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GRAPH OPERATIONS ON WEAKLY COVERED GRAPHS

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Abstract. Let G be a finite simple undirected graph. G is said to be a well covered graph if every maximal independent set is a maximum independent set. A graph is weakly well covered if every non-maximal independent set is contained in a maximum independent set. In an earlier paper, some results on weakly well covered graphs are derived. In this paper, a study of weakly well covered nature of $G \cup H$, $G + H$, $G \times H$, $G \square H$, $G \boxtimes H$ is made when G and H are weakly well covered graphs.

1. Introduction

(u_1, v_1), In graph theory an independent set is a set of vertices in a graph, no two of which are adjacent. A maximal independent set is either an independent set such that adding any other vertex to the set forces the set to contain an edge (or) the set of all vertices of the empty graph. A maximum independent set is an independent set of largest possible size for a given graph G, and the maximum cardinality is denoted by $\beta_0(G)$.

There are several ways to combine two graphs G_1 and G_2. Suppose that G_1 has vertex set V_1 and edge set E_1 and that G_2 has vertex set V_2 and edge set E_2. The union of the two graphs, written $G_1 \cup G_2$ will have vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The sum of two graphs G_1 and G_2, written $G_1 + G_2$ is obtained by first forming the union $G_1 \cup G_2$ and then making every vertex of G_1 adjacent to every vertex of G_2. The cartesian product $G \square H$ of graphs G and H is a graph such that, the vertex set of $G \square H$ is the cartesian product $V(G) \times V(H)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \square H$ if and only if either $u_1 = v_1$ and u_2 is adjacent with v_2 in H (or) $u_2 = v_2$ and u_1 is adjacent with v_1 in G. The tensor product $G \times H$ of graphs G and H is a graph such that, the vertex set of $G \times H$ is

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the cartesian product $V(G) \times V(H)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \times H$ if and only if u_2 is adjacent with v_2 and u_1 is adjacent with v_1.

The strong product $G \boxtimes H$ of graphs G and H is a graph such that, the vertex set of $G \boxtimes H$ is the cartesian product $V(G) \times V(H)$ and any two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \boxtimes H$ if and only if u_1 is adjacent with v_1 and $u_2 = v_2$ (or) $u_1 = v_1$ and u_2 is adjacent with v_2 (or) u_2 is adjacent with v_2 and u_1 is adjacent with v_1. In this paper we verify whether the sum, unoin, Cartesion product, Tensor product, Strong product of two weakly well covered graphs are weakly well covered are not.

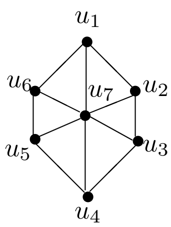
2. Weakly well covered graphs

Definition 1. *A graph G is said to be well covered if every maximal independent set of G is a maximum independent set of G.*

Example 2.
(i) C_4 is well covered.
(ii) $K_{m,n}$ is well covered.

Definition 3. *A graph G is said to be weakly well covered if every non-maximal independent set of G is contained in a maximum independent set of G.*

Example 4. *Let $G = W_7$. Here $\beta_0(G) = 3$. Also, $\{u_1, u_3, u_5\}$, $\{u_2, u_4, u_6\}$ are maximum independent sets of G. Clearly any non-maximal independent set of G is contained in a β_0-set of G. Therefore G is a weakly well covered graph.*



3. Graph operations on Weakly well covered graphs

Proposition 5. *Let G and H be weakly well covered. Then $G \cup H$ need not be weakly well covered.*

For example, let $G_1 = C_{10}$, $G_2 = C_{12}$. Both G_1 and G_2 are weakly well covered. Let $V(G_1) = \{u_1, u_2, \ldots, u_{10}\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_{12}\}$ and let $S_1 = \{u_1, u_3, u_5, u_7, u_9\}$ and $S_2 = \{v_1, v_4, v_7, v_{10}\}$. S_2 is not contained in any maximum independent set in G_2. $S_1 \cup S_2$ is a non-maximal independent set in $G_1 \cup G_2$, which is not contained in a maximum independent set of $G_1 \cup G_2$. Therefore $C_{10} \cup C_{12}$ is not weakly well covered.

Remark 6. *Let G and H be weakly well covered. let S be a non-maximal independent set in $G \cup H$. Let $S \cap V(G) = S_1$ and $S \cap V(H) = S_2$. Let S_1 and*

S_2 be non-maximal. Then $S_1 \cup S_2$ is contained in a maximum independent set of $G \cup H$.

PROOF. Let S_1 and S_2 be non-maximal. By hypothesis, there exist maximum independent sets T_1 and T_2 in G and H respectively such that $S_1 \subset T_1$ and $S_2 \subset T_2$. Clearly $T_1 \cup T_2$ is a maximum independent set in $G \cup H$ and $S_1 \cup S_2 = S \subset (T_1 \cup T_2)$. Therefore $G \cup H$ is weakly well covered. □

REMARK 7. *Let G be weakly well covered and H be well covered. Let S be a non-maximal independent set in $G \cup H$. Let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. Let S_1 be non-maximal. Then S is contained in a maximum independent set of $G \cup H$.*

PROOF. Let S_1 be non-maximal and S_2 be a maximal independent set. Since G is weakly well covered, there exists a maximum independent set T_1 in G such that $S_1 \subset T_1$. Since H is well covered, S_2 is a maximum independent set of H. Therefore $S \subset (T_1 \cup S_2)$ and $T_1 \cup S_2$ is a maximum independent set in $G \cup H$. Therefore $G \cup H$ is weakly well covered. □

REMARK 8. *If G and H are well covered, then $G \cup H$ is well covered and hence weakly well covered.*

THEOREM 9. *Let G and H be weakly well covered graphs and $\beta_0(G) = \beta_0(H)$. Then $G + H$ is weakly well covered.*

PROOF. Let S be a non-maximal independent set in $G + H$. Then S is either completely contained in $V(G)$ (or) completely contained in $V(H)$. Since S is non-maximal, either S is a non-maximal independent set in G (or) non-maximal independent set in H. Since G and H are weakly well covered and $\beta_0(G) = \beta_0(H)$. S is contained in a maximum independent set of $G+H$. Therefore $G+H$ is weakly well covered. □

EXAMPLE 10. *Let $G = W_7$ and $H = P_6$. Let $V(G) = \{u_1, u_2, \dots, u_7\}$ and $V(H) = \{v_1, v_2, \dots, v_6\}$. Let $S = \{u_1, u_3\}$. Then S is a non-maximal independent set in $G+H$. S is contained in a maximum independent set namely $T = \{u_1, u_3, u_5\}$ in G. T is also a maximum independent set in $G + H$.*

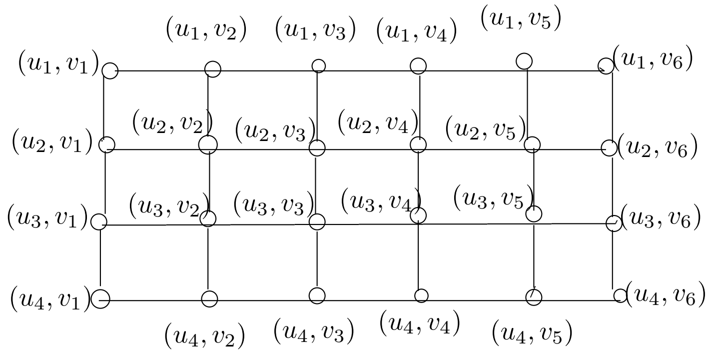
REMARK 11. *If G and H are weakly well covered and $\beta_0(G) \neq \beta_0(H)$, then $G + H$ need not be weakly well covered.*

For example, let Let $G = W_7$ and $H = P_8$. $\beta_0(G) = 3$ and $\beta_0(H) = 4$. $\beta_0(G+H) = 4$. Any non-maximal independent set in G is contained in a maximum independent set of G of cardinality 3. But $\beta_0(G+H) = 4$. Therefore $G+H$ is not weakly well covered.

PROPOSITION 12. *Let G and H be weakly well covered. Then*
(i) $G \Box H$ *need not be weakly well covered.*
(ii) $G \times H$ *need not be weakly well covered.*
(iii) $G \boxtimes H$ *need not be weakly well covered.*

Proof. **(i)** Let $G_1 = P_4$ and $G_2 = P_6$. Both G_1 and G_2 are weakly well covered. Let $V(G_1) = \{u_1, u_2, u_3, u_4\}$ and $V(G_2) = \{v_1, v_2, \ldots, v_6\}$.In $G_1 = P_4$, $\beta_0(P_4) = 2$. Also every non maximal independent set is contained in $\beta_0(P_4)$. That is P_4 is weakly well covered graph. Also $G_2 = P_6$, $\beta_0(P_6) = 3$, every non maximal independent set is contained in $\beta_0(P_6)$. That is P_6 is weakly well covered graph.

In $P_4\Box P_6$,



Here

$$S_1 = (u_1, v_1), (u_1, v_3), (u_1, v_5), (u_2, v_2), (u_2, v_4), (u_2, v_6), (u_3, v_1), (u_3, v_3),$$
$$(u_3, v_5), (u_4, v_2), (u_4, v_4), (u_4, v_6).$$
$$S_2 = (u_1, v_2), (u_1, v_4), (u_1, v_6), (u_2, v_1), (u_2, v_3),$$
$$(u_2, v_5), (u_3, v_2), (u_3, v_4), (u_3, v_6), (u_4, v_1), (u_4, v_3), (u_4, v_5)$$

are the maximum independent sets of cardinality 12. That is $\beta_0(P_4\Box P_6) = 12$.

Here $T = \{(u_1, v_2), (u_1, v_5), (u_2, v_4), (u_3, v_1), (u_3, v_6), (u_4, v_5)\}$ is a non-maximal independent set which is not contained in any β_0-set of $P_4\Box P_6$. Hence $P_4\Box P_6$ is not weakly well covered graph.

(ii) Let $G_1 = C_5$ and $G_2 = P_4$. Both G_1 and G_2 are weakly well covered. Let $V(G_1) \doteq \{u_1, u_2, u_3, u_4, u_5\}$ and $V(G_2) = \{v_1, v_2, v_3, v_4\}$.In $G_1 = C_5$, $\beta_0(C_5) = 2$. Also every non maximal independent set is contained in $\beta_0(C_5)$. That is C_5 is weakly well covered graph. Also $G_2 = P_4$, $\beta_0(P_4) = 2$, every non maximal independent set is contained in $\beta_0(P_4)$. That is P_4 is weakly well covered graph.

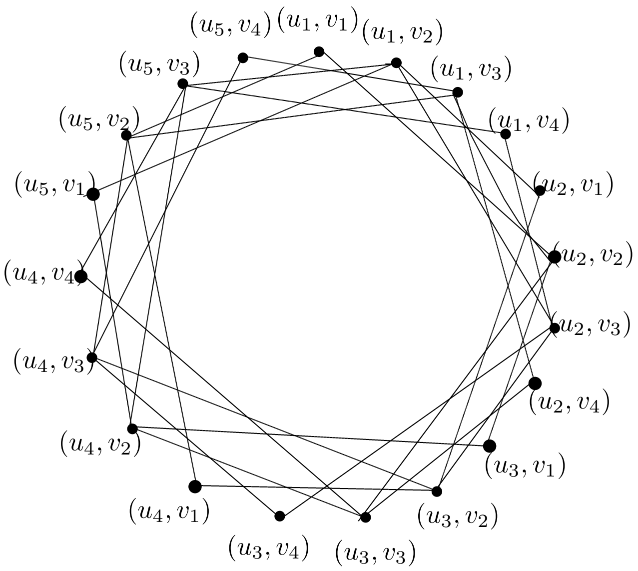
In $C_5 \times P_4$

(Here

$$S_1 = \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (u_3, v_1), (u_3, v_2), (u_3, v_3), (u_3, v_4)\}$$
$$S_2 = \{(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_4, v_1), (u_4, v_2), (u_4, v_3), (u_4, v_4)\}$$
$$S_3 = \{(u_3, v_1), (u_3, v_2), (u_3, v_3), (u_3, v_4), (u_5, v_1), (u_5, v_2), (u_5, v_3), (u_5, v_4)\}$$

are the maximum independent sets of cardinality 8. That is $\beta_0(C_5 \times P_4) = 8$.

Here $T = \{(u_1, v_1), (u_1, v_2), (u_4, v_1), (u_4, v_2)\}$ is a non-maximal independent set which is not contained in any β_0-set of $C_5 \times P_4$. Hence $C_5 \times P_4$ is not weakly well covered graph.)



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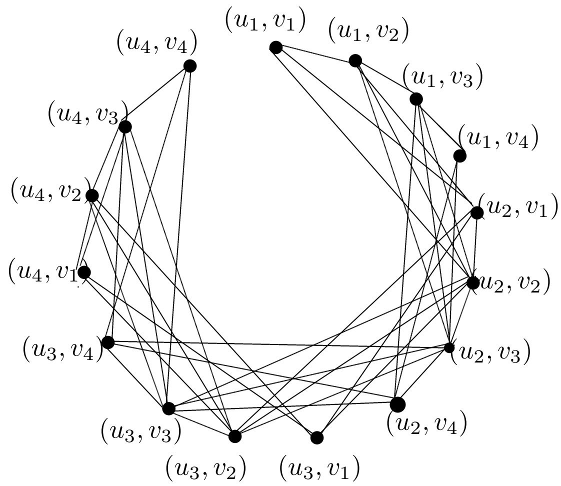
(iii) Let $G_1 = P_4$ and $G_2 = P_4$. Both G_1 and G_2 are weakly well covered. Let $V(G_1) = \{u_1, u_2, u_3, u_4\}$ and $V(G_2) = \{v_1, v_2, v_3, v_4\}$.

In $P_4 \boxtimes P_4$



Here

$$S_1 = \{(u_1, v_1), (u_1, v_3), (u_3, v_1), (u_3, v_2), (u_3, v_4)\}$$
$$S_2 = \{(u_1, v_2), (u_1, v_4), (u_3, v_1), (u_3, v_2), (u_3, v_4)\}$$
$$S_3 = \{(u_1, v_1), (u_2, v_4), (u_3, v_1), (u_3, v_2), (u_4, v_4)\}$$
$$S_4 = \{(u_1, v_1), (u_1, v_3), (u_3, v_1), (u_3, v_2), (u_4, v_4)\}$$

are the maximum independent sets of cardinality 5. That is $\beta_0(P_4 \boxtimes P_4) = 5$.

Here $T = \{(u_1, v_3), (u_2, v_1), (u_3, v_3)\}$ is a non-maximal independent set which is not contained in any β_0-set of $P_4 \boxtimes P_4$. Hence $P_4 \boxtimes P_4$ is not weakly well covered graph. □

4. 1-k-extendable graphs

DEFINITION 13. *A graph G is 1-k-extendable if for every $u \in V(G)$, $\beta_0(G-u) = \beta_0(G)$ and $G-u$ is k-extendable.*

EXAMPLE 14. *Let G be a component wise complete graph. Let $u \in V(G)$. Let $G = G_1 \cup G_2 \cup \ldots \cup G_r$, where each G_i is complete and $|V(G_i)| \geqslant 2$ for every i, $1 \leqslant i \leqslant r$. $\beta_0(G) = r$. $\beta_0(G-u) = r$ for any u in $V(G)$. G is 1-extendable. Therefore G is 1-1-extendable. G is also k-extendable, $1 \leqslant k \leqslant (r-1)$. Therefore G is $1-k$-extendable for $1 \leqslant k \leqslant (\beta_0(G)-1)$.*

EXAMPLE 15. *Let $G = C_n$. Then $\beta_0(G-u) = \beta_0(G)$ for every u in $V(G)$. If n is odd, P_{n-1} is 1-extendable. Therefore C_n is 1-1-extendable.*

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