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COMMON FIXED POINT THEOREMS IN MENGER SPACE FOR SIX SELF MAPPINGS USING AN IMPLICIT RELATION AND CLR/JCLR-PROPERTY

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Abstract. The aim of this paper is to prove, mainly, three common fixed point theorems for six self mappings of a Menger space using two weakly compatible pairs having CLR/JCLR-property and satisfying an implicit relation. These generalize several known results including those of Kohli et. al.

1. Introduction

Presently, an interesting area of research is proving results in Menger space. Menger [**4**] introduced the concept of probabilistic Menger space. Kohli et. al [**2**], Kumar et. al [**3**] proved interesting results in Menger space. Nagaraja Rao et. al[**6**] generalized the results of Kumar and Pant [**3**]. Sintauravant et. al [**10**] introduced the concept of CLR-property and this is further generalized as JCLR-property by Chauhan et. al [**1**]. Using these concepts, we generalized the above mentioned results. We observed that the conditions of closedness of the subspaces and continuity of the mappings are not needed in establishing our results.

As usual $\mathbb{R}$ stands for the set of all real numbers, $\mathbb{R}^+$ stands for the set of all non-negative real numbers and $\mathbb{N}$ stands for the set of all natural numbers.

2. Preliminaries

We hereunder give the following definitions and the result required in subsequent section.

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DEFINITION 2.1. ([**8**]) A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution if and only if it is nondecreasing, left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. The set of all distribution functions are denoted by $\mathfrak{L}$.

For example, Heaviside function $H : \mathbb{R} \to \mathbb{R}^+$, defined by

$$H(t) = \begin{cases} 0 & \text{if } t \leqslant 0, \\ 1 & \text{if } t > 0 \end{cases}$$

is a distribution function.

DEFINITION 2.2. ([**8**]) Probabilistic metric space (PM-space) is an ordered pair (X, F), where X is a non empty set and $F : X \times X \to \mathfrak{L}$ is defined by $(p, q) \to F_{p,q}$ where $\{F_{p,q} : p, q \in X\} \subseteq \mathfrak{L}$, and the functions $F_{p,q}$ satisfy the following:

(a) $F_{p,q}(t) = 1$ for all $t > 0$ if and only if $p = q$;
(b) $F_{p,q}(0) = 0$;
(c) $F_{p,q}(t) = F_{q,p}(t)$;
(d) $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$, then $F_{p,r}(t + s) = 1$.

DEFINITION 2.3. ([**8**]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (or t-norm) if

(a) $T(0, 0) = 0$ and $T(a, 1) = a$ for all $a \in [0, 1]$;
(b) $T(a, b) = T(b, a)$, for all $a, b \in [0, 1]$;
(c) $T(a, b) \leqslant T(c, d)$ for all $a, b, c, d \in [0, 1]$ with $a \leqslant c$ and $b \leqslant d$;
(d) $T(T(a, b), c) = T(a, T(b, c))$ for all $a, b, c, d \in [0, 1]$.

DEFINITION 2.4. ([**8**]) A Menger space is a triplet (X, F, T), where (X, F) is a Probabilistic metric space and T is a t-norm such that for all $p, q, r \in X$ and all $t, s \geqslant 0$,

$$F_{p,r}(s + t) \geqslant T(F_{p,q}(s), F_{q,r}(t)).$$

DEFINITION 2.5. ([**9**]) Self mappings f and g of a Menger space (X, F, T) are said to be weakly compatible if and only if for any $t > 0$, $F_{fx,gx}(t) = 1$ for some $x \in X$ implies $F_{fgx,gfx}(t) = 1$; i.e, $fx = gx$ for some $x \in X$ implies $fgx = gfx$.

DEFINITION 2.6. ([**7**]) A function $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}$ is said to be an implicit relation if

(i.) ϕ is continuous,
(ii.) ϕ is Monotonic increasing in the first argument and
(iii.) ϕ satisfies the following conditions:
 (a) for $x, y \geqslant 0$, $\phi(x, y, x, y) \geqslant 0$ or $\phi(x, y, y, x) \geqslant 0$ implies $x \geqslant y$,
 (b) $\phi(x, x, 1, 1) \geqslant 0$ implies $x \geqslant 1$.

EXAMPLE 2.1. Define $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}$ by $\phi(x_1, x_2, x_3, x_4) = ax_1+bx_2+cx_3+dx_4$ with $a + b + c + d = 0$,$a + b > 0$, $a + c > 0$ and $a + d > 0$. Clearly, ϕ is an implicit relation. In particular,

(i.) $\phi(x_1, x_2, x_3, x_4) = 6x_1 - 3x_2 - 2x_3 - x_4$, (ii.) $\phi(x_1, x_2, x_3, x_4) = 5x_1 - 3x_3 - 2x_4$

are implicit relations.

Notation: Let Φ be the class of all implicit relations.

DEFINITION 2.7. ([**5**]) Let (X, F, T) be a Menger space, where T is continuous t-norm.

(a) A sequence $\{p_n\}$ in X is said to converge to a point p in X (written as $p_n \to p$) if for every $\in > 0$ and $\lambda > 0$, there exists a positive integer $M(\in, \lambda)$ such that $F_{p_n,p}(\in) > 1 - \lambda$ for all $n \geqslant M(\in, \lambda)$.

(b) A sequence $\{p_n\}$ in X is said to be Cauchy if for each every $\in > 0$ and $\lambda > 0$, there is a positive integer $M(\in, \lambda)$ such that $F_{p_n,p_m}(\in) > 1 - \lambda$ for all $n, m \in \mathbb{N}$ with $n, m \geqslant M(\in, \lambda)$.

(c) A Menger space (X, F, T) is said to be complete if every Cauchy sequence in X converges to a point of it.

LEMMA 2.1 ([**9**]). *Let $(X, F, *)$ be a sequence in a Menger space (X, F, T). If there is a $k \in (0, 1)$ such that*

$$F_{x,y}(kt) \geqslant F_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then $y = x$.

DEFINITION 2.8. Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, g, h, k be self mappings on X.

(a) The ordered pairs (f, g) and (h, k) are said to satisfy the "common limit in the range of g" (CLR_g-) property if and only if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} F_{fx_n,gx}(t) = \lim_{n\to\infty} F_{gx_n,gx}(t) = \lim_{n\to\infty} F_{hx_n,gx}(t) = \lim_{n\to\infty} F_{kx_n,gx}(t) = 1,$$

for some $x \in X$ and for all $t > 0$.

(b) The ordered pairs (f, g) and (h, k) are said to satisfy the " joint common limit in the ranges of g and k" $(JCLR_{gh}-)$ property if and only if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $ku = gu$ and

$$\lim_{n\to\infty} F_{fx_n,gu}(t) = \lim_{n\to\infty} F_{gx_n,gu}(t) = \lim_{n\to\infty} F_{hx_n,gu}(t) = \lim_{n\to\infty} F_{kx_n,gu}(t) = 1,$$

for some $u \in X$ and for all $t > 0$.

3. Main theorem

THEOREM 3.1. *Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, g, h, k, p and q be self mappings of X, satisfying:*

(i) *$p(X) \subseteq fg(X)$ and $q(X) \subseteq hk(X)$;*

(ii) *the pairs $\{p, hk\}$ and $\{q, fg\}$ be weakly compatible;*

(iii) *the ordered pairs* (p, hk) *and* (q, fg) *share either*
 (a) CLR_p*-property or*
 (b) CLR_q*-property;*

(iv) $\phi(F_{px,qy}(\alpha t), F_{hkx,fgy}(t), F_{px,hkx}(t), F_{qy,fgy}(\alpha t)) \geqslant 0$,
for all $x, y \in X$ *&* $t > 0$ *and for some* $\phi \in \Phi$ *&* $\alpha \in (0,1)$*;*

(v) h *commutes with* k *and 'either* p *commutes with* h *or with* k*';*

(vi) f *commutes with* g *and 'either* q *commutes with* f *or with* g*'.*

Then f, g, h, k, p *and* q *have a unique common fixed point in* X.

Proof. **Case I**: Suppose $(iii)(a)$ holds.
By definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} px_n = \lim_{n\to\infty} hkx_n = \lim_{n\to\infty} qy_n = \lim_{n\to\infty} fgy_n = pu, \text{for some } u \in X.$$

Since $p(X) \subseteq fg(X)$, there is a $v \in X$ such that $pu = fgv$. By taking $x = x_n$ and $y = v$ in (iv), we get that

$$\phi(F_{px_n,qv}(\alpha t), F_{hkx_n,fgv}(t), F_{px_n,hkx_n}(t), F_{qv,fgv}(\alpha t)) \geqslant 0.$$

As $n \to \infty$, the above becomes

$$\phi(F_{pu,qv}(\alpha t), F_{pu,fgv}(t), F_{pu,pu}(t), F_{qv,fgv}(\alpha t)) \geqslant 0.$$

So, by the property of ϕ, $F_{pu,qv}(\alpha t) \geqslant F_{pu,qv}(t)$. By Lemma(2.1), $pu = qv$. Since $q(X) \subseteq hk(X)$, there is a $w \in X$ such that $qv = hkw$. By taking $x = w$ and $y = v$ in (iv), we get that

$$\phi(F_{pw,qv}(\alpha t), F_{hkw,fgv}(t), F_{pw,hkw}(t), F_{qv,fgv}(\alpha t)) \geqslant 0$$

i.e,

$$\phi(F_{pw,qv=hkw}(\alpha t), 1, F_{pw,hkw}(t), 1) \geqslant 0$$

So, by the property of ϕ, we have

$$\phi(F_{pw,qv=hkw}(t), 1, F_{pw,hkw}(t), 1) \geqslant 0 \Rightarrow F_{pw,hkw}(t) \geqslant 1 \Rightarrow pw = hkw.$$

Thus $pw = hkw = pu = fhv = qu = z$(say).

Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible, we have $p(hk)w = hk(p)w$ and $q(fg)v = fg(q)v$. $i.e, pz = hkz$ and $qz = fgz$.

By putting $x = z$ and $y = v$ in (iv), we get that

$$\phi(F_{pz,qv=z}(\alpha t), F_{hkz=pz,fgv=z}(t), F_{pz,hkz=pz}(t), F_{qv=z,fgv=z}(\alpha t)) \geqslant 0$$

i.e,

$$\phi(F_{pz,z}(\alpha t), F_{pz,z}(t), 1, 1) \geqslant 0 \Rightarrow \phi(F_{pz,z}(t), F_{pz,z}(t), 1, 1) \geqslant 0$$
$$\Rightarrow F_{pz,z}(t) \geqslant 1 \Rightarrow pz = z.$$

Similarly, by taking $x = w$ and $y = z$ in (iv), we get that $qz = z$. Thus $pz = hkz = z = qz = fgz$.

Since h commutes with k, we have $hk(hz) = h(hkz) = hz$. Suppose p commutes with h, so $p(hz) = h(pz) = hz$; by taking $x = hz$ and $y = z$ in (iv), we get that

$$\phi(F_{phz=hz,qz=z}(\alpha t), F_{hkhz=hz,fgz=z}(t), F_{phz=hz,hkhz=hz}(t), F_{qz=z,fgz=z}(\alpha t)) \geqslant 0$$

$$\Rightarrow \phi(F_{hz,z}(\alpha t), F_{hz,z}(t), F_{hz,hz}(t), F_{z,z}(\alpha t)) \geqslant 0$$
$$\Rightarrow \phi(F_{hz,z}(t), F_{hz,z}(t), 1, 1) \geqslant 0$$
$$\Rightarrow F_{hz,z}(t) \geqslant 1 \Rightarrow hz = z.$$

Since $hkz = z$, follows that $kz = z$. Thus $hz = kz = pz = z$. Suppose p commutes with k, so $p(kz) = k(pz) = kz$. Since h commutes with k, we have $hk(kz) = k(hkz) = kz$. By taking $x = kz$ and $y = z$ in (iv), we get that $kz = z$. Since $hkz = z$, follows that $hz = z$. Thus $hz = kz = pz = z$.

Now, (vi) is similar to (v) when p, h, k are replaced by q, f, g respectively. Hence as above, we get $z = fz = gz = qz$. Thus $fz = gz = hz = kz = pz = qz = z$. Hence z is a common fixed point of f, g, h, k, p and q.

Case II: Suppose $(iii)(b)$ holds. By definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} px_n = \lim_{n\to\infty} hkx_n = \lim_{n\to\infty} qy_n = \lim_{n\to\infty} fgy_n = qv$$

for some $v \in X$. Since $q(X) \subseteq hk(X)$, there is a $u \in X$ such that $qv = hku$. By taking $x = u$ and $y = y_n$ in (iv), we get that $qv = pu$. Since $p(X) \subseteq fg(X)$, there is a $w \in X$ such that $pu = fgw$. By taking $x = u$ and $y = w$ in (iv), we get that $qw = fgw$. Thus $pu = hku = qv = fgw = qw = z$(say). Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible, $p(hk)u = hk(p)u$ and
$q(fg)w = fg(q)w$. $i.e, pz = hkz$ and $qz = fgz$. From this stage, the proof is the same given in the previous case. Thus, z is a common fixed point of f, g, h, k, p and q.

Uniqueness: If w is also a common fixed point of f, g, h, k, p and q. By taking $x = z$ and $y = w$ in (iv), we get that

$$\phi(F_{pz,qw}(\alpha t), F_{hkz,fgw}(t), F_{pz,hkz}(t), F_{qw,fgw}(\alpha t)) \geqslant 0$$

i.e,

$$\phi(F_{z,w}(\alpha t), F_{z,w}(t), F_{z,z}(t), F_{w,w}(\alpha t)) \geqslant 0$$

So, by property of ϕ, $F_{z,w}(\alpha t) \geqslant F_{z,w}(t)$. By lemma(2.1), we get that $w = z$. Hence z is the unique common fixed point of f, g, h, k, p and q. This completes the proof of the theorem. ☐

NOTE 3.1. Theorem (3.1) is also valid if

(a) (iv) is replaced by $\phi(F_{px,qy}(\alpha t), F_{hkx,fgy}(t), F_{px,hkx}(\alpha t), F_{qy,fgy}(t)) \geqslant 0$.

(b) (i) is replaced by $q(X) \subseteq hk(X)$ and (iii) is replaced by (p, hk) and (q, fg) share $CLR_{(fg)}$-property.

(c) (i) is replaced by $p(X) \subseteq fg(X)$ and (iii) is replaced by (p, hk) and (q, fg) share $CLR_{(hk)}$-property.

Now we give the following example in support of our Theorem (3.1).

EXAMPLE 3.1. Let $X = [0, \infty)$, $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Then $(X, F, *)$ is a Menger space.

Define self mappings f, g, h, k, p and q on X by $fx = x^2, gx = x^{\frac{1}{2}}$, $hx = x^3, kx = x$,

$$p(x) = \begin{cases} 0 & \text{if } x \leqslant 1, \\ \frac{1}{2} & \text{if } x > 1, \end{cases}$$

$qx = 0$, for all $x \in X$. Define $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}$ by

$$\phi(x_1, x_2, x_3, x_4) = 6x_1 - 3x_2 - 2x_3 - x_4.$$

Then ϕ is an implicit relation.

For $x \leqslant 1$ and $y \in X$, we have

$$\begin{aligned} &\phi(F_{0,0}(\alpha t), F_{x^3,y}(t), F_{0,x^3}(t), F_{0,y}(\alpha t)) \\ &= 6 - 3\tfrac{t}{t+|x^3-y|} - 2\tfrac{t}{t+x^3} - \tfrac{\alpha t}{\alpha t+y} \\ &\geqslant 6 - 3 - 2 - 1 = 0. \end{aligned}$$

For $x > 1$ and $y \in X$, we have

$$\begin{aligned} &\phi(F_{\frac{1}{2},0}(\alpha t), F_{x^3,y}(t), F_{\frac{1}{2},x^3}(t), F_{0,y}(\alpha t)) \\ &= 6\tfrac{\alpha t}{\alpha t+\frac{1}{2}} - 3\tfrac{t}{t+|x^3-y|} - 2\tfrac{t}{t+x^3-\frac{1}{2}} - \tfrac{\alpha t}{\alpha t+y} \\ &\geqslant 6 - 3 - 2 - 1 = 0. \end{aligned}$$

The other conditions of the Theorem are trivially satisfied. Clearly $'0'$ is the unique common fixed point of f, g, h, k, p and q in X.

Now, taking $g = k = I$(the identity mapping on X) in Theorem (3.1), we have the following:

Corollary 3.1. *Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, h, p and q be self mappings of X, satisfying:*

(i) *$p(X) \subseteq f(X)$ and $q(X) \subseteq h(X)$;*

(ii) *the pairs $\{p, h\}$ and $\{q, f\}$ are weakly compatible;*

(iii) *the ordered pairs (p, h) and (q, f) share either (a) CLR_p-property or (b) CLR_q-property;*

(iv) *$\phi(F_{px,qy}(\alpha t), F_{hx,fy}(t), F_{px,hx}(t), F_{qy,fy}(\alpha t)) \geqslant 0$, for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$.*

Then f, g, h, k, p and q have a unique common fixed point in X.

Now, we prove the following:

Theorem 3.2. *Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, g, h, k, p and q be self mappings of X, satisfying:*

(i) *the pairs $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible;*

(ii) *the ordered pairs (p, hk) and (q, fg) share $JCLR_{(hk)(fg)}$-property;*

(iii) *$\phi(F_{px,qy}(\alpha t), F_{hkx,fgy}(t), F_{px,hkx}(t), F_{qy,fgy}(\alpha t)) \geqslant 0$, for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$;*

(iv) *h commutes with k and 'either p commutes with h or with k';*

(v) *f commutes with g and 'either q commutes with f or with g'.*

Then f, g, h, k, p and q have a unique common fixed point in X.

PROOF. Suppose (p, hk) and (q, fg) share $JCLR_{(hk)(fg)}$-property, by definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} px_n = \lim_{n\to\infty} hkx_n = \lim_{n\to\infty} qy_n = \lim_{n\to\infty} fgy_n = hku = fgu,$$

for some $u \in X$. By taking $x = u$ and $y = y_n$ in (iii), we get that

$$\phi(F_{pu,qy_n}(\alpha t), F_{hku,fgy_n}(t), F_{pu,hkx}(t), F_{qy_n,fgy_n}(\alpha t)) \geqslant 0$$

As $n \to \infty$, we get that

$$\phi(F_{pu,hku}(\alpha t), F_{hku,hku}(t), F_{pu,hku}(t), F_{hku,hku}(\alpha t)) \geqslant 0$$

i.e,

$$\begin{gathered}\phi(F_{pu,hku}(\alpha t), 1, F_{pu,hku}(t), 1) \geqslant 0\\ \Rightarrow \phi(F_{pu,hku}(t), 1, F_{pu,hku}(t), 1) \geqslant 0\\ \Rightarrow F_{pu,hku}(t) \geqslant F_{pu,hku}(t) \Rightarrow pu = hku \text{ (by lemma(2.1))}.\end{gathered}$$

By taking $x = x_n$ and $y = u$ in (iii), we get that

$$\phi(F_{px_n,qu}(\alpha t), F_{hkx_n,fgu}(t), F_{px_n,hkx_n}(t), F_{qu,fgu}(\alpha t)) \geqslant 0$$

As $n \to \infty$, we get that

$$\begin{gathered}\phi(F_{fgu,qu}(\alpha t), F_{fgu,fgu}(t), F_{fgu,fgu}(t), F_{qu,fgu}(\alpha t)) \geqslant 0\\ \Rightarrow F_{fgu,qu}(\alpha t) \geqslant 1 \Rightarrow fgu = qu.\end{gathered}$$

Thus $fgu = qu = pu = hku = z$ (say). Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible, $p(hk)u = hk(p)u$ and $q(fg)u = fg(q)u$. i.e, $pz = hkz$ and $qz = fgz$. From this stage, the proof is the same given in the Theorem (3.1). Hence, we get that z is a common fixed point of f, g, h, k, p and q.

Uniqueness follows trivially. □

NOTE 3.2. Theorem (3.2) is also valid if (iii) is replaced by

$$\phi(F_{px,qy}(\alpha t), F_{hkx,fgy}(t), F_{px,hkx}(\alpha t), F_{qy,fgy}(t)) \geqslant 0.$$

We now give the following example in support of Theorem (3.2).

EXAMPLE 3.2. Let $X = [0, \infty)$, $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Then $(X, F, *)$ is a Menger space.

Define self mappings f, g, h, k, p and q on X by $fx = x^4, gx = x^{\frac{1}{2}}$, $hx = x^5, kx = x$,

$$p(x) = \begin{cases} 0 & \text{if } x \leqslant 3,\\ 2 & \text{if } x > 3,\end{cases}$$

$qx = 0$, for all $x \in X$. Define $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}$ by $\phi(x_1, x_2, x_3, x_4) = 5x_1 - 3x_2 - 2x_4$. Then ϕ is an implicit relation.

For $x \leqslant 3$ and $y \in X$, we have

$$\begin{aligned}\phi(F_{0,0}(\alpha t), F_{x^5,y^2}(t), F_{0,x^5}(t), F_{0,y^2}(\alpha t)) &= 5 - 3\tfrac{t}{t+|x^5-y^2|} - \tfrac{\alpha t}{\alpha t+y^2}\\ &\geqslant 5 - 3 - 2 = 0.\end{aligned}$$

For $x > 3$ and $y \in X$, we have

$$\phi(F_{2,0}(\alpha t), F_{x^5,y^2}(t), F_{2,x^5}(t), F_{0,y^2}(\alpha t)) = 5\tfrac{\alpha t}{\alpha t+2} - 3\tfrac{t}{t+|x^5-y^2|} - 2\tfrac{\alpha t}{\alpha t+y^2}$$
$$\geqslant 5 - 3 - 2 = 0.$$

The other conditions of the Theorem are trivially satisfied. Clearly $'0'$ is the unique common fixed point of f, g, h, k, p and q in X.

By taking $g = k = I$ (the identity mapping on X) in Theorem (3.2), we have the following:

Corollary 3.2. *Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, h, p and q be self mappings of X, satisfying:*

(i) *the pairs $\{p, h\}$ and $\{q, f\}$ are weakly compatible;*
(ii) *the ordered pairs (p, h) and (q, f) share $JCLR_{hf}$-property;*
(iii) $\phi(F_{px,qy}(\alpha t), F_{hx,fy}(t), F_{px,hx}(t), F_{qy,fy}(\alpha t)) \geqslant 0$,
for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$.

Then f, h, p and q have a unique common fixed point in X.

Now we prove the following:

Theorem 3.3. *Let (X, F, T) be a Menger space, where T denotes a continuous t-norm and f, g, h, k, p and q be self mappings of X, satisfying:*

(i) *the pairs $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible;*
(ii) *the ordered pairs (p, hk) and (q, fg) share $JCLR_{pq}$-property;*
(iii) *one of the following holds:*
either (a) $\phi(F_{hkx,fgy}(\alpha t), F_{px,fgy}(t), F_{hkx,qy}(t), F_{px,qy}(\alpha t)) \geqslant 0$,
or (b) $\phi(F_{hkx,fgy}(\alpha t), F_{px,fgy}(t), F_{hkx,qy}(\alpha t), F_{px,qy}(t)) \geqslant 0$,
for all $x, y \in X$ & $t > 0$ and for some $\phi \in \Phi$ & $\alpha \in (0, 1)$;
(iv) *h commutes with k and 'either p commutes with h or with k';*
(v) *f commutes with g and 'either q commutes with f or with g'.*

Then f, g, h, k, p and q have a unique common fixed point in X.

Proof. Since (p, hk) and (q, fg) share $JCLR_{pq}$-property, by definition, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} px_n = \lim_{n\to\infty} hkx_n = \lim_{n\to\infty} qy_n = \lim_{n\to\infty} fgy_n = pu = qu,$$

for some $u \in X$.

Case I: Suppose $(iii)(a)$ holds. By taking $x = x_n$ and $y = u$ in $(iii)(a)$, we get that

$$\phi(F_{hkx_n,fgu}(\alpha t), F_{px_n,fgu}(t), F_{hkx_n,qu}(t), F_{px_n,qu}(\alpha t)) \geqslant 0.$$

As $n \to \infty$, we get that

$$\phi(F_{pu=qu,fgu}(\alpha t), F_{pu=qu,fgu}(t), F_{pu=qu,qu}(t), F_{pu=qu,qu}(\alpha t)) \geqslant 0$$

i.e,

$$\begin{gathered}\phi(F_{qu,fgu}(\alpha t), F_{qu,fgu}(t), 1, 1) \geqslant 0\\ \Rightarrow \phi(F_{qu,fgu}(t), F_{qu,fgu}(t), 1, 1) \geqslant 0\\ \Rightarrow F_{qu,fgu}(t) \geqslant 1 \Rightarrow qu = fgu\end{gathered}$$

By taking $x = u$ and $y = y_n$ in $(iii)(a)$, we get that

$$\phi(F_{hku,fgy_n}(\alpha t), F_{pu,fgy_n}(t), F_{hku,qy_n}(t), F_{pu,qy_n}(\alpha t)) \geqslant 0.$$

As $n \to \infty$, we get that

$$\phi(F_{hku,pu=qu}(\alpha t), F_{pu,pu=qu}(t), F_{hku,pu=qu}(t), F_{pu,pu=qu}(\alpha t)) \geqslant 0$$

i.e,

$$\begin{gathered}\phi(F_{hku,pu}(\alpha t), F_{pu,pu}(t), F_{hku,pu}(t), F_{pu,pu}(\alpha t)) \geqslant 0\\ \Rightarrow \phi(F_{hku,pu}(\alpha t), 1, F_{hku,pu}(t), 1) \geqslant 0\\ \Rightarrow \phi(F_{hku,pu}(t), 1, F_{hku,pu}(t), 1) \geqslant 0 \text{ (by the property of } \phi)\\ \Rightarrow F_{hku,pu}(t) \geqslant 1 \Rightarrow hku = pu.\end{gathered}$$

Thus $fgu = qu = hku = pu = z$(say)

Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible,

$$p(hk)u = hk(p)u \text{ and } q(fg)u = fg(q)u$$

i.e, $pz = hkz$ and $qz = fgz$.

From this stage, the proof is the same given in the Theorem(3.1). Hence, we get that z is a common fixed point of f, g, h, k, p and q.

Case II: Suppose $(iii)(b)$ holds: By taking $x = x_n$ and $y = u$ in $(iii)(b)$, we get that

$$\phi(F_{hkx_n,fgu}(\alpha t), F_{px_n,fgu}(t), F_{hkx_n,qu}(\alpha t), F_{px_n,qu}(t)) \geqslant 0.$$

As $n \to \infty$, we get that

$$\phi(F_{pu=qu,fgu}(\alpha t), F_{pu=qu,fgu}(t), F_{pu=qu,qu}(\alpha t), F_{pu=qu,qu}(t)) \geqslant 0$$

i.e,

$$\begin{gathered}\phi(F_{qu,fgu}(\alpha t), F_{qu,fgu}(t), 1, 1) \geqslant 0\\ \Rightarrow \phi(F_{qu,fgu}(t), F_{qu,fgu}(t), 1, 1) \geqslant 0\\ \Rightarrow F_{qu,fgu}(t) \geqslant 1 \Rightarrow qu = fgu.\end{gathered}$$

By taking $x = u$ and $y = y_n$ in $(iii)(b)$, we get that

$$\phi(F_{hku,fgy_n}(\alpha t), F_{pu,fgy_n}(t), F_{hku,qy_n}(\alpha t), F_{pu,qy_n}(t)) \geqslant 0.$$

As $n \to \infty$, we get that

$$\phi(F_{hku,pu=qu}(\alpha t), F_{pu,pu=qu}(t), F_{hku,pu=qu}(\alpha t), F_{pu,pu=qu}(t)) \geqslant 0$$

i.e,

$$\phi(F_{hku,pu}(\alpha t), F_{pu,pu}(t), F_{hku,pu}(\alpha t), F_{pu,pu}(t)) \geqslant 0$$
$$\Rightarrow \phi(F_{hku,pu}(\alpha t), 1, F_{hku,pu}(\alpha t), 1) \geqslant 0$$
$$\Rightarrow F_{hku,pu}(\alpha t) \geqslant 1 \text{ (by the property of } \phi)$$
$$\Rightarrow hku = pu.$$

Thus $fgu = qu = hku = pu = z$(say).

Since $\{p, hk\}$ and $\{q, fg\}$ are weakly compatible,

$$p(hk)u = hk(p)u \text{ and } q(fg)u = fg(q)u$$

i.e, $pz = hkz$ and $qz = fgz$. From this stage, the proof is the same given in the Theorem(3.1). Hence, we get that z is a common fixed point of f, g, h, k, p and q.

Uniqueness follows trivially. □

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