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ON FUZZY WEAKLY BAIRE SPACES

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Abstract. In this paper, the concepts of fuzzy σ-boundary set, fuzzy co-σ-boundary set, in fuzzy topological spaces are introduced and studied. By means of fuzzy σ-boundary sets, the concept of fuzzy weakly Baire space is defined and several characterizations of fuzzy weakly Baire spaces are studied.

1. Introduction

In 1965, L.A. Zadeh [**15**] introduced the concept of fuzzy sets as a new approach for modeling uncertainties. In 1968, C.L. Chang [**5**] defined fuzzy topological spaces by using fuzzy sets introduced by L.A. Zadeh. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Several mathematicians have tried all the pivotal concepts of general topology for extension to the fuzzy setting.

In 1899, Rene Louis Baire [**2**] introduced the concepts of first and second category sets in his doctoral thesis. In 1913, Arnaud Denjoy [**6**] introduced residual sets which are the complements of first category sets. The concept of Baire space was first advanced by Rene-Louis Baire. The study of Baire spaces and related concepts became popular among topologists interested in the completeness. The concepts of Baire spaces have been studied extensively in classical topology by R.C.Haworth and R.A.McCoy [**7**]. The concept of Baire space in fuzzy setting was introduced and studied by G.Thangaraj and S. Anjalmose in [**13**]. The concepts of σ-boundary sets in classical topology was defined and studied by J.Martinez and W.W.McGovern [**8**]. In this paper, the concepts of fuzzy σ-boundary set, fuzzy co-σ-boundary set in fuzzy topological spaces are introduced and studied. By means of fuzzy σ-boundary sets, the concept of fuzzy weakly Baire space is defined and

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several characterizations of fuzzy weakly Baire spaces are studied. The conditions under which fuzzy weakly Baire spaces become fuzzy Baire spaces are also obtained in this paper.

2. Preliminaries

In order to make the exposition self-contained, we give some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we denote a fuzzy topological space due to Chang (1968). Let X be a non-empty set and I, the unit interval [0,1]. A fuzzy set λ in X is a function from X into I.

Definition 2.1. [**5**] Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The fuzzy interior and the fuzzy closure of λ are defined respectively as follows:
(i). $int(\lambda) = \vee\{\mu/\mu \leqslant \lambda, \mu \in T\}$.
(ii). $cl(\lambda) = \wedge\{\mu/\lambda \leqslant \mu, 1-\mu \in T\}$.
More generally, for a family $\{\lambda_i/i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i(\lambda_i)$ and the intersection $\delta = \wedge_i(\lambda_i)$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = int_i\{\lambda_i(x), x \in X\}$.

Lemma 2.1. [**1**] *For a fuzzy set λ of a fuzzy topological space X,*
(i). $1 - int(\lambda) = cl(1-\lambda)$,
(ii). $1 - cl(\lambda) = int(1-\lambda)$.

Lemma 2.2. [**1**] *For a family $A = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leqslant cl\big(\vee(\lambda_\alpha)\big)$. In case A is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leqslant int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.2. [**12**] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is., $int[cl(\lambda)] = 0$, in (X, T).

Definition 2.3. [**12**] A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.4. [**12**] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category.

Definition 2.5. [**3**] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy F_σ-set in (X, T) if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$.

Definition 2.6. [**3**] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.

Definition 2.7. [**10**] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy P - space, if countable intersection of fuzzy open sets in (X, T) is fuzzy open. That is., every non-zero fuzzy G_δ-set, is fuzzy open in (X, T).

DEFINITION 2.8. [**12**] A fuzzy topological space (X, T) is called fuzzy first category if the fuzzy set 1_X is a fuzzy first category set in (X, T). That is, $1_X = \vee_{i=1}^{\infty}(\lambda_i)$ where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X, T). Otherwise (X, T) will be called a fuzzy second category space.

DEFINITION 2.9. [**13**] Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a fuzzy residual set in (X, T).

DEFINITION 2.10. [**1**] Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a
(i) fuzzy regular open set, if $int[cl(\lambda)] = \lambda$,
(ii) fuzzy regular closed set, if $cl[int(\lambda)] = \lambda$.

LEMMA 2.3. [**1**] *(i). The closure of a fuzzy open set is a fuzzy regular closed set. (ii). The interior of a fuzzy closed set is a fuzzy regular open set.*

DEFINITION 2.11. [**4**] Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a
(i) fuzzy pre-open set if $\lambda \leqslant int[cl(\lambda)]$,
(ii) fuzzy pre-closed set if $cl[int(\lambda)] \leqslant \lambda$.

DEFINITION 2.12. [**9**] Let λ be a fuzzy set in a fuzzy topological space (X, T). The fuzzy boundary of λ is defined as $Bd(\lambda) = cl(\lambda) \wedge cl(1 - \lambda)$.

DEFINITION 2.13. [**14**] A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy σ-nowhere dense set, if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

DEFINITION 2.14. [**13**] Let (X,T) be a fuzzy topological space. Then (X, T) is called a fuzzy Baire space if $int(\vee_{i=1}^{\infty}(\lambda_i)) = 0$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T).

DEFINITION 2.15. [**12**] A fuzzy topological space (X, T) is called a fuzzy open hereditarily irresolvable space if $int[cl(\lambda)] \neq 0$, then $int(\lambda) \neq 0$, for any non-zero fuzzy set λ in (X, T).

THEOREM 2.1. [**13**] *Let (X, T) be a fuzzy topological space. Then the following are equivalent:*

(1) (X, T) *is a fuzzy Baire space.*

(2) $int(\lambda) = 0$ *for every fuzzy first category set* λ *in* (X, T).

(3) $cl(\mu) = 1$ *for every fuzzy residual set* μ *in* (X, T).

3. Fuzzy σ - boundary set

DEFINITION 3.1. A fuzzy set λ in a fuzzy topological space (X, T) is called a fuzzy σ - boundary set, if $\lambda = \vee_{i=1}^{\infty}(\mu_i)$ where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T).

EXAMPLE 3.1. Let $X = \{a, b, c\}$ and λ, μ, δ be the fuzzy sets defined on X as follows:

$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.6; \lambda(b) = 0.4; \lambda(c) = 0.5.$
$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.5; \mu(b) = 0.7; \mu(c) = 0.6.$
$\delta : X \to [0,1]$ is defined as $\delta(a) = 0.4; \delta(b) = 0.5; \delta(c) = 0.7.$

Then $T = \{0, \lambda, \mu, \delta, (\lambda \vee \mu), (\lambda \vee \delta), (\mu \vee \delta), (\lambda \wedge \mu), (\lambda \wedge \delta), (\mu \wedge \delta), [\lambda \vee (\mu \wedge \delta)], [\delta \vee (\lambda \wedge \mu)], [\mu \wedge (\lambda \vee \delta)], \lambda \vee \mu \vee \delta, 1\}$ is a fuzzy topology on I. Now λ, $\lambda \wedge \mu$, $\lambda \wedge \delta$ and 1 are the fuzzy regular open sets in (X,T) and on computation, we have

$\mu_1 = cl(\lambda) \wedge (1 - \lambda) = 1 - \lambda$
$\mu_2 = cl(\lambda \wedge \mu) \wedge (1 - \lambda \wedge \mu) = 1 - [\lambda \wedge \mu]$
$\mu_3 = cl(\lambda \wedge \delta) \wedge (1 - [\lambda \wedge \delta]) = 1 - \lambda.$

Then, $(\mu_1 \vee \mu_2 \vee \mu_3) = 1 - (\lambda \wedge \mu)$, implies that $[1 - (\lambda \wedge \mu)]$ is a fuzzy σ-boundary set in (X,T).

Proposition 3.1. *If λ is a fuzzy σ - boundary set in a fuzzy topological space (X,T), then λ is a fuzzy F_σ - set in (X,T).*

Proof. Let λ be a fuzzy σ - boundary set in a fuzzy topological space (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\mu_i)$ where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T). Since λ_i is a fuzzy regular open set in (X,T), $(1 - \lambda_i)$ is a fuzzy regular closed set in (X,T). This implies that $(1-\lambda_i)$ is a fuzzy closed set in (X,T) and hence $cl(1 - \lambda_i) = 1 - \lambda_i$ in (X,T). Then $\mu_i = cl(\lambda_i) \wedge cl(1 - \lambda_i)$ is a fuzzy closed set in (X,T). Thus $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $(\mu_i)'s$ are fuzzy closed sets in (X,T), implies that λ is a fuzzy F_σ - set in (X,T). □

Proposition 3.2. *If $\mu = cl(\lambda) \wedge (1 - \lambda)$, where λ is a fuzzy regular open set in a fuzzy topological space (X,T), then μ is a fuzzy pre-closed set in (X,T).*

Proof. Let λ be a fuzzy regular open set in (X,T). Then $int[cl(\lambda)] = \lambda$ in (X,T). Now, $\mu = cl(\lambda) \wedge (1 - \lambda) = cl(int[cl(\lambda)]) \wedge (1 - int[cl(\lambda)]) = cl(int[cl(\lambda)]) \wedge cl[int(1 - \lambda)] \geqslant cl[(int[cl(\lambda)]) \wedge int(1 - \lambda)] = clint[cl(\lambda) \wedge (1 - \lambda)] = clint(\mu)$. Thus $clint(\mu) \leqslant \mu$ in (X,T) implies that μ is a fuzzy pre-closed set in (X,T). □

Proposition 3.3. *If $\mu = cl(\lambda) \wedge (1 - \lambda)$, where λ is a fuzzy regular open set in (X,T), then $int(\mu) = \lambda \wedge int[1 - \lambda]$ in (X,T).*

Proof. Suppose that $\mu = cl(\lambda) \wedge (1 - \lambda)$. Then $int(\mu) = intcl(\lambda) \wedge (1 - \lambda) = intcl(\lambda) \wedge int(1 - \lambda) = intcl(\lambda) \wedge [1 - cl(\lambda)] = \lambda \wedge [1 - cl(\lambda)]$, since λ is a fuzzy regular open set in (X,T), $intcl(\lambda) = \lambda$. Then $int(\mu) = \lambda \wedge [1 - cl(\lambda)] = \lambda \wedge [int(1 - \lambda)]$ in (X,T). □

Proposition 3.4. *If $\mu = cl(\lambda) \wedge (1 - \lambda)$, where λ is a fuzzy regular open set in (X,T), then $cl(\mu) \leqslant bd(\lambda)$ where $bd(\lambda)$ is the fuzzy boundary of λ in (X,T).*

Proof. Suppose that $\mu = cl(\lambda) \wedge (1 - \lambda)$, where λ is a fuzzy regular open set in (X,T). Then $cl(\mu) = cl[cl(\lambda) \wedge (1 - \lambda)] \leqslant clcl(\lambda) \wedge cl(1 - \lambda) = cl(\lambda) \wedge cl(1 - \lambda)$. This implies that $cl(\mu) \leqslant [cl(\lambda) \wedge cl(1 - \lambda)] = bd(\lambda)$. Thus $cl(\mu) \leqslant bd(\lambda)$, in (X,T). □

Proposition 3.5. *If $bd(\lambda)$ is a fuzzy nowhere dense set, for a fuzzy regular open set λ in (X,T), then $\mu = cl(\lambda) \wedge (1 - \lambda)$ is a fuzzy nowhere dense set in (X,T).*

Proof. Let λ be a fuzzy regular open set in (X,T) such that $intcl[bd(\lambda)] = 0$. Suppose that $\mu = cl(\lambda)\wedge(1-\lambda)$ in (X,T). Then by proposition 3.4., $cl(\mu) \leqslant bd(\lambda)$, in (X,T). Then $intcl[cl(\mu)] \leqslant intcl[bd(\lambda)]$ in (X,T) and hence $intcl(\mu) \leqslant intcl[bd(\lambda)]$ in (X,T). Since $intcl[bd(\lambda)] = 0$, $intcl(\mu) \leqslant 0$ in (X,T). That is., $intcl(\mu) = 0$ in (X,T) and hence μ is a fuzzy nowhere dense set in (X,T). □

Proposition 3.6. *If λ is a fuzzy σ - boundary set in a fuzzy topological space (X,T), then $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $(\mu_i)'s$ are fuzzy pre-closed sets in (X,T).*

Proof. Let λ be a fuzzy σ - boundary set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i = cl(\lambda_i) \wedge (1-\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T). By proposition 3.2., $(\mu_i)'s$ are fuzzy pre-closed sets in (X,T), and hence $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $(\mu_i)'s$ are fuzzy pre-closed sets in (X,T). □

Definition 3.2. A fuzzy set λ in a fuzzy topological space (X,T) is called a fuzzy pre-F_σ set, if $\lambda = \vee_{i=1}^{\infty}(\mu_i)$ where $(\mu_i)'s$ are fuzzy pre-closed sets in (X,T).

Proposition 3.7. *If λ is a fuzzy σ-boundary set in a fuzzy topological space (X,T), then λ is a fuzzy pre F_σ set in (X,T).*

Proof. The proof follows from definition 3.2., and proposition 3.6. □

Proposition 3.8. *If λ is a fuzzy σ-boundary set in (X,T), then $1-\lambda$ is a fuzzy G_δ set in (X,T) and $1-\lambda = \wedge_{i=1}^{\infty}(\gamma_i)$, where $\gamma_i = int(1-\lambda_i) \vee (\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T).*

Proof. Let λ be a fuzzy σ- boundary set in (X,T). Then $\lambda = \vee_{i=1}^{\infty}(\mu_i)$ where $\mu_i = cl(\lambda_i) \wedge (1-\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T). Then $1-\lambda = 1 - \vee_{i=1}^{\infty}(\mu_i) = \wedge_{i=1}^{\infty}(1-\mu_i)$. Now, $1-\mu_i = 1 - [cl(\lambda_i) \wedge (1-\lambda_i)] = [1-cl(\lambda_i)] \vee [1-(1-\lambda_i)] = int(1-\lambda_i) \vee (\lambda_i)$, in (X,T).

Since $(\lambda_i)'s$ are fuzzy regular open sets in (X,T), $(\lambda_i)'s$ are fuzzy open sets in (X,T). Thus $1-\mu_i = int(1-\lambda_i) \vee (\lambda_i)$ implies that $(1-\mu_i)$ is a fuzzy open set in (X,T). Thus $1-\lambda = \wedge_{i=1}^{\infty}(1-\mu_i)$, where $(1-\mu_i)'s$ are fuzzy open sets in (X,T), implies that $1-\lambda$ is a fuzzy G_δ set in (X,T). Let $\gamma_i = 1-\mu_i$, then $1-\lambda = \wedge_{i=1}^{\infty}(\gamma_i)$, where $\gamma_i = int(1-\lambda_i) \vee (\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T). □

Definition 3.3. A fuzzy set γ in a fuzzy topological space (X,T) is called a fuzzy co-σ-boundary set, if $\gamma = \wedge_{i=1}^{\infty}(\gamma_i)$, where $\gamma_i = int(1-\lambda_i) \vee \lambda_i$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T).

Proposition 3.9. *If γ is a fuzzy co-σ-boundary set in (X,T), then $1-\gamma$ is a fuzzy σ-boundary set in (X,T).*

Proof. Let γ be a fuzzy co-σ-boundary set in (X,T). Then $\gamma = \wedge_{i=1}^{\infty}(\gamma_i)$, where $\gamma_i = int(1-\lambda_i) \vee \lambda_i$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T). Now, $1-\gamma = 1 - \wedge_{i=1}^{\infty}(\gamma_i) = \vee_{i=1}^{\infty}(1-\gamma_i)$, in (X,T).

Also $1-\gamma_i = 1 - [int(1-\lambda_i) \vee \lambda_i] = [1 - int(1-\lambda_i)] \wedge (1-\lambda_i) = \{1 - [1 - cl(\lambda_i)]\} \wedge (1-\lambda_i)$ and hence $1-\gamma_i = cl(\lambda_i) \wedge (1-\lambda_i)$. Let $\mu_i = 1-\gamma_i$, then $1-\gamma = \vee_{i=1}^{\infty}(\mu_i)$ where $\mu_i = cl(\lambda_i) \wedge (1-\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X,T). Thus $1-\gamma$ is a fuzzy σ-boundary set in (X,T). □

4. Fuzzy weakly Baire space

DEFINITION 4.1. A fuzzy topological space (X, T) is called a fuzzy weakly Baire if $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$, where $\mu_i = cl(\lambda_i) \wedge (1-\lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T).

EXAMPLE 4.1. Let μ_1, μ_2 and μ_3 be fuzzy sets of I defined as

$$\mu_1(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 1/2; \\ 2x-1, & 1/2 \leqslant x \leqslant 1. \end{cases}$$

$$\mu_2(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1/4; \\ -4x+2, & 1/4 \leqslant x \leqslant 1/2; \\ 0 & 1/2 \leqslant x \leqslant 1. \end{cases}$$

and

$$\mu_3(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 1/4; \\ 1/3\ (4x-1) & 1/2 \leqslant x \leqslant 1. \end{cases}$$

Clearly, $T = \{0, \mu_1, \mu_2, (\mu_1 \vee \mu_2), 1\}$ is a fuzzy topology on I. On computation, it follows that $cl(\mu_1) = 1-\mu_2, cl(\mu_2) = 1-\mu_1, cl(\mu_1 \vee \mu_2) = 1, int(1-\mu_1) = \mu_2, int(1-\mu_2) = \mu_1$, and $int[1-(\mu_1 \vee \mu_2)] = 0$. Now, $int[cl(\mu_1)] = int(1-\mu_2) = \mu_1$; $int[cl(\mu_2)] = int(1-\mu_1) = \mu_2$; and $int[cl(\mu_1 \vee \mu_2)] = int(1) \neq \mu_1 \vee \mu_2$.

Thus μ_1 and μ_2 are fuzzy regular open sets in the fuzzy topological space (I, T). Now, $\delta_1 = cl(\mu_1) \wedge (1-\mu_1) = (1-\mu_2) \wedge (1-\mu_1)$; $\delta_2 = cl(\mu_2) \wedge (1-\mu_2) = (1-\mu_1) \wedge (1-\mu_2)$, then $int(\delta_1 \vee \delta_2) = int[(1-\mu_2) \wedge (1-\mu_1)] = int[1-(\mu_1 \vee \mu_2)] = 1 - cl(\mu_1 \vee \mu_2) = 1-1 = 0$. Hence (I, T) is a fuzzy weakly Baire space.

EXAMPLE 4.2. Let $X = \{a, b, c\}$ and λ, μ, δ be the fuzzy sets defined on X as follows:
$\lambda : X \to [0,1]$ is defined as $\lambda(a) = 0.4; \lambda(b) = 0.6; \lambda(c) = 0.5$,
$\mu : X \to [0,1]$ is defined as $\mu(a) = 0.7; \mu(b) = 0.5; \mu(c) = 0.4$,
$\delta : X \to [0,1]$ is defined as $\delta(a) = 0.3; \delta(b) = 0.4; \delta(c) = 0.6$.

Then, $T = \{0, \lambda, \mu, \delta, (\lambda \vee \mu), (\lambda \vee \delta), (\mu \vee \delta), (\lambda \wedge \mu), (\lambda \wedge \delta), (\mu \wedge \delta), [\lambda \wedge (\mu \vee \delta)], [\delta \vee (\lambda \wedge \mu)], [\mu \vee (\lambda \wedge \delta)], [\lambda \vee \mu \vee \delta], 1\}$ is a fuzzy topology on X. Now the fuzzy regular open sets in (X, T) are $\mu, \delta, (\lambda \vee \mu), (\lambda \wedge \mu), (\lambda \wedge \delta), (\mu \wedge \delta), [\delta \vee (\lambda \wedge \mu)], [\lambda \wedge (\mu \vee \delta)], \lambda \vee \mu \vee \delta$ and 1. Now on computation, we have
$\mu_1 = cl(\mu) \wedge (1-\mu) = 1-(\mu \vee \delta)$,
$\mu_2 = cl(\delta) \wedge (1-\delta) = 1-(\mu \vee \delta)$,
$\mu_3 = cl(\lambda \vee \mu) \wedge (1-[\lambda \vee \mu]) = 1-(\lambda \vee \mu)$,
$\mu_4 = cl(\lambda \wedge \mu) \wedge (1-[\lambda \wedge \mu]) = 1-[\delta \vee (\lambda \wedge \mu)]$,
$\mu_5 = cl(\lambda \wedge \delta) \wedge (1-[\lambda \wedge \delta]) = 1-(\lambda \vee \mu)$,
$\mu_6 = cl(\mu \wedge \delta) \wedge [1-(\mu \wedge \delta)] = 1-[\lambda \vee \mu \vee \delta]$,
$\mu_7 = cl[\delta \vee (\lambda \wedge \mu)] \wedge \{1-[\delta \vee (\lambda \wedge \mu)]\} = 1-[\delta \vee (\lambda \wedge \mu)]$,
$\mu_8 = cl(\lambda \wedge [\mu \vee \delta]) \wedge \{1-(\lambda \wedge [\mu \vee \delta])\} = 1-[\lambda \wedge (\mu \vee \delta)]$ and
$\mu_9 = cl(\lambda \vee [\mu \vee \delta]) \wedge \{1-(\lambda \vee [\mu \vee \delta])\} = 1-[\lambda \vee \mu \vee \delta]$.

Now, $int\{\mu_1 \vee \mu_3 \vee \mu_4 \vee \mu_6 \vee \mu_8 \vee \mu_9\} = [\lambda \wedge (\mu \vee \delta)] \neq 0$. Hence (X, T) is not a fuzzy weakly Baire space.

Proposition 4.1. *Let (X, T) be a fuzzy topological space. Then, the following are equivalent:*

(i). (X, T) is a fuzzy weakly Baire space.

(ii). $int(\lambda) = 0$, for each fuzzy σ-boundary set λ in (X, T).

(iii). $cl(\gamma) = 1$, for each fuzzy co-σ-boundary set γ in (X, T).

Proof. $(i) \Rightarrow (ii)$ Let λ be a fuzzy σ-boundary set in (X, T). Then $\lambda = \vee_{i=1}^{\infty}(\mu_i)$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T). Then $int(\lambda) = int[\vee_{i=1}^{\infty}(\mu_i)] = 0$, since (X, T) is a fuzzy weakly Baire space. Thus, for the fuzzy σ-boundary set λ in (X, T), $int(\lambda) = 0$ in (X, T).

$(ii) \Rightarrow (iii)$ Let γ be a fuzzy co-σ-boundary set in (X, T), then by proposition 3.9., $1 - \gamma$ is a fuzzy σ-boundary set in (X, T). Then, by hypothesis, $int(1 - \gamma) = 0$ in (X, T). Now $cl(\gamma) = 1 - [1 - cl(\gamma)] = 1 - [int(1 - \gamma)] = 1 - 0 = 1$. Thus $cl(\gamma) = 1$, for the fuzzy co-σ-boundary set γ in (X, T).

$(iii) \Rightarrow (i)$ Let γ be a fuzzy co-σ-boundary set in (X, T). Then $1 - \gamma$ is a fuzzy σ-boundary set in (X, T) and hence $1 - \gamma = \vee_{i=1}^{\infty}(\mu_i)$ where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T). Then $int(1 - \gamma) = int[\vee_{i=1}^{\infty}(\mu_i)]$. Now, by hypothesis $cl(\gamma) = 1$ in (X, T). Since $int(1 - \gamma) = 1 - cl(\gamma) = 1 - 1 = 0$, $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$ where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T). This implies that (X, T) is a fuzzy weakly Baire space. □

Proposition 4.2. *If (X, T) is a fuzzy weakly Baire space, then for any regular open set λ in (X, T), $[\lambda \wedge int(1 - \lambda)] = 0$ in (X, T).*

Proof. Let (X, T) be a fuzzy weakly Baire space. Then $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T). By lemma 2.2., $\vee_{i=1}^{\infty} int(\mu_i) \leqslant int[\vee_{i=1}^{\infty}(\mu_i)]$ in (X, T). Then $\vee_{i=1}^{\infty} int(\mu_i) \leqslant 0$ in (X, T). That is., $\vee_{i=1}^{\infty} int(\mu_i) = 0$ in (X, T). This implies that $int(\mu_i) = 0$ in (X, T). Then from proposition 3.3., $[\lambda_i \wedge int(1 - \lambda_i)] = 0$, in (X, T). Thus, if λ is a fuzzy regular open set in (X, T), then $\lambda \wedge int(1 - \lambda) = 0$ in (X, T). □

Remark 4.1. A fuzzy Baire space need not be a fuzzy weakly Baire space. For, consider the following example:

Example 4.3. Let $X = \{a, b, c\}$ and λ, μ, δ, η be the fuzzy sets defined on X as follows:

$\lambda : X \to [0, 1]$ is defined as $\lambda(a) = 0.6; \lambda(b) = 0.4; \lambda(c) = 0.5$,

$\mu : X \to [0, 1]$ is defined as $\mu(a) = 0.5; \mu(b) = 0.7; \mu(c) = 0.6$,

$\delta : X \to [0, 1]$ is defined as $\delta(a) = 0.4; \delta(b) = 0.5; \delta(c) = 0.7$,

$\eta : X \to [0, 1]$ is defined as $\eta(a) = 0.6; \eta(b) = 0.5; \eta(c) = 0.5$.

Then, $T = \{0, \lambda, \mu, \delta, (\lambda \vee \mu), (\lambda \vee \delta), (\mu \vee \delta), (\lambda \wedge \mu), (\lambda \wedge \delta), (\mu \wedge \delta), [\lambda \vee (\mu \wedge \delta)], [\delta \vee (\lambda \wedge \mu)], [\mu \wedge (\lambda \vee \delta)], \lambda \vee \mu \vee \delta, 1\}$ is a fuzzy topology on X. On computation, the fuzzy regular open sets in (X, T) are $\lambda, \lambda \wedge \mu, \lambda \wedge \delta$ and

$\mu_1 = cl(\lambda) \wedge (1 - \lambda) = 1 - \lambda$,

$\mu_2 = cl(\lambda \wedge \mu) \wedge (1 - [\lambda \wedge \mu]) = 1 - (\lambda \wedge \mu)$,

$\mu_3 = cl(\lambda \wedge \delta) \wedge (1 - [\lambda \wedge \delta]) = 1 - \lambda$.

and $int\{\mu_1 \vee \mu_2 \vee \mu_3\} = int(1 - [\lambda \wedge \mu]) = (\lambda \wedge \mu) \neq 0$. Hence (X, T) is not a fuzzy weakly Baire space. The fuzzy nowhere dense sets in (X, T) are $1 - \mu, 1 - \delta, 1 - (\lambda \vee \mu), 1 - (\lambda \vee \delta), 1 - (\mu \vee \delta), 1 - (\mu \wedge \delta), 1 - [\lambda \vee (\mu \wedge \delta)], 1 - [\delta \vee (\lambda \wedge \mu)], 1 - [\mu \wedge (\lambda \vee \delta)], 1 - [\lambda \vee \mu \vee \delta]$. Here, $1 - (\mu \wedge \delta)$ is a fuzzy first category set in (X, T) and $int(1 - [\mu \wedge \delta]) = 0$. Hence (X, T) is a fuzzy Baire space.

The following proposition gives a condition under which a fuzzy weakly Baire space is a fuzzy Baire space.

Proposition 4.3. *If $bd(\lambda)$ is a fuzzy nowhere dense set, for each fuzzy regular open set λ in a fuzzy weakly Baire space (X, T), then (X, T) is a fuzzy Baire space.*

Proof. Let (X, T) be a fuzzy weakly Baire space. Then $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$, where $\mu_i = cl(\lambda_i) \wedge (1 - \lambda_i)$ and $(\lambda_i)'s$ are fuzzy regular open sets in (X, T). By hypothesis, $bd(\lambda_i)$ is a fuzzy nowhere dense set in (X, T). Then by proposition 3.5., $(\mu_i)'s$ are fuzzy nowhere dense sets in (X, T). Thus we have, $int[\vee_{i=1}^{\infty}(\mu_i)] = 0$, where $(\mu_i)'s$ are fuzzy nowhere dense sets in (X, T) and hence (X, T) is a fuzzy Baire space. □

Remark 4.2. In example 4.1., the fuzzy regular open sets in (I, T) are μ_1 and μ_2. On computation, we have $bd(\mu_1) = cl(\mu_1) \wedge cl(1 - \mu_1) = (1 - \mu_2) \wedge (1 - \mu_1) = 1 - (\mu_1 \vee \mu_2)$; and $bd(\mu_2) = cl(\mu_2) \wedge cl(1 - \mu_2) = (1 - \mu_1) \wedge (1 - \mu_2) = 1 - (\mu_1 \vee \mu_2)$. Now $int[clbd(\mu_1)] = intcl[1 - (\mu_1 \vee \mu_2)] = int[1 - (\mu_1 \vee \mu_2)] = 0$ and $int[clbd(\mu_2)] = intcl[1 - (\mu_1 \vee \mu_2)] = int[1 - (\mu_1 \vee \mu_2)] = 0$. Thus $bd(\mu_1)$ and $bd(\mu_2)$ are fuzzy nowhere dense sets in (I, T). Then by proposition 4.3., (I, T) is a fuzzy Baire space.

Theorem 4.1 (13). *If the fuzzy topological space (X, T) is a fuzzy Baire space, then (X, T) is a fuzzy second category space.*

Proposition 4.4. *If $bd(\lambda)$ is a fuzzy nowhere dense set, for each fuzzy regular open set λ in a fuzzy weakly Baire space (X, T), then (X, T) is a fuzzy second category space.*

Proof. Let $(\lambda_i)'s$ be fuzzy regular open sets in a fuzzy weakly Baire space (X, T) such that $int(cl[bd(\lambda_i)]) = 0$. Then by proposition 4.3., (X, T) is a fuzzy Baire space. By theorem 4.1., (X, T) is a fuzzy second category space. □

Proposition 4.5. *If (X, T) is a fuzzy weakly Baire space, then $int(\lambda) \wedge int(1 - \lambda) = 0$ for any fuzzy set λ defined on X.*

Proof. Let λ_i be a fuzzy set defined on X. Then $cl(\lambda_i)$ is a fuzzy closed set in (X, T). Then, by lemma 2.3., $int[cl(\lambda_i)]$ is a fuzzy regular open set in (X, T).

Let $\mu_i = cl\{int(cl[(\lambda_i)])\} \wedge \{1 - int[cl(\lambda_i)]\} = cl\{int[cl(\lambda_i)]\} \wedge cl[int(1 - \lambda_i)] \geqslant cl\{int[cl(\lambda_i)] \wedge int(1 - \lambda_i)\} = clint[cl(\lambda_i) \wedge (1 - \lambda_i)]$. Thus,

$$clint[cl(\lambda_i) \wedge (1 - \lambda_i)] \leqslant \mu_i \tag{4.1}$$

Since (X, T) is a fuzzy weakly Baire space,

$$int[\vee_{i=1}^{\infty}(\mu_i)] = 0 \tag{4.2}$$

where $\mu_i = cl(int[cl(\lambda_i)]) \wedge (1 - int[cl(\lambda_i)])$ and $int[cl(\lambda_i)]$ is a fuzzy regular open set in (X, T).

Now, from Eqn., 4.1, we have $\vee_{i=1}^{\infty} clint[cl(\lambda_i) \wedge (1 - \lambda_i)] \leqslant \vee_{i=1}^{\infty}(\mu_i)$ and then $int[\vee_{i=1}^{\infty} clint[cl(\lambda_i) \wedge (1 - \lambda_i)]] \leqslant int[\vee_{i=1}^{\infty}(\mu_i)]$.

From Eqn., 4.2, we have $int \vee_{i=1}^{\infty} clint[cl(\lambda_i) \wedge (1 - \lambda_i)] \leqslant 0$ in (X, T). Thus $int[\vee_{i=1}^{\infty} clintcl(\lambda_i) \wedge (1 - \lambda_i)] = 0$ in (X, T). By lemma 2.2., $\vee_{i=1}^{\infty} int(clint[cl(\lambda_i) \wedge (1 - \lambda_i)]) \leqslant int\{\vee_{i=1}^{\infty} clint[cl(\lambda_i) \wedge (1 - \lambda_i)]\}$ and hence $\vee_{i=1}^{\infty} intcl[int[cl(\lambda_i)] \wedge (1 - \lambda_i)] = 0$. Then, $\vee_{i=1}^{\infty} intint[cl(\lambda_i) \wedge (1 - \lambda_i)] = 0$. Thus we have $\vee_{i=1}^{\infty} int\{cl(\lambda_i) \wedge (1 - \lambda_i)\} = 0$ in (X, T), for any fuzzy set defined on X.

This implies that $int[cl(\lambda_i) \wedge (1 - \lambda_i)] = 0$ in (X, T) and hence $int[cl(\lambda_i)] \wedge int[(1 - \lambda_i)] = 0$ in (X, T). Since $int(\lambda_i) \wedge int(1 - \lambda_i) \leqslant int[cl(\lambda_i)] \wedge int(1 - \lambda_i)$ in (X, T), we have $int(\lambda_i) \wedge int(1 - \lambda_i) = 0$ in (X, T). □

Remark 4.3. If (X, T) is a fuzzy weakly Baire space, then $cl(\lambda) \vee cl(1 - \lambda) = 1$, for any fuzzy set λ defined on X. For, if λ is a fuzzy set defined on the fuzzy weakly Baire space (X, T), by proposition 4.5., $int(\lambda) \wedge int(1 - \lambda) = 0$ and hence $1 - [int(\lambda) \wedge int(1 - \lambda)] = 1$. Thus, $[1 - int(\lambda)] \wedge [1 - int(1 - \lambda)] = 1$ and this implies that, $cl(1 - \lambda) \vee cl(\lambda) = 1$ in (X, T).

Proposition 4.6. *If λ is a fuzzy σ-boundary set in a fuzzy weakly Baire space (X, T), then λ is a fuzzy σ-nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy σ-boundary set in (X, T). Then by proposition 3.1., λ is a fuzzy F_σ-set in (X, T). Since (X, T) is a fuzzy weakly Baire space, by proposition 4.1., $int(\lambda) = 0$, for the fuzzy σ-boundary set λ in (X, T). Thus λ is a fuzzy F_σ-set such that $int(\lambda) = 0$, in (X, T). Hence λ is a fuzzy σ-nowhere dense set in (X, T). □

Theorem 4.2 (14). *In a fuzzy topological space (X, T) a fuzzy set λ is a fuzzy σ-nowhere dense set in (X, T) if and only if $1 - \lambda$ is a fuzzy dense and fuzzy G_δ-set in (X, T).*

Proposition 4.7. *If λ is a fuzzy σ-boundary set in a fuzzy weakly Baire space (X, T), then $1 - \lambda$ is a fuzzy dense and fuzzy G_δ-set in (X, T).*

Proof. Let λ be a fuzzy σ-boundary set in the fuzzy weakly Baire space (X, T). Then, by proposition 4.6., λ is a fuzzy σ-nowhere dense set in (X, T). Now by theorem 4.2., $(1 - \lambda)$ is a fuzzy dense and fuzzy G_δ-set in (X, T). □

Theorem 4.3 (13). *If λ is a fuzzy dense and fuzzy open set in (X, T), then $1 - \lambda$ is a fuzzy nowhere dense set in (X, T).*

Proposition 4.8. *If λ is a fuzzy σ-boundary set in a fuzzy weakly Baire and fuzzy P - space (X, T), then λ is a fuzzy nowhere dense set in (X, T).*

Proof. Let λ be a fuzzy σ-boundary set in the fuzzy weakly Baire space (X, T). Then by proposition 4.7., $(1 - \lambda)$ is a fuzzy dense and fuzzy G_δ set in (X, T). Since (X, T) is a fuzzy P - space, then the fuzzy G_δ set $(1 - \lambda)$ is a fuzzy open set in (X, T) and hence $(1 - \lambda)$ is a fuzzy dense and fuzzy open set in (X, T).

By theorem 4.3., $1-(1-\lambda)$ is a fuzzy nowhere dense set in (X,T). That is., λ is a fuzzy nowhere dense set in (X,T). □

PROPOSITION 4.9. *If $(\lambda_i)'s$ $(i = 1to\infty)$ are fuzzy σ-boundary sets in a fuzzy weakly Baire and fuzzy P - space (X,T), then $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X,T).*

PROOF. Let $(\lambda_i)'s$ be fuzzy σ-boundary sets in (X,T). Since (X,T) is a fuzzy weakly Baire and fuzzy P - space, by proposition 4.8., $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X,T). Then $\vee_{i=1}^{\infty}(\lambda_i)$ is a fuzzy first category set in (X,T). □

PROPOSITION 4.10. *If $(\lambda_i)'s$ are fuzzy σ-boundary sets in a fuzzy weakly Baire and fuzzy P - space (X,T) such that $int[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, then (X,T) is a fuzzy Baire space.*

PROOF. Let $(\lambda_i)'s$ be fuzzy σ-boundary sets in (X,T). Since (X,T) is a fuzzy weakly Baire and fuzzy P - space, then by proposition 4.8, $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X,T). Thus $int[\vee_{i=1}^{\infty}(\lambda_i)] = 0$, where $(\lambda_i)'s$ are fuzzy nowhere dense sets in (X,T) implies that (X,T) is a fuzzy Baire space. □

THEOREM 4.4 (11). *If the fuzzy topological space (X,T) is a fuzzy open hereditarily irresolvable space, then $int(\lambda) = 0$, for any non-zero fuzzy set λ in (X,T) implies that $int[cl(\lambda)] = 0$.*

PROPOSITION 4.11. *If (X,T) is a fuzzy weakly Baire and fuzzy open hereditarily irresolvable space, then (X,T) is a fuzzy Baire space.*

PROOF. Let $(\eta_i)'s$ $(i = 1to\infty)$ be fuzzy σ-boundary sets in a fuzzy weakly Baire space (X,T). By proposition 4.1., $int(\eta_i) = 0$ in (X,T), then $int[\vee_{ij=1}^{\infty}(\mu_{ij})] = 0$ where $(\mu_{ij})'s$ are fuzzy sets defined on X, such that $\mu_{ij} = cl(\lambda_j)\wedge(1-\lambda_j)$ and $(\lambda_j)'s$ are fuzzy regular open sets in (X,T). Now by lemma 2.2., we have $\vee_{ij=1}^{\infty}int(\mu_{ij}) \leqslant [\vee_{ij=1}^{\infty}(\mu_{ij})]$ in (X,T). Then, $\vee_{ij=1}^{\infty}int(\mu_{ij}) \leqslant 0$. That is., $\vee_{ij=1}^{\infty}int(\mu_{ij}) = 0$ and hence $int(\mu_{ij}) = 0$, in (X,T). Since (X,T) is a fuzzy open hereditarily irresolvable space, $int(\mu_{ij}) = 0$ implies, by theorem 4.4., that $int[cl(\mu_{ij})] = 0$ in (X,T). Hence $(\mu_{ij})'s$ are fuzzy nowhere dense sets in (X,T). Then $int[\vee_{ij=1}^{\infty}(\mu_{ij})] = 0$ where $(\mu_{ij})'s$ are fuzzy nowhere dense sets in (X,T), implies that (X,T) is a fuzzy Baire space. □

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