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SOLITARY WAVE SOLUTION OF THE VARIABLE COEFFICIENT KdV - BURGERS EQUATION

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Abstract. In this paper, we find exact solitary wave solutions for the variable coefficient KdV Burgers equation of the form $u_t + uu_x + f(t)u_{xx} + g(t)u_{xxx} = 0$. We construct a transformation of variables which is applied in order to obtain a constant coefficient KdV Burgers equation and also we obtain certain solitary wave solutions with a constraint on $f(t)$ and $g(t)$.

1. Introduction

At present many phenomena in modern mathematical physics have been modeled in terms of a variety of nonlinear partial differential equations. In order to understand these nonliear phenomena, many mathematician and physicists do strive in seeking more exact solutions for them. Therefore it is still a very important and essential task to search for the explicit and exact solutions to nonlinear partial differential equations in modern science. Several powerful methods have been proposed in order to obtain exact solution nonlinear evolution equations, such as homogeneous balance method [1, 2], the tanh-function method [3], the sech-function method[4], the Jacobi elliptic function expansion method [5, 6] and so on. However not all the above approaches are applicable for solving all kinds of nonlinear evolution equations directly.

The standard form of the Korteweg-de Vries - Burgers (Kdv-Burgers) equation is

$$u_t + \alpha uu_x + \beta u_{xx} + su_{xxx} = 0, \tag{1.1}$$

where α, β and s are real constants with $\alpha, \beta, s \neq 0$.

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The KdV-Burgers equation (1.1) is the simplest form of the wave equation in which the nonlinearity (uu_x), the dispersion u_{xxx} and the dissipation u_{xx} all occur. It arises from many physical contexts, for example, the propagation of undular bores in a shallow water [7, 8], the flow of liquids containing gas bubbles[9], the propagation of waves in an elastic tube filled with a viscous fluid [10], weakly nonlinear plasma waves with certain dissipative effects [11, 12], and the cascading down process of turbulence[13]. It is also widely used as a nonlinear governing model in the crystal lattice theory, the nonlinear circuit theory and the atmospheric dynamics.

Recently as per demand of several physical situations it has become of considerable interest to find the exact solutions of a nonlinear partial differential equation when the parameters depend explicitly on time. In particular the KdV equation with variable coefficients has been studied in the context of ocean waves, there the spatio-temporal variability of the coefficients are due to the changes in the water depth and other physical conditions.

A simple generalization of the KdV-Burgers equation, which bears more realistic physical importance, is the variable-coefficient KdV-Burgers (VCKdVB) equation

$$u_t + uu_x + f(t)u_{xx} + g(t)u_{xxx} = 0. \tag{1.2}$$

where $g(t)$ and $f(t)$ stands for the dispersion u_{xxx} and the dissipation u_{xx} respectively. The purpose of this paper is to study (1.2) for exact solitary wave solution and a more general result is obtained. It does not seem that this new result has been presented previously for VCKdVB equation.

The structure of paper is as follows: In section 2, the variable coefficients KdV-Burgers equation is transformed into the standard form of the KdV-Burgers equation under the generalized transformation. In section 3, we obtain the exact solitary wave solutions of the variable coefficients KdV-Burgers equation through the standard form of the KdV-Burgers equation by using complex tanh method. Section 4 contains conclusion of this study.

2. Transforming the variable coefficients KdV-Burgers equation via the generalized transformation

In order to look for solutions to KdV-Burgers equation

$$u_t + uu_x + f(t)u_{xx} + g(t)u_{xxx} = 0, \tag{2.1}$$

we apply the generalized transformation given by

$$u(x,t) = A(x,t)G(\tau,z) + B(x,t), \quad z = \alpha(t)x + \beta(t), \tau = \tau(t), \tag{2.2}$$

where $A(x,t) \neq 0$, $B(x,t), \alpha(t), \beta(t)$ and $\tau(t)$ are some functions to be determined later.

We use the above ansatz into (2.1) to obtain,

$$\begin{aligned}
&A\tau' G_\tau + \alpha A^2 G G_z + [f\alpha^2 A + 3g\alpha^2 A_x] G_{zz} + \alpha^3 A g G_{zzz} \\
&+[A_t + AB_x + A_x B + fA_{xx} + gA_{xxx}]G + AA_x G^2 \\
&+[\alpha' A_x + \beta' A + \alpha AB + 2\alpha f A_x + 3\alpha g A_{xx}]G_z \\
&+[B_t + BB_x + fB_{xx} + gB_{xxx}] \ = \ 0, \qquad (2.3)
\end{aligned}$$

Now we assume the following conditions,

$$\frac{\alpha A^2}{A\tau'} \ = \ 1, \qquad (2.4)$$

$$\frac{f\alpha^2 A + 3g\alpha^2 A_x}{A\tau'} \ = \ -\gamma, \qquad (2.5)$$

$$\frac{\alpha^3 g}{\tau'} \ = \ \mu, \qquad (2.6)$$

$$A_t + AB_x + A_x B + fA_{xx} + gA_{xxx} \ = \ 0, \qquad (2.7)$$

$$AA_x \ = \ 0, \qquad (2.8)$$

$$\alpha' A_x + \beta' A + \alpha AB + 2\alpha f A_x + 3\alpha g A_{xx} \ = \ 0, \qquad (2.9)$$

$$B_t + BB_x + fB_{xx} + gB_{xxx} \ = \ 0, \qquad (2.10)$$

where γ and μ are non zero real constants.

We use the above conditions (2.4)-(2.10), equation (2.1) reduces to the KdV-Burgers equation,

$$G_\tau + GG_z - \gamma G_{zz} + \mu G_{zzz} \ = \ 0, \qquad (2.11)$$

Solving the equation (2.8), we get

$$A \ = \ A(t), \qquad (2.12)$$

Subtitute $A \ = \ A(t)$ in equation (2.7), we get $B_x \ = \ -\frac{A'}{A}$ and

$$B \ = \ \frac{-A'}{A}x + C_1(t), \qquad (2.13)$$

where $C_1(t)$ is a constant. Using $A \ = \ A(t)$ in equation(2.9),

$$B \ = \ \frac{-\alpha'}{\alpha}x - \frac{\beta'}{\alpha} \qquad (2.14)$$

Using the equation(2.14), B_{xx} and B_{xxx} are vanishes and from the equation (2.10), we get

$$B_t + BB_x \ = \ 0, \qquad (2.15)$$

Inserting (2.14) into (2.15), then we obtain

$$\left[-\left(\frac{\alpha'}{\alpha}\right)' + \left(\frac{\alpha'}{\alpha}\right)^2\right]x + \left[-\left(\frac{\beta'}{\alpha}\right)' + \left(\frac{\beta'}{\alpha}\right)\left(\frac{\alpha'}{\alpha}\right)\right] \ = \ 0, \qquad (2.16)$$

Equating the x coefficients in equation (2.16) to zero, we get

$$-\left(\frac{\alpha'}{\alpha}\right)' + \left(\frac{\alpha'}{\alpha}\right)^2 = 0, \tag{2.17}$$

We take $y = \frac{\alpha'}{\alpha}$ and from equation (2.17),

$$-y' + y^2 = 0. \tag{2.18}$$

Solving the equation (2.18), $y = -1/(t + C_0)$. Therfore $\alpha = C_2/(t + C_0)$.

Equating the constant coefficients in equation (2.16) to zero, we get

$$-\left(\frac{\beta'}{\alpha}\right)' + \left(\frac{\beta'}{\alpha}\right)\left(\frac{\alpha'}{\alpha}\right) = 0, \tag{2.19}$$

Next we take $s = \frac{\beta'}{\alpha}$ and from equation (2.19) gives

$$-s' + s\frac{\alpha'}{\alpha} = 0, \tag{2.20}$$

Solving the equation(2.20), $s = C_3(t + C_0)^{-1}$ where C_3 is a constant.
Use $s = C_3(t + C_0)^{-1}$, we get

$$\beta = \frac{-C_4}{(t + C_0)} + C_5, \tag{2.21}$$

where $C_4 = C_2C_3$ and C_5 are constants. On solving equations (2.13) and (2.14), we obtain $C_1(t) = \frac{-\beta'}{\alpha}$ Therefore

$$C_1(t) = -\frac{C_3}{t + C_0},$$

From equations (2.13) and (2.14), we get

$$\frac{\alpha'}{\alpha} = \frac{A'}{A}, \tag{2.22}$$

Solving the equation (2.22), $A = \frac{C_6}{t+C_0}$.

Now (2.4) gives $\tau' = A\alpha$, and its solution is,

$$\tau = \frac{-C_7}{t + C_0} + C_8,$$

After inserting for α and τ in (2.6), we have

$$g = \frac{\mu C_7}{C_2^3}(t + C_0).$$

Using $A = A(t)$, (2.5) gives $f\alpha^2 = -\gamma\tau'$, and its solution is, (after inserting for α^2, τ) We get $f = -\gamma\frac{C_7}{C_2^2}$.

Substituting all the known terms into (2.2), we find that

$$u = \frac{C_6}{t + C_0}G(\tau, z) + \frac{x}{t + C_0} - \frac{C_3}{t + C_0}.$$

where $z = \frac{C_2}{t+C_0}x - \frac{C_4}{t+C_0} + C_5$ and $\tau = -\frac{C_7}{t+C_0} + C_8$.

3. Solitary wave solutions

In order to construct the solitary solution of equation (2.11) by the complex $\tan h$ method[14], we use the wave transformation

$$G(z,\tau) = g(\xi), \quad \xi = i(z+\lambda\tau) \tag{3.1}$$

Substituting (3.1) into (2.11) and integrating once with respect to ξ and setting the constant of integration to be zero, we obtain

$$\lambda i g + \frac{i}{2}g^2 + \gamma g' - i\mu g'' = 0. \tag{3.2}$$

For the tanh method [14], we introduce the new independent variable $y = \tan h\xi$ which leads to the change of variables

$$\begin{aligned} \frac{d}{d\xi} &= (1-y^2)\frac{d}{dy}, \\ \frac{d^2}{d\xi^2} &= -2y(1-y^2)\frac{d}{dy} + (1-y^2)^2\frac{d^2}{dy^2}. \end{aligned}$$

Using the transformation $y = \tan h\xi$, equation (3.2) becomes,

$$\lambda g + \frac{1}{2}g^2 - i\gamma(1-y^2)\frac{dg}{dy} + \mu\left[2y(1-y^2)\frac{dg}{dy} - (1-y^2)^2\frac{d^2g}{dy^2}\right] = 0. \tag{3.3}$$

Now, we balance the linear term of highest order with the highest order non-linear terms in equation (3.3), we obtain $n = 2$. Hence we assume that

$$G(z,\tau) = g(\xi) = a_0 + a_1 y + a_2 y^2. \tag{3.4}$$

Substituting equation (3.4) into equation (3.3), and equating the coefficient of $y^i, i = 0,1,2,3,4$ leads to the following system of algebraic equation:

$$\begin{aligned} \frac{a_0^2}{2} + a_0\lambda - ia_1\gamma - 2a_2\mu &= 0 \\ a_0a_1 + a_1\lambda + 2a_1\mu - 2ia_2\gamma &= 0 \\ a_0a_2 + \frac{a_1^2}{2} + ia_1\gamma + a_2\lambda + 8a_2\mu &= 0 \\ a_1a_2 - 2a_1\mu + 2ia_2\gamma &= 0 \\ \frac{a_2^2}{2} - 6a_2\mu &= 0 \end{aligned} \tag{3.5}$$

Solving the system of equations (3.5), we get two classes of the solutions

$I\ : a_0 = -(\lambda + 12\mu),\ a_1 = -\frac{12\gamma}{5}i,\ a_2 = 12\mu,\ \lambda = \pm 24\mu,\ \gamma = \pm 10i\mu$

$II : a_0 = -(\lambda + 8\mu),\ a_1 = 0,\ a_2 = 12\mu,\ \lambda = \pm 4\mu, \gamma = 0.$

Therefore the solutions of the equation (2.11) are

$$G(z,\tau) = -(\lambda + 12\mu) - \frac{12\gamma}{5}\tan(z+\lambda\tau) - 12\mu\tan^2(z+\lambda\tau), \tag{3.6}$$

with $\lambda = \pm 24\mu, \gamma = \pm 10i\mu$ and

$$G(z,\tau) = -(\lambda + 8\mu) - 12\mu\tan^2(z+\lambda\tau), \text{ with } \lambda = \pm 4\mu, \gamma = 0,$$

Hence the solutions of the equation (1.2) with

$$f = -\gamma \frac{C_7}{C_2^2} \text{ and } g = \frac{\mu C_7}{C_2^3}(t + C_0)$$

are

$$\begin{aligned} u(x,t) = \quad & -(\lambda + 12\mu) - \tfrac{12\gamma}{5} \tan\left[\tfrac{c_2 x - c_4 - \lambda c_7}{t + c_7} + (c_5 + \lambda c_8)\right] \\ & -12\mu \tan^2\left[\tfrac{c_2 x - c_4 - \lambda c_7}{t + c_7} + (c_5 + \lambda c_8)\right], \end{aligned}$$

with $\lambda = \pm 24\mu, \gamma = \pm 10 i\mu$,

$$u(x,t) = \quad -(\lambda + 8\mu) - 12\mu \tan^2\left[\tfrac{c_2 x - c_4 - \lambda c_7}{t + c_7} + (c_5 + \lambda c_8)\right],$$

with $\lambda = \pm 4\mu, \gamma = 0$.

4. Conclusion

In this paper we reduce the problem of finding solution to a VCKdVB equation to that of solving a similar equation with constant coefficients. This approach simplifies the tedious algebraic computations and allows us to use known results. First the variable coefficients KdV-Burgers equation is transformed into the standard form of the KdV-Burgers equation under the generalized transformation and we obtain the exact solutions of the variable coefficients KdV-Burgers equation through the standard form of the KdV-Burgers equation by using complex tanh method. We think that the results we presented in this work are new in the literature for VCKdVB equation.

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