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A STUDY ON THE INHERENT INJ-EQUITABLE GRAPHS

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Abstract. Let G be a graph. The inherent Inj-equitable graph of a graph G ($IIE(G)$) is the graph with the same vertices as G and any two vertices u and v are adjacent in $IIE(G)$ if they are adjacent in G and $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$, where for any vertex $w \in V(G)$, $\deg_{in}(w) = \left|\{w^{'} \in V : N(w^{'}) \cap N(w) \neq \phi\}\right|$ [**2**]. In this paper, inherent Inj-equitable graph of some graphs are obtained, some properties and results are established. We define iterated Inj-equitable graph of a graph, complete Inj-equitable graph and we define the Inj-equitable graph.

1. Introduction

All graphs considered in this paper are finite, undirected without loops or multiple edges. Let $G = (E, V)$ be a graph with vertex set $V = \{v_1, v_2, ...v_n\}$. Thus $|V| = n$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{u\}$, respectively. The degree of a vertex v in G is $\deg(v) = |N(v)|$. $\Delta(G)$ and $\delta(G)$ are the maximum and minimum vertex degree of G respectively. The distance $d(u, v)$ between any two vertices u and v in a graph G is the number of the edges in a shortest path. The eccentricity of a vertex u in a connected graph G is $e(u) = \max\{d(u, v), v \in V\}$. The diameter of G is the value of the greatest eccentricity, and the radius of G is the value of the smallest eccentricity. The Inj-neighborhood of a vertex $u \in V(G)$ denoted by $N_{in}(u)$ is defined as $N_{in}(u) = \{v \in V(G) : |\Gamma(u, v)| \geqslant 1\}$, where $|\Gamma(u, v)|$ is the number of common neighborhood between the vertices u and v. The cardinality of $N_{in}(u)$ is called injective degree of the vertex u and is denoted by $\deg_{in}(u)$ in G and $N_{in}[u] = N_{in}(u) \cup \{u\}$. Let G and H be any two graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$, respectively. Then the union $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The join $G \vee H$, is the graph obtained by taking the disjoint union of G and H and adding all edges

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$\{uv : u \in V(G),\, v \in V(H)\}$. The corona product $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leqslant i \leqslant |V(G)|$. The cartesian product $G \times H$ is a graph with vertex set $V(G) \times V(H)$ and edge set $E(G \times H) = \{((u, u^{'}), (v, v^{'})) : u = v$ and $u^{'}, v^{'}) \in E(H)$, or $u^{'} = v^{'}$ and $(u, v) \in E(G)\}$. For more terminologies and notations, we refer the reader to [**2**], [**4**], [**6**] and [**8**]. A strongly regular graph with parameters (n, k, λ, μ) is a k-regular graph with n vertices such that any two adjacent vertices have λ common neighbors, and any two non-adjacent vertices have μ common neighbors, [**5**].

Definition 1.1 ([**1**]). *Let $G = (V, E)$ be a graph. The inherent injective equitable graph of G, denoted by $IIE(G)$ is defined as the graph with vertex set $V(G)$ and two vertices u and v are adjacent in $IIE(G)$ if and only if they are adjacent in G and $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$. An edge $e = uv \in G$ is called injective equitable edge if $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$ and we say that u and v are Inj-equitable adjacent.*

The adjacency matrix of the graph G is the symmetric square matrix $A = A(G) = \|a_{ij}\|$ of order n whose (i, j)-entry is defined as:

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

The equitable graph of a graph G is the graph with vertex set $V(G)$ and two vertices u, v are adjacent if and only if $|\deg(u) - \deg(v)| \leqslant 1$, [**7**]. The adjacency matrix of equitable graph is the symmetric square matrix $A_e = A_e(G) = \|b_{ij}\|$ whose (i, j)-entry is defined as:

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent and } |\deg(v_i) - \deg(v_j)| \leqslant 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

The adjacency matrix of the congraph, defined in [**3**], is the symmetric matrix $\left\|a^{'}_{ij}\right\|$ whose (i, j)-entry is defined as:

$$a^{'}_{ij} = \begin{cases} 1 & \text{if } |\Gamma(v_i, v_j)| \geqslant 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Where $\Gamma(v_i, v_j)$ is the set of vertices, different from v_i and v_j, that are adjacent to both v_i and v_j.

Bearing in mind equations 1.2 and 1.3 as a sort of compromise, we introduce a new symmetric square matrix $A_{IIE} = \|d_{ij}\|$ of order n, whose (i, j)-entry is defined as:

$$d_{ij} = \begin{cases} 1 & \text{if} v_i \text{ and } v_j \text{ are adjacent and } |\deg_{in}(v_i) - \deg_{in}(v_j)| \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

This matrix can be viewed as the adjacency matrix of the inherent injective equitable graph. In this paper, the benefit of graph characterization to study the properties and the structure of graphs motivated us to introduce and study new graphs called inherent injective equitable graph and complete inherent injective equitable graph.

2. The Inherent Inj-equitable Graph of a Graph

In this section, we discuss some properties of the inherent injective equitable graph of a graph and the inherent injective equitable graph of some graph's families is found.

PROPOSITION 2.1. *For any graph G, $G \cong IIE(G)$ if and only if every edge is an Inj-equitable edge.*

THEOREM 2.1. *Let G be a complete graph or k-regular triangle-free graph with diameter 2, then $IIE(G) \cong G$.*

PROOF. If G is complete graph, then obviously $IIE(G) \cong G$. Suppose that G is k-regular triangle-free graph with diameter 2. We know that $IIE(G)$ is a subgraph of G for any graph G. Since G is k-regular with diameter 2, then for any vertex v, $\deg(v) = k$ and $\deg_{in}(v) = n - k - 1$. So, any adjacent vertices in G is also Inj-equitable adjacent. Hence, $IIE(G) \cong G$. $\square$

COROLLARY 2.1. *For any strongly regular graph without triangle G, $IIE(G) \cong G$.*

PROPOSITION 2.2. *For any strongly regular graph with parameters (n, k, λ, η) , $IIE(G)$ is also a strongly regular graph with the same parameters.*

PROOF. Let G be a strongly regular graph with parameters (n, k, λ, η). Then for any two adjacent vertices u and v, $\deg_{in}(u) = \deg_{in}(v) = \lambda$. Therefore,

$$|\deg_{in}(u) - \deg_{in}(v)| = 0.$$

Hence $IIE(G) \cong G$. $\square$

REMARK 2.1. *It is not true in general that for any regular graph G, $IIE(G) \cong G$. For example, one can see Figure 1.*

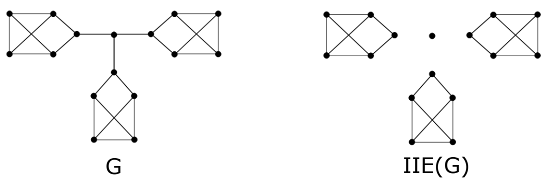

FIGURE 1. A regular graph G with $IIE(G) \ncong G$

PROPOSITION 2.3. *The following holds:*

(i) For any path P_n, $IIE(P_n) \cong P_n$.

(ii) For any cycle C_n, $IIE(C_n) \cong C_n$.
(iii) For any wheel W_n, $IIE(W_n) \cong W_n$.

PROPOSITION 2.4. *For any complete bipartite graph $K_{r,s}$, where $r+s \geqslant 4$,*

$$IIE(K_{r,s}) \cong \begin{cases} K_{r,s} & \text{if} \quad |r-s| \leqslant 1; \\ \overline{K}_{r+s} & \text{if} \quad |r-s| \geqslant 2. \end{cases}$$

PROOF. Let $G \cong K_{r,s}$ be a complete bipartite graph with partite sets A and B such that $|A| = r$, $|B| = s$. Clearly for any vertex v from A, $\deg_{in}(v) = r-1$ and for any vertex u from B, $\deg_{in}(u) = s-1$. Therefore, u and v are Inj-equitable adjacent if $|(s-1)-(r-1)| = |r-s| \leqslant 1$. Otherwise, they are not Inj-equitable adjacent. Hence,

$$IIE(K_{r,s}) \cong \begin{cases} K_{r,s} & \text{if} \quad |r-s| \leqslant 1; \\ \overline{K}_{r+s} & \text{if} \quad |r-s| \geqslant 2. \end{cases}$$

□

A firefly graph $F_{r,s,t}$ is a graph on $2r+2s+t+1$ vertices that consists of r triangles, s pendant paths of length 2 and t pendant edges sharing a common vertex.

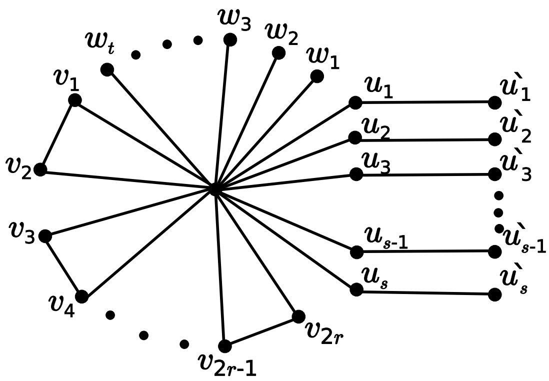

FIGURE 2. Firefly Graph

THEOREM 2.2. *For any firefly graph $F_{r,s,t}$, where $r,s,t \geqslant 1$,*

$$IIE(F_{r,s,t}) \cong \begin{cases} F_{r,0,s+1} \cup \overline{K}_s & \text{if } t = 1; \\ rK_2 \cup K_{1,s+2} \cup \overline{K}_s & \text{if } t = 2; \\ \overline{K}_{2s+t+1} \cup rK_2 & \text{if } t > 2. \end{cases}$$

PROOF. Let G be a firefly graph $F_{r,s,t}$ as in Figure 2., where $r \geqslant 1, s \geqslant 1$ and $t \geqslant 1$. Let v be the center vertex, v_i where $i = 1, 2, ...2r$ be any vertex from the triangle other than v, w_i where $i = 1, 2, ...t$ be any end vertex in the pendant edge, u_i and $u_i^{\backslash}$ where $i = 1, 2, ..., s$ be any end vertex and internal vertex respectively in the pendant path. Then, $\deg_{in}(v) = 2r + s$, $\deg_{in}(v_i) = 2r + s + t$, $\deg_{in}(w_i) = 2r + s + t - 1$, $\deg_{in}(u_i^{\backslash}) = 2r + s + t - 1$ and $\deg_{in}(u_i) = 1$. We have three cases:

Case 1. Suppose that, $t = 1$. Since $|\deg_{in}(v) - \deg_{in}(v_i)| = 1$ and

$$\left|\deg_{in}(u_i^{\backslash}) - \deg_{in}(w_i)\right| = 0,$$

then $IIE(F_{r,s,t}) \cong F_{r,o,s+1} \cup \overline{K}_s$.

Case 2. Suppose that, $t = 2$. Then, $\deg_{in}(v) = 2r + s$, $\deg_{in}(v_i) = 2r + s + 2$, $\deg_{in}(w_i) = 2r+s+1$, $\deg_{in}(u_i^{\backslash}) = 2r+s+1$. Hence, $IIE(F_{r,s,t}) \cong rK_2 \cup K_{1,s+2} \cup \overline{K}_s$.

Case 3. Suppose that $t > 2$. , then the only injective edges are $e_1 = v_1v_2$, $e_2 = v_2v_3...e_r = v_{2r-1}v_{2r}$. Hence, $IIE(F_{r,s,t}) \cong \overline{K}_{2s+t+1} \cup rK_2$. □

PROPOSITION 2.5.

(i) *For any firefly graph* $G \cong F_{r,0,0}$, $IIE(G) \cong G$.
(ii) *For any firefly graph* $F_{0,s,0}$,

$$IIE(F_{0,s,0}) \cong \begin{cases} F_{0,2,0} & \text{if } s = 2; \\ K_{1,s} \cup \overline{K}_s & \text{if } s > 2. \end{cases}$$

(iii) *For any firefly graph* $F_{r,s,t}$, *where* $r = s = 0$

$$IIE(F_{r,s,t}) \cong \begin{cases} F_{0,0,t} & \text{if } t \leqslant 2; \\ \overline{K}_{t+1} & \text{if } t \geqslant 3. \end{cases}$$

(iv) *For any firefly graph* $F_{r,s,t}$, *where* $r = 0$, $s \geqslant 1$, $t \geqslant 1$,

$$IIE(F_{r,s,t}) \cong \begin{cases} P_4 & \text{if } s = t = 1; \\ K_{t+s} \cup \overline{K}_s & \text{if } t \leqslant 2; \\ \overline{K}_{t+2s+1} & \text{if } t \geqslant 3. \end{cases}$$

PROPOSITION 2.6. *For any bipartite graph* G, $IIE(G)$ *is also bipartite graph.*

PROOF. Let G be a bipartite graph. Suppose that $IIE(G)$ is not bipartite graph. Then it contains at least one odd cycle say C_m. Since $IIE(G)$ is a subgraph of G, then G contains odd cycle which contradicts that G is bipartite graph. Hence $IIE(G)$ is bipartite graph. □

PROPOSITION 2.7. *Let* G *be a graph such that* $G \cong P_m \times P_2$. *Then* $IIE(G) \cong P_m \times P_2$.

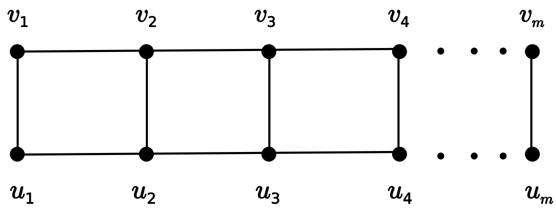

FIGURE 3. $P_m \times P_2$.

PROOF. Let G be a graph such that $G \cong P_m \times P_2$. Then we have four cases:
Case 1. If $m = 2$, then $G \cong C_4$. Therefore, $IIE(G) \cong G$.

Case 2. If $m = 3$, then $\deg_{in}(v) = 2$ for all $v \in V(G)$. Therefore, for any adjacent vertices u and v, $|\deg_{in}(u) - \deg_{in}(v)| = 0$. Hence, $IIE(G) \cong P_m \times P_2$.

Case 3. If $m = 4$, then for all $v \in V(G)$, either $\deg_{in}(v) = 2$ or $\deg_{in}(v) = 3$. Therefore, for any two adjacent vertices u and v, $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$. Hence, $IIE(G) \cong P_m \times P_2$.

Case 4. If $m \geqslant 5$, let G be labeling as in Figure 3. Then, $\deg_{in}(v_1) = \deg_{in}(v_m) = \deg_{in}(u_1) = \deg_{in}(u_m) = 2$, $\deg_{in}(v_2) = \deg_{in}(v_{m-1}) = \deg_{in}(u_2) = \deg_{in}(u_{m-1}) = 3$ and for $i = 3, 4, ...m - 2$, $\deg_{in}(v_i) = \deg_{in}(u_i) = 4$. Therefore, for any two adjacent vertices u and v in G, $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$. Hence, $IIE(G) \cong P_m \times P_2$.

□

PROPOSITION 2.8. *Let G be a graph such that $G \cong P_m \times P_3$, where $m \geqslant 4$. Then $IIE(G) \cong P_m \times P_3 - \{e_1, e_2\}$, where e_1 and e_2 are the edges which are not Inj-equitable edges in G.*

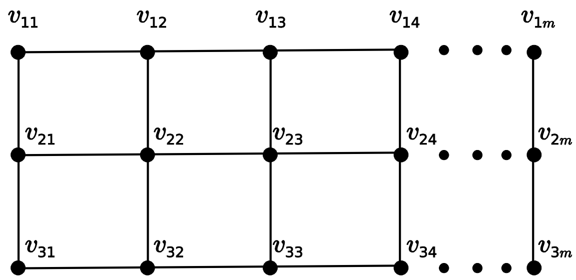

FIGURE 4. $P_m \times P_3$

PROOF. Suppose that, $G \cong P_m \times P_3$ be labeling as in Figure 4. Then we have two cases:
Case 1. If $m = 4$, then all the vertices have Inj-degree 3 or 4 or 5 except v_{22} and v_{23} have Inj-degree 5. Therefore all the edges are Inj-equitable edges except $e_1 = v_{21}v_{22}$ and $e_2 = v_{23}v_{24}$. Hence, $IIE(G) \cong P_m \times P_3 - \{e_1, e_2\}$.

Case 2. If $m \geqslant 5$, then all the edges are Inj-equitable edges except $e_1 = v_{21}v_{22}$ and $e_2 = v_{2m-1}v_{2m}$. Hence, $IIE(G) \cong P_m \times P_3 - \{e_1, e_2\}$.

□

For the generalized case, we have the following result:

PROPOSITION 2.9. *Let G be a graph such that $G \cong P_m \times P_n$, where $m, n \geqslant 5$. Then $IIE(G) \cong C_{2m+2n-4} \cup (P_{m-2} \times P_{n-2})$.*

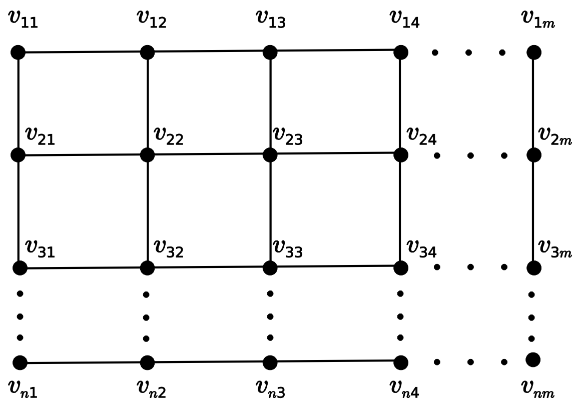

FIGURE 5. $P_m \times P_n$

PROOF. Suppose that $G \cong P_m \times P_n$ be labeling as in Figure 5. Then

$$\deg_{in}(v_{11}) = \deg_{in}(v_{1m}) = \deg_{in}(v_{n1}) = \deg_{in}(v_{nm}) = 3$$

and

$$\deg_{in}(v_{12}) = \deg_{in}(v_{(1)(m-1)}) = \deg_{in}(v_{21}) = \deg_{in}(v_{2m}) = \deg_{in}(v_{(n-1)(1)}) = \deg_{in}(v_{(n-1)(m)}) = \deg_{in}(v_{n2}) = \deg_{in}(v_{(n)(m-2)}) = 4.$$

Also, for $i = 3, 4, ...m-2$,

$$\deg_{in}(v_{1i}) = \deg_{in}(v_{ni}) = 5$$

and for $i = 3, 4, ...m$,

$$\deg_{in}(v_{i1}) = \deg_{in}(v_{im}) = 5.$$

For $i, j = 2, m-1$,

$$\deg_{in}(v_{ij}) = 6.$$

For $i = 3, 4, ...m-2$,

$$\deg_{in}(v_{2i}) = \deg_{in}(v_{(n-1)(i)}) = 7$$

and for $i = 3, 4, ...n-1$,

$$\deg_{in}(v_{i2}) = \deg_{in}(v_{(i)(m-1)}) = 7.$$

For $i, j = 3, 4, ...m-2$,

$$\deg_{in}(v_{ij}) = 8.$$

Therefore, all the edges are Inj-equitable edges except

$$v_{21}v_{22},\ v_{(2)(m-1)}v_{2m},\ v_{(n-1)(1)}v_{(n-1)(2)},\ v_{(n-1)(m-1)}v_{(n-1)(m)}$$

and for $j = 2, 3, ...m-1$, $v_{1j}v_{2j}$, $v_{nj}v_{(n-1)(j)}$. Hence $IIE(P_m \times P_n) \cong C_{2m+2n-4} \cup (P_{m-2} \times P_{n-2})$.

□

PROPOSITION 2.10. *Let G be a generalized petersen graph $GP(m, 1)$. Then $IIE(G) \cong GP(m, 1)$.*

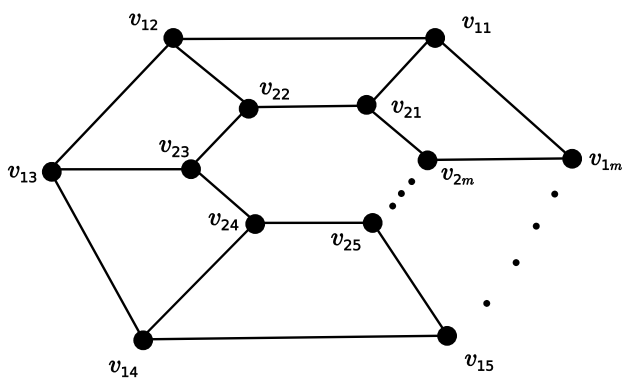

FIGURE 6. $GP(m, 1)$

PROOF. Let G be a generalized petersen graph $GP(m, 1)$. Then $G \cong C_m \times P_2$. We have three cases:

Case 1. If $m = 3$, then for all $v \in V(G)$, $\deg_{in}(v) = 4$. Therefore, all the edges are Inj-equitable edge. Hence, $IIE(G) \cong C_m \times P_2$.

Case 2. If $m = 4$, then for all $v \in V(G)$, $\deg_{in}(v) = 3$. Therefore, all the edges are Inj-equitable edge. Hence, $IIE(G) \cong C_m \times P_2$.

Case 3. If $m \geqslant 5$, let $G \cong C_m \times P_2$ be labeling as in Figure 6. Then $\deg_{in}(v_i) = 4$ and $\deg_{in}(u_i) = 4$, for $i = 1, 2, ...m$. Therefore, for any adjacent vertices u and v in G, $|\deg_{in}(u) - \deg_{in}(v)| = 0$. Therefore, $IIE(G) \cong C_m \times P_2$. Hence, $IIE(GP(m, 1)) \cong GP(m, 1)$. □

PROPOSITION 2.11. *Let $G \cong C_m \times P_3$. Then $IIE(G) \cong C_m \times P_3$.*

PROOF. Let $G \cong C_m \times P_3$ be labeling as in Figure 7. We have three cases:

Case 1. If $m = 3$, then for all $v \in V(G)$, $\deg_{in}(v) = 5$. Therefore, all the edges are Inj-equitable edges. Hence, $IIE(G) \cong C_m \times P_3$.

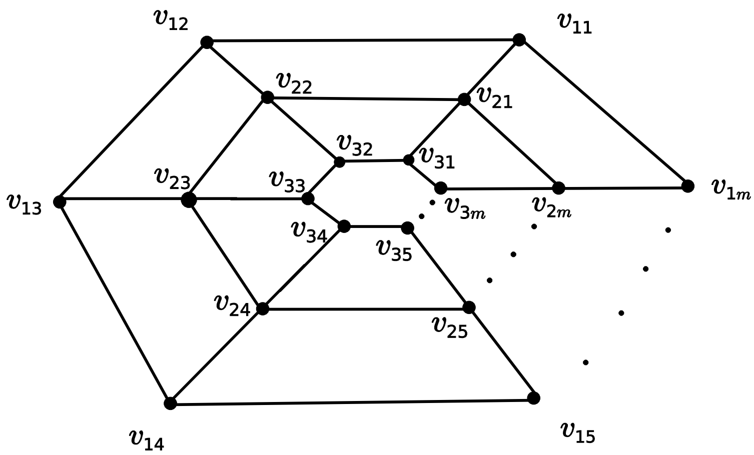

FIGURE 7. $C_m \times P_3$

Case 2. If $m = 4$, then for $i = 1, 2, ...4$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{3i}) = 4$ and $\deg_{in}(v_{2i}) = 5$. Therefore, all the edges are Inj-equitable edges. Hence, $IIE(G) \cong C_m \times P_3$.

Case 3. If $m \geqslant 5$, then for $i = 1, 2, ...m$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{3i}) = 5$ and $\deg_{in}(v_{2i}) = 6$. Therefore, for any adjacent vertices u and v in G, $|\deg_{in}(u) - \deg_{in}(v)| = 0$. Hence, $IIE(G) \cong C_m \times P_3$. $\square$

THEOREM 2.3. *For any graph G such that $G \cong C_m \times P_n$, $IIE(G) \cong 2C_m \cup (C_m \times P_{n-2})$, where $n \geqslant 5$.*

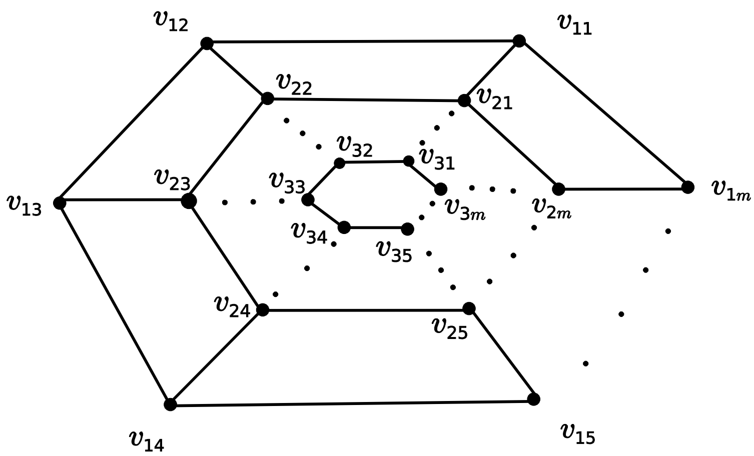

FIGURE 8. $C_m \times P_n$

PROOF. Suppose that $G \cong C_m \times P_n$ be labeling as in Figure 8. We have three cases:

Case 1. If $m = 3$, then for $i = 1, 2, 3$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{ni}) = 5$, $\deg_{in}(v_{2i}) = \deg_{in}(v_{n-1\ i}) = 7$ and for $i = 3, 4, ..n-2$, $j = 1, 2, 3$, $\deg_{in}(v_{ij}) = 8$. Hence $IIE(G) \cong 2C_m \cup (C_m \times P_{n-2})$.

Case 2. If $m = 4$, then for $i = 1, 2, ...4$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{ni}) = 4$, $\deg_{in}(v_{2i}) = \deg_{in}(v_{n-1\ i}) = 6$ and for $i = 3, 4, ..n-2$, $j = 1, 2, ...4$, $\deg_{in}(v_{ij}) = 7$. $IIE(G) \cong 2C_m \cup (C_m \times P_{n-2})$.

Case 3. if $m \geqslant 5$, then as in Figure 8., for $i = 1, 2, ..m$, $\deg_{in}(v_{1i}) = \deg_{in}(v_{ni}) = 5$, $\deg_{in}(v_{2i}) = \deg_{in}(v_{n-1\ i}) = 7$ and for $i = 3, 4, ..n-2$, $j = 1, 2, ..m$, $\deg_{in}(v_{ij}) = 8$. $IIE(G) \cong 2C_m \cup (C_m \times P_{n-2})$. □

THEOREM 2.4. *For any graph G, such that $G \cong C_n \times C_m$, $IIE(G) \cong C_n \times C_m$.*

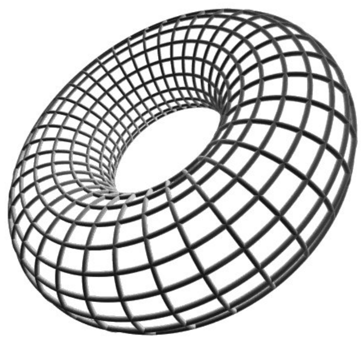

FIGURE 9. $C_n \times C_m$

PROOF. Let G be any graph such that $G \cong C_n \times C_m$. From Figure 9., for all $v \in V(G)$, $\deg_{in}(v) = 8$. Therefore, all the edges are Inj-equitable edges. Hence $IIE(G) \cong C_n \times C_m$. □

PROPOSITION 2.12. *For any two graphs G_1 and G_2, $IIE(G_1 \vee G_2) = G_1 \vee G_2$.*

PROOF. Let G_1 and G_2 be any two graphs. Since every edge in $G_1 \vee G_2$ is injective equitable edge, then $IIE(G_1 \vee G_2) = G_1 \vee G_2$. □

PROPOSITION 2.13. *For any cycle graph C_n and any totally disconnected graph $\overline{K}_m$, where $m > 2$, $IIE(C_n \circ \overline{K}_m) \cong C_n \cup \overline{K}_{nm}$.*

PROOF. Let $\{u_1, u_2, ...u_n\}$ be the vertex set of the cycle graph C_n and let $\{v_1, v_2, ...v_m\}$ be the vertex set of $\overline{K_m}$. Then for $i = 1, 2, ...n$, $\deg_{in}(u_i) = 2(m+1)$ and for $j = 1, 2, ...m$, $\deg_{in}(v_j) = m + 1$. Therefore, $|\deg_{in}(u_i) - \deg_{in}(v_j)| = m + 1 > 1$. Hence, $IIE(C_n \circ \overline{K}_m) \cong C_n \cup \overline{K}_{nm}$. □

THEOREM 2.5. *For any graph G with $\delta \geqslant 2$, if G is $k-$regular or $(k, k+1)-$biregular, then*

$$IIE(G \circ \overline{K_m}) = IIE(G) \cup \overline{K}_{nm}.$$

where n is the number of vertices in G.

PROOF. Let G be a $k-$regular graph with n vertices and $\delta \geqslant 2$. Suppose that $\{u_1, u_2, ...u_n\}$ and $\{v_{1,}v_2, ...v_m\}$ is the vertex set of G and $\overline{K_m}$, respectively. Therefore, Then for $i = 1, 2, ...n$, $\deg_{in}(u_i) = k(m+1)$ and for $j = 1, 2, ...m$, $\deg_{in}(v_j) = k + m - 1$. Therefore, $|\deg_{in}(u_i) - \deg_{in}(v_j)| = m(k-1) + 1 > 1$. Hence, $IIE(G \circ \overline{K_m}) = IIE(G) \cup \overline{K}_{nm}$. Similarly, we can prove if G is $(k, k+1)$-biregular, then $IIE(G \circ \overline{K_m}) = IIE(G) \cup \overline{K}_{nm}$. □

3. Complete inherent Inj-equitable graphs

DEFINITION 3.1. *A graph G is called complete inherent injective equitable graph ($CIIE-$graph) if for any two adjacent vertices u and v, $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$.*

EXAMPLE 3.1. $C_n \times C_m$ is $CIIE$-graph.

PROPOSITION 3.1. *Any complete graph is $CIIE$-graph but the converse is not always true. For example, paths and cycles are $CIIE$-graph but not complete.*

PROPOSITION 3.2. *Let G be any graph. $IIE(G) \cong G$ if and only if G is $CIIE$-graph.*

PROPOSITION 3.3. *Let H be a $CIIE$-graph and let G be a subgraph of H. Then $IIE(G)$ is a subgraph of $IIE(H)$.*

PROOF. Let H be a $CIIE$-graph and let G be a subgraph of H. Let e be an edge in $IIE(G)$. Then $e \in G$. Therefore $e \in H$. So, $e \in IIE(H)$. Hence, $IIE(G)$ is a subgraph of $IIE(H)$.

□

PROPOSITION 3.4. *For any $CIIE$-graph G, $IIE(G)$ is also $CIIE$-graph.*

PROOF. Let $e = uv$ be any edge in $IIE(G)$. Then e is an edge in G. Since G is $CIIE-$graph, then $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$ in G. Therefore, $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$ in $IIE(G)$. So, $IIE(G)$ is $CIIE$-graph. □

THEOREM 3.1. *A graph G is $CIIE-$graph if and only if $A(G) = A_{IIE}(G)$, where $A(G)$ and $A_{IIE}(G)$ are the adjacency matrix of G and adjacency matrix of the inherent injective equitable graph of G respectively.*

PROOF. Suppose that G is $CIIE-$graph. Then for any two adjacent vertices v_i and v_j, $|\deg_{in}(v_i) - \deg_{in}(v_j)| \leqslant 1$. Therefore, $A(G) = A_{IIE}(G)$. Similarly, if $A(G) = A_{IIE}(G)$ then G is $CIIE-$graph. □

PROPOSITION 3.5. *Let $G \cong \bigcup_{i=1}^{m} G_i$. If $G_i, i = 1, 2, ...m$, are $CIIE-$graphs, then G is $CIIE-$graph.*

PROOF. Let u and v be any two adjacent vertices in G. Therefore, u and v are adjacent vertices in a graph G_i, $i = 1, 2, ...n$. But G_i is $CIIE-$graph. Then, $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$. Hence, G is $CIIE-$graph. □

DEFINITION 3.2. *A graph G which is $CIIE-$graph is called strong $CIIE-$graph if $\overline{G}$ is also $CIIE-$graph.*

EXAMPLE 3.2. Any cycle C_n with n vertices is strong $CIIE-$graphs. Similarly, any path P_n with n vertices is strong $CIIE-$graphs.

PROPOSITION 3.6. *For any graph $G \cong K_{m,n}$ such that $|m - n| \leqslant 1$, G is strong $CIIE-$graph.*

PROOF. Let u and v be any two adjacent vertices in $G \cong K_{m,n}$. Then $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$. Therefore, G is $CIIE-$graph. Also, since $\overline{G} \cong K_m \cup K_n$, then $|\deg_{in}(u) - \deg_{in}(v)| \leqslant 1$ for any two adjacent vertices u and v. Hence, G is strong $CIIE-$graph. □

PROPOSITION 3.7. *For any graph G, $IIE(\overline{G})$ is a subgraph of $\overline{IIE(G)}$.*

PROOF. Let e be any edge in $IIE(\overline{G})$. Then, $e \in \overline{G}$. Therefore, $e \notin G$. So, $e \notin IIE(G)$. Then, $e \in \overline{IIE(G)}$. Hence $IIE(\overline{G})$ is a subgraph of $\overline{IIE(G)}$. □

PROPOSITION 3.8. *$\overline{IIE(G)}$ is subgraph of $\overline{G}$.*

THEOREM 3.2. *For any strong $CIIE-$graph G, $\overline{IIE(G)} = IIE(\overline{G})$.*

PROOF. Let e be any edge in $\overline{IIE(G)}$. Then $e \notin IIE(G)$. Since G is strong $CIIE-$graph, then $e \notin G$. Therefore, $e \in \overline{G}$ which implies that $e \in IIE(\overline{G})$, since G is strong $CIIE-$graph . So, $\overline{IIE(G)} \subseteq IIE(\overline{G})$. Henc by propostion 3.7 , $\overline{IIE(G)} = IIE(\overline{G})$. □

THEOREM 3.3. *Let G be a graph with adjacency matrix $A = \|a_{ij}\|$. Let $B_{IIE} = \|b_{ij}\|$, where $b_{ij} = \begin{cases} 1 & \text{if } |\deg_{in}(v_i) - \deg_{in}(v_j)| \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$*

Then

$$A_{IIE} = \|h_{ij}\| = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & ... & a_{1n}b_{1n} \\ . & & & \\ . & & & \\ a_{n1}b_{n1} & a_{n2}b_{n2} & ... & a_{nn}b_{nn} \end{bmatrix},$$

where A_{IIE} is the adjacency matrix of the inherent injective equitable graph of G.

PROOF. Suppose that G is a graph with adjacency matrix A and suppose $B_{IIE} = \|b_{ij}\|$, where $b_{ij} = \begin{cases} 1 & \text{if } |\deg_{in}(v_i) - \deg_{in}(v_j)| \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$

Let

$$C_{IIE} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & ... & a_{1n}b_{1n} \\ . & & & \\ . & & & \\ a_{n1}b_{n1} & a_{n2}b_{n2} & ... & a_{nn}b_{nn} \end{bmatrix}.$$

Then for $i, j = 1, 2, ...m$, $a_{ij}b_{ij} = 0$ if $a_{ij} = 0$ or $b_{ij} = 0$, i.e, v_i and v_j are not adjacent or $|\deg_{in}(v_i) - \deg_{in}(v_j)| > 1$. For $i, j = 1, 2, ...m$, $a_{ij}b_{ij} = 1$ if $a_{ij} = b_{ij} = 1$, i.e, v_i and v_j are Inj-equitable adjacent. Therefore,

$$C_{IIE} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are Inj-equitable adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

Hence $C_{IIE} = A_{IIE}$. □

4. Iterated inherent Inj-equitable graphs

DEFINITION 4.1. *We consider iterated inherent Inj-equitable graph, i.e., those obtained from a graph G as follows: $IIE^0(G) = G$ and $IIE^k = IIE(IIE^{k-1}(G))$, for $k \in \mathbb{N}$.*

THEOREM 4.1. *For any graph G, there exists a positive integer k such that $IIE^k(G)$ is $CIIE-$graph for some k.*

PROOF. If G is $CIIE-$graph, then $IIE(G) \cong G$ and then, $IIE(G)$ is $CIIE-$graph. If G is not $CIIE-$graph, then there exists an edge $e = uv$ such that $|\deg_{in}(u) - \deg_{in}(v)| > 1$. Therefore $e \notin IIE(G)$ and all the edges in $IIE(G)$ are Inj-equitable edge in G. If $IIE(G)$ is $CIIE-$graph, then $IIE^2(G)$ is $CIIE$-graph. If it is not $CIIE-$graph, then there exist an edge e in $IIE(G)$ such that e is not Inj-equitable edge and therefore, $e \notin IIE^2(G)$ and all the edges in $IIE^2(G)$ are Inj-equitable edge in $IIE(G)$. Continues in the same way until we get $CIIE-$graph or totally disconnected graph. Hence there exists $k \geqslant 1$ such that $IIE^k(G)$ is $CIIE-$graph. □

DEFINITION 4.2. *For any graph G, the completeness injective inherent equitable number is the smallest postive integer k such that $IIE^k(G)$ is $CIIE$-graph and denoted by $c_{iie}(G)$.*

PROPOSITION 4.1.

(i) *If G is $CIIE$-graph, then $c_{iie}(G) = 0$.*
(ii) *If $G \cong C_m \times C_n$, then $c_{iie}(G) = 0$.*

5. Inherent Inj-equitable graphs

DEFINITION 5.1. *A graph G is said to be inherent Inj-equitable graph (IIE-graph) if there exists a graph H such that $IIE(H) \cong G$.*

For example, any path, cycle and complete graph are IIE-graph. The family of graphs H which satisfy the condition $IIE(H) \cong G$ is called the inherent Inj-equitable family of G and denoted by

$$G_{IIE} = \{H : IIE(H) \cong G\}.$$

REMARK 5.1. *The inherent Inj-equitable graph is not unique.*

THEOREM 5.1. *For any Complete bipartite graph $G \cong K_{1,p}$, G is not IIE-graph, where $p \geqslant 3$.*

PROOF. Suppose to the contrary that $G \cong K_{1,p}$ is IIE-graph. So, there exists at least a graph H such that $IIE(H) \cong G$. Therefore, H contains at least the same number of edges as G or more. Clearly $H \ncong K_{1,p}$ and the number of edges in H will be more than the number of edges in $K_{1,p}$. So any edge in H other than the edges of $K_{1,p}$ is Inj-equitable edge which is contradiction to $IIE(H) \cong G$. Hence G is not IIE-graph. □

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