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2−BONDAGE NUMBER OF A FUZZY GRAPH

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ABSTRACT. In this paper, 2−bondage set of a fuzzy graph G is defined. The 2−bondage number, $b_2(G)$ is the minimum cardinality among all 2−bondage sets of G. The condition for a 2−bondage set of a fuzzy graph to be a bondage set is also given. The exact values of $b_2(G)$ is determined for several classes of fuzzy graphs.

1. Introduction

Euler first introduced the concept of graph theory, in the year 1736. Cockayne and Hedetniemi **[2]** introduced the domination number and the independent domination number of graphs but the concept of dominating sets in graphs was introduced by Ore and Berge **[1, 10]**. The concept of the bondage number in graphs was introduced by Fink, Jacobson, Kinch and Roberts **[3]** in the year 1990. The concept of 2−bondage number in graph theory was discussed by Krzywkowski **[4]** in the year 2012.

The concept of fuzzy relation was introduced by Zadeh **[13]** in his classical paper in 1965. Rosenfeld **[11]** introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as paths, cycles and connectedness. Somasundram and Somasundram **[12]** discussed domination in fuzzy graphs using effective edges in 1998. Gani and Chandrasekaran **[5]** discussed domination in fuzzy graph using strong arcs. Gani and Vadivel **[6]** discussed domination, independent domination and irredundance in fuzzy graphs using strong arcs. And the concept of bondage and non bongage number of a fuzzy graph was discussed by Gani, Devi and Akram **[7]** in the year 2015. Gani and Devi **[9]** also discussed 2−dominating set in fuzzy graphs.

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2. Preliminaries

The 2− *bondage number* of a graph G is the minimum cardinality of a set of edges of G whose removal from G results in a graph with 2−domination number larger than that of G.

A *fuzzy graph* $G =< \sigma, \mu >$ is a pair of functions $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$, where for all $x, y \in V$, we have $\mu(x,y) \leqslant \sigma(x) \wedge \sigma(y)$. The *underlying crisp graph* of a fuzzy graph $G =< \sigma, \mu >$ is denoted by $G^* =< \sigma^*, \mu^* >$, where $\sigma^* = \{v_i \in V/\sigma(v_i) > 0\}$ and $\mu^* = \{(v_i, v_j) \in V \times V/\mu(v_i, v_j) > 0\}$. An edge in G is called an *isolated edge* if it is not adjacent to any edge in G. A *path* with n vertices in a fuzzy graph is denoted as P_n. A fuzzy graph $G =< \sigma, \mu >$ is a *complete fuzzy graph* if $\mu(v_i, v_j) = \sigma(v_i) \wedge \sigma(v_j)$ for all $v_i, v_j \in \sigma^*$. An arc (x, y) in a fuzzy graph $G =< \sigma, \mu >$ is said to be *strong* if $\mu^\infty(x,y) = \mu(x,y)$.

A subset D of V is called a *dominating set* of a fuzzy graph G if for every $v \in V - D$, there exist $u \in D$ such that u dominates v. The *domination number*, $\gamma(G)$, is the smallest number of nodes in any dominating set of G. A subset D of V is called a 2−*dominating set* of G if for every node $v \in V - D$ there exist atleast two strong neighbours in D. The 2−*domination number* of a fuzzy graph G denoted by $\gamma_2(G)$, is the minimum cardinality of a 2−dominating set of G.

3. 2-Bondage number

In this section we discuss about the 2−bondage set and 2−bondage number of a fuzzy graph. The condition for a 2−bondage set of a fuzzy graph to be a bondage set is also given.

Definition 3.1. *A set $X \subseteq S$ is said to be a 2−**bondage set** of the fuzzy graph if $\gamma_2(G - X) > \gamma_2(G)$, where S is the set of all strong arcs in G.*

Definition 3.2. *The 2−**bondage number**, $b_2(G)$, of a fuzzy graph G is the minimum cardinality among all 2−bondage sets of G. The 2−bondage set of the fuzzy graph G having cardinality equal to $b_2(G)$ is called the **minimum** 2−**bondage set** of G.*

In other words, the 2−***bondage number*** of a fuzzy graph G is the minimum number of strong arcs whose removal from G increases the 2−domination number of G.

Example 3.1. ***(G):***

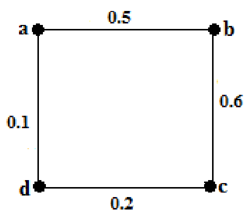


Fig. 3.1

Here $S = \{ab, bc, cd\}$ *and* $\gamma_2(G) = 3$. $X = \{ab, cd\}$ *and* $\gamma_2(G - X) = 4$ *Thus* X *is a* $2-$*bondage set and it is minimum. Therefore* $b_2(G) = 2$.

Theorem 3.1. *Every fuzzy end node of G is in every* $2-$*domination set of G.*

Proof. Let u be a fuzzy end node of G. Let D be any $2-$dominating set of G.

Since u has atmost one strong neighbour in G, it is dominated by atmost one strong neighbour and by itself. And since D is a $2-$dominating set, for every $v \in V - D$, there exist atleast $2-$strong neighbours in D.

Suppose $u \notin D$ then u must be dominated by atleast 2 strong neighbours in D, which is a contradiction.

Therefore, $u \in D$, for every $2-$dominating set D of G. Thus every fuzzy end node of G is in every $2-$dominating set of G. □

Theorem 3.2. *If* e *is an isolated edge in G then* e *does not belongs to any minimum* $2-$*bondage set of G.*

Proof. Let $e = (u, v)$ be an isolated edge in G. Since u and v are fuzzy end nodes in G and $| N_S(u) |=| N_S(v) |= 1$, we have u and v both belongs to every $2-$dominating set of G.

Thus deleting e from G does not increases the $2-$domination number of G. Therefore e does not belongs to any minimum $2-$bondage set of G. □

Theorem 3.3. *If* $\gamma_2(G) = n$*, where G is a fuzzy graph with* n *nodes then* $b_2(G) = 0$.

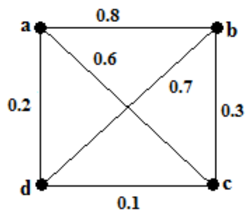
Proof. Let G be a fuzzy graph with n nodes and $\gamma_2(G) = n$. i.e., all the nodes of G are in the minimum $2-$dominating set of G.

Suppose deletion of an arc e from G increase the $2-$domination number of G. i.e., $\gamma_2(G - e) > n$, which is not possible.

Thus deletion of any arc e from G does not increases $\gamma_2(G)$.

Therefore G does not have a $2-$bondage set. Hence $b_2(G) = 0$. □

Remark 3.1. *The converse of the above theorem is not true.*



(G)

Fig. 3.2

Here $S = \{ab, ac, bd\}$. Thus $\{a, c, d\}$ is a minimum $2-$dominating set of G and $\gamma_2(G) = 3$. Deletion of all arcs of S does not increases the domination number of G. i.e., $\gamma_2(G - S) = \gamma_2(G) \Rightarrow$G does not have a $2-$bondage set i.e., $b_2(G) = 0$. But $\gamma_2(G) = 3 < 4$. Thus $b_2(G) = 0$ but $\gamma_2(G) \neq n$.

THEOREM 3.4. *A minimum* $2-$*bondage set* X *of a fuzzy graph* G *is a bondage set of* G *if* $\gamma_2(G-X)=\gamma(G-X)$.

PROOF. Let G be a fuzzy graph and X be the minimum $2-$bondage set of G with $\gamma_2(G-X)=\gamma(G-X)$. Therefore $\gamma_2(G)\geqslant\gamma(G)$ and given $\gamma_2(G-X)=\gamma(G-X)$.

Since X is a minimum $2-$bondage set of G then we have $\gamma_2(G-X)>\gamma_2(G)$.

Thus $\gamma_2(G-X)>\gamma_2(G)\geqslant\gamma(G)$.

$\gamma_2(G-X)>\gamma(G)$.

$\Rightarrow\gamma(G-X)>\gamma(G)$

Therefore X is a bondage set of G. Hence the proof. □

THEOREM 3.5. *A minimum* $2-$*bondage set* X *of a fuzzy graph* G *is a bondage set of* G *if* $\gamma(G-X)>\gamma_2(G)$.

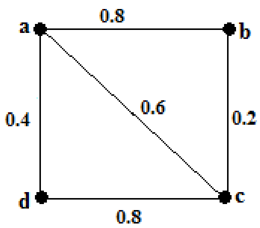
PROOF. Let G be a fuzzy graph and X be the minimum $2-$bondage set of G with $\gamma(G-X)>\gamma_2(G)$. We know that $\gamma_2(G)\geqslant\gamma(G)$ and $\gamma_2(G-X)>\gamma_2(G)$.

Therefore $\gamma(G-X)>\gamma_2(G)\geqslant\gamma(G)$.

$\Rightarrow\gamma(G-X)>\gamma(G)$

Therefore X is a bondage set of G. Hence the proof. □

REMARK 3.2. *Every* $2-$*bondage set of a fuzzy graph need not be a bondage set of G.*



(G)

Fig. 3.3

Here $S=\{ab,ac,cd\}$, $\gamma_2(G)=3$ and $\gamma_2(G-S)=4\Rightarrow S$ is a $2-$bondage set of G. $\gamma_2(G-S)=2$ and $\gamma(G)=2\Rightarrow S$ is not a $2-$bondage set of G. Thus every $2-$bondage set of a fuzzy graph G need not be a bondage set of G.

4. 2-Bondage number for specific fuzzy graphs

THEOREM 4.1. *Let* G *be a fuzzy graph and* $G^*=nK_2$ *then* $b_2(G)=0$.

PROOF. Let G be a fuzzy graph and $G^*=nK_2$. Then G has $2n$ nodes and $|N_S(v)|=1$, for all $v\in V(G)$. Thus each node of G is a fuzzy end node and $\gamma_2(G)=2n$.

Since G has $2n$ nodes and $\gamma_2(G)=2n$ we get $b_2(G)=0$. □

THEOREM 4.2. *Let* G *be a fuzzy graph with* $n+1$ *nodes,* $n\geqslant 2$ *and* G^* *is a star then* $b_2(G)=n-1$.

Proof. Let G be a fuzzy graph with $n+1$ nodes and G^* is a star i.e., one node is as centre and all other nodes as its leaves in G^*. Thus $\gamma_2(G) = n$.

Let v be the centre node and $v_1, v_2, v_3, ..., v_n$ be its leaves. All the n leaves i.e., $v_1, v_2, v_3, ..., v_n$ will form a 2−dominating set of G. Thus v has n strong neighbours in D. Thus deleting the $n-1$ arcs in G will increase the 2−domination number by 1.

Therefore, $b_2(G) = n-1$. □

Remark 4.1. .

(1) *If G is a complete fuzzy graph with n nodes* $(n \geqslant 3)$ *then* $b_2(G) = \lfloor 2n/3 \rfloor$.
(2) *Let G be a fuzzy graph with n nodes and* G^* *be a cycle,*
   - *If n is odd then* $b_2(G) = 2$
   - *If n is even then*

$$b_2(G) = \{ \begin{matrix} 1, & if \quad G \quad has \quad more \quad than \quad one \quad weakest \quad arc \\ 2, & if \quad G \quad has \quad only \quad one \quad weakest \quad arc \end{matrix}$$

(3) *If* P_n *is a fuzzy path with* $n(\geqslant 3)$ *nodes then* $b_2(P_n) = 1$.

5. Conclusion

We discussed about the 2−bondage set and 2−bondage number of fuzzy graphs. We have given some results on 2−bondage set. We also given 2−bondage number of a complete fuzzy graph. Some future work are to find the upper bound and lower bound for the 2−bondage number of the fuzzy graph.

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