge partition of $L(B_3R_k)$

Consequently, we get $H_e(B_3R_k, x) = 8(3k-2)x^6 + 2(6k+4)x^5 + 4(6k-3)x^4 + 18x^3$ *and* $H_e(B_3R_k) = \frac{696k+67}{70}$.

PROPOSITION 2.11. *Let* B_nR_k *be a bridge graph of [k]-polyphenylene, then*

(1) $H_e(B_nR_k, x) = (6n-20)x^7 + (8kn-4n)x^6 + (4kn-2n+12)x^5 + (8kn-4n)x^4 + 6nx^3$

(2) $H(B_nR_k) = \frac{3n}{2} + \frac{8kn-4n}{5} + \frac{4kn-2n+12}{6} + \frac{8kn-4n}{7} + \frac{6n-20}{8}$

PROOF. The line graph L(B_nR_k) for $n = 6$ and $k = 3$ is shown in Figure 8. it is easy to see that in line graph $L(B_nR_k)$, the total number of vertices are $7nk+n-1$ out of which $2kn+2n$ vertices are of degree 2, $4kn-2n+4$ vertices are of degree 3 , and $n(k+1)-5$ vertices are of degree 4 as shown in Figure 8. It is easily seen from Lemma 1.2 that the total number of edges in $L(B_nR_k)$ are $10kn+n-4$. The edge partition of $E(G)$ based on the degree of the vertices is shown in Table 11.

(d_e, d_f)	$(2,2)$	$(2,3)$	$(3,3)$	$(3,4)$	$(4,4)$
$Numberofedges$	$3n$	$4kn-2n$	$2kn-n+6$	$4kn-2n$	$3n-10$

TABLE 11. The edge partition of $L(B_nR_k)$

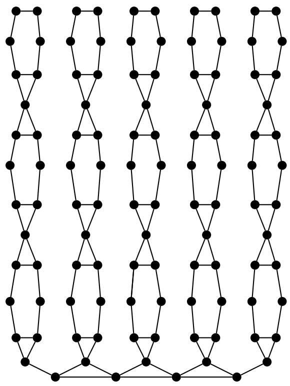

FIGURE 8. The line graph of B_5R_3

consequently we get $H_e(B_nR_k, x) = (6n-20)x^7+(8kn-4n)x^6+(4kn-2n+12)x^5+(8kn-4n)x^4+6nx^3$ and $H_e(B_nR_k) = \frac{3n}{2} + \frac{8kn-4n}{5} + \frac{4kn-2n+12}{6} + \frac{8kn-4n}{7} + \frac{6n-20}{8}$. $\square$

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University of Management and Technology (UMT), Lahore Pakistan
E-mail address: rabianazir006@gmail.com
E-mail address: shoaibsardar093@gmail.com
E-mail address: sohailahmad04@gmail.com
E-mail address: zohaib_zahid@hotmail.com