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SOME INEQUALITIES FOR THE BETA FUNCTION

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Abstract. In this paper we establish some inequalities involving the Euler beta function. We use the ideas and methods that were used by A. McD. Mercer in his paper [**1**] for Bessel functions.

1. Introduction

We first recall the Euler's beta function [**5**], which has the integral representation

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x>0, y>0. \tag{1.1}$$

Euler generalized the factorial function from the domain of natural numbers to the gamma function and defined Euler's gamma function [**5**]

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt, \quad x>0. \tag{1.2}$$

In 1994 Chaudary and Zubair [**3**] have introduced the following extension of gamma function

$$\Gamma_b(x)=\int_0^\infty t^{x-1}e^{-t-\frac{b}{t}}dt, \quad b>0 \tag{1.3}$$

In 1997, Chaudary et al. [**2**] introduced the following extension of Euler's beta function

$$B(x,y,b)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{\frac{-b}{t(1-t)}}dt, \quad x>0, y>0, b>0. \tag{1.4}$$

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In 2011, Y. S. Kim et al. [**4**] introduced the following extension of Euler's beta function

$$B_b(x,y,m)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-b/t^m(1-t)^m}dt,\quad x>0,y>0,b>0. \tag{1.5}$$

Note that

$$B_0(x,y)=B(x,y,0)=B(x,y)$$

And it is easy to see that

THEOREM 1.1. *The following integral representation for beta function (1.1), (1.4) and (1.5) holds true :*

$$B(x,y)=2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta d\theta,\quad x>0,y>0. \tag{1.6}$$

$$B(x,y,b)=2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta e^{-b\sec^2\theta\csc^2\theta}d\theta,\quad x>0,y>0,b>0. \tag{1.7}$$

and

$$B_b(x,y,m)=2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta e^{-b\sec^{2m}\theta\csc^{2m}\theta}d\theta,\quad x>0,y>0,b>0. \tag{1.8}$$

PROOF. Equations (1.6), (1.7) and (1.8) can be obtained by the transformation $t=\cos^2\theta$. □

In section 2, we prove that some inequalities for the beta function (1.1) and the extended beta functions (1.4) and (1.5). We use the ideas and methods that were used by A. McD. Mercer in his paper [**1**].

2. Main Results

In this section, we prove the some inequalities for the beta function (1.1),(1.4) and (1.5), we also prove inequality between beta and the extended beta function.

THEOREM 2.1. *Holds*

$$|\tfrac{1}{\pi}B(x_1,y_1)+\tfrac{1}{\pi}B(x_2,y_2)|\leqslant 1+\tfrac{1}{\pi}B(x_1+x_2,y_1+y_2),$$

for all $x_1,y_1,x_2,y_2\geqslant 1$, *and* $0<\theta<\pi/2$.

PROOF. Let $\varepsilon_1=\pm1$ and $\varepsilon_2=\pm1$. We have

$$(1+\varepsilon_1\cos^{2x_1-1}\theta\sin^{2y_1-1}\theta)(1+\varepsilon_2\cos^{2x_2-1}\theta\sin^{2y_2-1}\theta)>0.$$

Multiplying out, integrating over $[0,\pi/2]$ and from (1.6), we get

$$\pi+\varepsilon_2B(x_2,y_2)+\varepsilon_1B(x_1,y_1)+\varepsilon_1\varepsilon_2B(x_1+x_2,y_1+y_2)>0.$$

Take $\varepsilon_1=\varepsilon_2=1$ and $\varepsilon_1=\varepsilon_2=-1$, and we get

$$-(\pi+B(x_1+x_2,y_1+y_2)<B(x_1,y_1)+B(x_2,y_2)<(\pi+B(x_1+x_2,y_1+y_2)$$

which is the desired result. □

Clearly, other, similar results can be obtained by changing the factors of the type $(1+\varepsilon\cos^{2x_1-1}\theta\sin^{2y_1-1}\theta)$ used in the above analysis. The simplest result uses a single factor and proceeds from the inequality

$$(1+\varepsilon\cos^{2x_1-1}\theta\sin^{2y_1-1}\theta)>0, \varepsilon=\pm 1$$

yielding the trivial result

$$|B(x_1,y_1)|<\pi$$

Similarly, we prove the inequality for the beta functions (1.4) and (1.5), which are stated in the next two theorems.

THEOREM 2.2. *Holds*

$$|\tfrac{1}{\pi}B(x_1,y_1,b_1)+\tfrac{1}{\pi}B(x_2,y_2,b_2)|\leqslant 1+\tfrac{1}{\pi}B(x_1+x_2,y_1+y_2,b_1+b_2),$$

for all $x_1,y_1,x_2,y_2\geqslant 1$, $b_1,b_2\geqslant 0$ *and* $0<\theta<\pi/2$.

THEOREM 2.3. *Holds*

$$|\tfrac{1}{\pi}B_{b_1}(x_1,y_1,m)+\tfrac{1}{\pi}B_{b_2}(x_2,y_2,m)|\leqslant 1+\tfrac{1}{\pi}B_{b_1+b_2}(x_1+x_2,y_1+y_2,m),$$

for all $x_1,y_1,x_2,y_2\geqslant 1$, $b_1,b_2,m\geqslant 0$ *and* $0<\theta<\pi/2$.

Next, we prove inequality between beta and the extended beta functions.

THEOREM 2.4. *For* $x,y,b>0$, *the* $B_b(x,y,m)\leqslant B(x,y,b)\leqslant B(x,y)$ *and equality holds if and only if* $b=0$.

PROOF. Since we have

$$e^{-b\sec^{2m}\theta\csc^{2m}\theta}\leqslant e^{-b\sec^{2}\theta\csc^{2}\theta}\leqslant 1,$$

multiplying with $2\cos^{2x-1}\theta\sin^{2y-1}\theta$ and integrating over $[0,\pi/2]$, we get

$$\begin{aligned}&2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta e^{-b\sec^{2m}\theta\csc^{2m}\theta}\mathrm{d}\theta\\ &\leqslant 2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta e^{-b\sec^{2}\theta\csc^{2}\theta}\mathrm{d}\theta\\ &\leqslant 2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\mathrm{d}\theta\end{aligned}$$

and from (1.6), (1.7) and (1.8), we have

$$B_b(x,y,m)\leqslant B(x,y,b)\leqslant B(x,y).$$

$\square$

Equality holds when $b=0$.

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