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ir-EXCELLENT GRAPHS

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Abstract. Terasa W. Haynes et. al. [**7**], introduced the concept of irredundance in graphs. A subset S of $V(G)$ is called an irredundant set of G if for every vertex $u \in S$, $pn[u, S] \neq \phi$. The minimum (maximum)cardinality of a maximal irredundant set of G is called the irredundance number of G (upper irredundance number of G) and is denoted by $ir(G)(IR(G))$. A subset $V(G)$ is called an ir-set if it is an irredundant set of G of cardinality $ir(G)$. A vertex $u \in V(G)$ is called ir-good if u belongs to an ir-set of G. G is said to be ir-excellent if every vertex of G is ir-good. In this paper, a study of the excellent graphs with respect to irredundance is initiated.

1. Introduction

We consider the graphs which are finite, undirected, non - trivial without loops or multiple edges. Let $G = (V, E)$ be a simple graph. For graph theoretic terminology, we refer to [**1**]. A subset S of V is a dominating set of G if every vertex in $V - S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. For a set S of vertices in a graph G, the closed neighborhood $N[S]$ of S is defined $N[S] = \bigcup\limits_{v\in S} N[v]$. Each vertex in $N[v]N[Sv]$ is referred to as a private neighbour of $v \in G$ and is denoted by $pn(v, S)$. In [**7**], a subset S of $V(G)$ is called an ir-set if it is an irredundant set of cardinality $ir(G)$ ($ir(G)$ is the minimum cardinality of a maximal irredundant set). Any non empty subset of an irredundant set is irredundant. Hence, the property of irredundance is hereditary.

Let μ be a parameter of a graph. A vertex $v \in V(G)$ is said to be μ-good if v belongs to a μ-minimum (μ-maximum) set of G according as μ is a super hereditary (hereditary) parameter. v is said to be -bad if it is not μ-good. A graph G is said to

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be μ-excellent if every vertex of G is μ-good. Excellence with respect to domination and total domination were studied in [**2**]. In a social network , we may exchange any node inside the network by a node in outside the network, gives a better status in the form of a new group. Such a situation can be modelled as a set S of vertices in the graph G representing the social network such that for every $y \in V(G) - S$ there exists $x \in S$ such that the new social group $S = (S - \{x\}) \cup \{y\}$ has the same property as that of S and is possibly better in terms of external connections as well as its internal organization. This is the motivation for studying excellent graphs with various graph parameters.N. Sridharan and Yamuna [**4, 5, 6**], have defined various types of excellence.

2. ir-excellent graphs

In this section, we define and study a new type of graph, namely ir-excellent graph.

Definition 2.1. Let $G = (V, E)$ be a simple graph. Then G is said to be an ir-excellent graph if every vertex belongs to an ir-set of G.

Example 2.1. **ir-excellent graphs.**

(1) K_n
(2) $\overline{K_n}$
(3) C_n
(4) $K_{n,n},\ n \geqslant 2$
(5) $K_{m,n},\ m, n \geqslant 2,\ m < n$
(6) $D_{r,s}$ is ir-excellent if $r = s = 1$.

Example 2.2. **Graphs which are not ir-excellent**

1. $K_{1,n}$
2. $D_{r,s}$ for $r, s \geqslant 2(\ ir(D_{r,s}) = 2,\ IR(D_{r,s}) = r + s)$.

Let $V(D_{r,s}) = \{u_1, u_2, \cdots, u_r, u, v, v_1, v_2, v_s\}$ where u is the support of the pendent vertices $u_1, u_2, \cdots, u_r\}$ and v is the support of the pendent vertices $\{v_1, v_2, \cdots, v_s\}$. Let $S = \{u, v\}$. Then S the only ir-set of $D_{r,s}$, since all the pendent vertices are not in any ir-set of $D_{r,s}$.

Proposition 2.1. *If G is vertex transitive then G is ir-excellent.*

Proof. Let D be an ir-set of G. Let $u \notin D$. Let $v \in D$. Then there exists an automorphism ϕ such that $\phi(v) = u$. Then $u \in \phi(D)$.

Claim 1: $\phi(D)$ is irredundant.

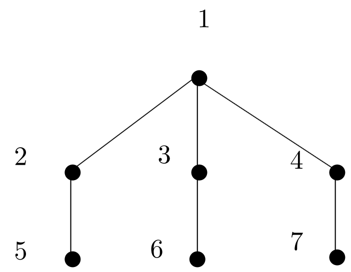
For: Let $w \in \phi(D)$. Then $w = \phi(y)$ for some $y \in D$. If y is the private neighbourhood of itself with respect to D, then y is an isolate of D, which implies $\phi(y)$ is an isolate of $\phi(D)$. Therefore w is a private neighbourhood of itself with respect to $\phi(D)$. If y_1 is a private neighbourhood of y with respect to D, then y_1 is not adjacent to any vertex of D other than y. Therefore $\phi(y_1)$ is not adjacent to any vertex of $\phi(D)$ other than $\phi(y) = w$. Hence w has a private neighbourhood $\phi(y)$ with respect to $\phi(D)$. Therefore $\phi(D)$ is irredundant.

Claim 2: $\phi(D)$ is a maximal irredundant set.
Suppose not. Then there exists $S \subset V(G)$ such that $\phi(D) \subsetneqq S$ and S is irredundant. Let $x \in S - \phi(D)$. Let $\phi^{-1}(x) = t$. Then $t \in \phi^{-1}(S)$ and $t \notin D$. Therefore $D \subsetneqq \phi^{-1}(S)$ and $\phi^{-1}(S)$ is irredundant, a contradiction to maximality of D. Therefore $\phi(D)$ is a maximal irredundant set of G. $|D| = |\phi(D)| = ir(G)$ and hence $\phi(D)$ is an ir-set of G containing u. Therefore u is ir-good and G is ir-excellent. $\square$

Observation 2.1. *Let $\gamma(G) = ir(G)$. If G is γ-excellent, then G is ir-excellent.*

Observation 2.2. *There exists a graph G in which $\gamma(G) = ir(G)$, G is ir-excellent but not γ-excellent.*

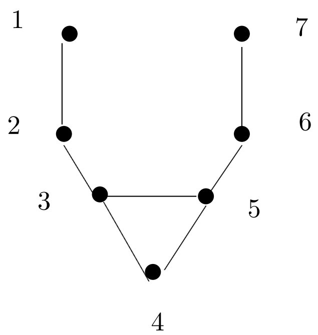
Example 2.3.



γ-sets of G are: $\{2,3,4\}, \{2,4,6\}, \{2,6,7\}, \{4,5,6\}, \{5,3,7\}$
ir-sets of G are: $\{1,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,5,7\}, \{1,5,6\}, \{1,6,7\},$
$\{3,5,7\}, \{2,3,4\}, \{2,3,7\}, \{3,4,5\}, \{2,4,6\}, \{5,6,7\}$
1 does not belong to any γ-set. Therefore G is not γ-excellent. But G is ir-excellent.

Observation 2.3. *There exists a graph G in which $ir(G) < \gamma(G)$, G is not ir-excellent but γ-excellent.*

Consider the Allan Laskar graph(A. L. graph), which is shown below:



ir-set: $\{3,5\}$

γ-sets : $\{1,3,7\},\{2,4,6\},\{5,2,7\}$.
The graph is γ-excellent but not ir-excellent.

In general the above type of graphs with a subgraph as a complete graph have the property $ir < \gamma,\ ir = 2$ *and* $\gamma = 3$. Such type of graphs are γ-excellent graphs but not ir-excellent.

PROPOSITION 2.2. *For any path P_n, $ir(P_n) = \gamma(P_n)$.*

PROOF. $\Delta(P_n) = 2$. From [**1**], we have $\frac{n}{(2\Delta(G)-1)} \leqslant ir(G)$. Therefore $\frac{2n}{3\times 2} \leqslant ir(P_n)$, that is $\frac{n}{3} \leqslant ir(P_n)$. Therefore $\lceil \frac{n}{3} \rceil \leqslant ir(P_n)$, which means $\gamma(P_n) \leqslant ir(P_n)$. But $ir(P_n) \leqslant \gamma(P_n)$. Therefore $\gamma(P_n) = ir(P_n)$. □

PROPOSITION 2.3.
(1) *P_{3n+1} is ir-excellent for all n.*
(2) *P_{3n+2} is not ir-excellent for $n \geqslant 3$.*
(3) *P_{3n} is not ir-excellent for all n.*

PROOF. (1). Let $V(P_{3n+1}) = \{u_1, u_2, \cdots, u_{3n+1}\}$. $\gamma(P_{3n+1}) = ir(P_{3n+1}) = n+1$.

$$D_1 = \{u_1, u_4, u_7, \cdots, u_{3n+1}\},\ D_2 = \{u_2, u_5, \cdots, u_{3n-1}, u_{3n} \textit{ or } u_{3n+1}\},$$
$$D_3 = \{u_1, u_3, u_6, \cdots, u_{3n}\}$$

are minimum dominating sets and hence also ir-sets of P_{3n+1}. Therefore P_{3n+1} is ir-excellent.

(2). Let $V(P_{3n+2}) = \{u_1, u_2, \cdots, u_{3n+2}\}$, $n \geqslant 3$. $\gamma(P_{3n+2}) = ir(P_{3n+2}) = n+1$.

$$D_1 = \{u_1, u_4, \cdots, u_{3n+1}\},\ D_2 = \{u_2, u_5, u_8, \cdots, u_{3n+2}\},$$
$$D_3 = \{u_3, u_4, u_7, \cdots, u_{3n+1}\},\ D_4 = \{u_2, u_5, u_8, \cdots, u_{3n-1}, u_{3n}\}$$

are all ir-sets and hence $u_i,\ i \neq 6, 9, \cdots, 3n-3$ are not ir-good.

When $n = 2$, both u_3 and u_6 will be ir-good and hence P_8 is ir-excellent.

When $n = 1$, u_3 is ir-good and hence P_5 is ir-excellent. Therefore P_{3n+2} is not ir-excellent if $n \geqslant 3$.

(3). Let $V(P_{3n}) = \{u_1, u_2, \cdots, u_{3n}\}$.
When $n = 1$, P_3 is not ir-excellent since $ir(P_3) = 1$ and u_1 is not in any ir-set.

When $n = 2$, we get P_6. Again u_1 is not in any ir-set, since the minimum cardinality of an irredundant set containing u_1 is 3 and $ir(P_6) = 2$. $ir(P_{3n}) = n$ and the minimum cardinality of an irredundant set containing u_1 is $n+1$. ($\{u_1, u_3, u_6, \cdots, u_{3n}\}$ or $\{u_1, u_4, u_7, \cdots, u_{3n-2}, u_{3n}\}$ are irredundant sets containing u_1 of minimum cardinality). Therefore P_{3n} is not ir-excellent. □

PROPOSITION 2.4. *If $ir(G) < \gamma(G)$, then any independent set is not an ir-set.*

PROOF. Let S be an independent set of G. Suppose S is an ir-set of G. Then S is a maximal independent set of G. Therefore S is a minimal dominating set of G. Therefore $ir(G) < \gamma(G) \leqslant |S| = ir(G)$, a contradiction. Therefore S is not an ir-set of G. □

Corollary 2.1. *If $ir(G) < \gamma(G)$, then for any ir-set S of G, number of private neighbours of S lying in $V - S$ is greater than or equal to 2.*

Proposition 2.5. *For any graph G, G^+ is both ir-excellent and γ-excellent.*

Proof. Let S be an ir-set of G^+. Suppose $|S| < n$. Then there exists $u \in V(G)$ such that $u, u^{'} \notin S$ where $u^{'}$ is the pendent of u. Then $S \cup \{u\}$ is an irredundant set of G^+, since $u^{'}$ is the private neighbour of u, a contradiction. Therefore $|S| \geqslant n$. Since $\gamma(G^+) = n$, $|S| = n$. Since any γ-set of G^+ is also an ir-set of G^+, G^+ is ir-excellent. □

Observation 2.4. *Any graph G is an induced graph of an ir-excellent graph.*

Proposition 2.6. *Let G be a non-ir-excellent graph with a unique ir-bad vertex. Then there exists an ir-excellent graph H such that*
(i). G is an induced subgraph of H.
(ii). $ir(H) = ir(G) + 1$.

Proof. Let u be the unique ir-bad vertex of G. Let H be the graph obtained from G by adding a new vertex v and making it adjacent with only u in G.

Claim: $ir(H) = ir(G) + 1$.

Let $ir(G) = k$. Note that for any ir-set S of G, $u \notin S$. Hence $S \cup \{v\}$ is an irredundant set of H. Clearly it is a maximal irredundant set of H.

Suppose $ir(H) = k' \leqslant k$. Let T be an ir-set of H. If $v \notin T$, then $T \cup \{v\}$ is an irredundant set of H if $u \notin T$, a contradiction. Since T is a maximal irredundant set of H, $u \in T$. Since $T \subseteq V(G)$, T is a maximal irredundant set of G and $k = ir(G) \leqslant |T| = k' \leqslant k$. Therefore $|T| = k$. T is an irredundant set of G containing u, a contradiction, since u is an ir-bad vertex of G. Therefore $v \in T$. Let $T_1 = T - \{v\}$. Then $T_1 \subseteq V(G)$. $|T_1| = k' - 1 < k$. Clearly T_1 being a subset of an irredundant set of H is irredundant in H.

Case 1: $u \notin T_1$. Then T_1 is an irredundant set of G. Suppose T_1 is a maximal irredundant set of G. Then $k = ir(G) \leqslant |T_1| < k$, a contradiction. Therefore T_1 is not a maximal irredundant set of G. Therefore there exists $w \in G$ such that $T_1 \cup \{w\}$ is an irredundant set of G. Suppose $w \neq u$. Then $T \cup \{w\} = T_1 \cup \{w\} \cup \{v\}$ is an irredundant set of H contradicting the maximality of T. Therefore $w = u$. Therefore $T_1 \cup \{u\}$ is an irredundant set of G. If $T_1 \cup \{u\}$ is a maximal irredundant set of G, Then $k = ir(G) \leqslant |T_1| + 1 = k' \leqslant k$. Therefore $|T_1| + 1 = k$ and hence $T_1 \cup \{u\}$ is an ir-set of G implying u is ir-good, a contradiction. Therefore $T_1 \cup \{u\}$ is not a maximal irredundant set of G. Thus there exists $z \in V(G) - (T_1 \cup \{u\})$ Such that $T_1 \cup \{u\} \cup \{z\}$ is an irredundant set of G. Therefore $T_1 \cup \{z\}$ is an irredundant set of G. Therefore $T_1 \cup \{z\} \cup \{v\}$ is an irredundant set of H. Thus $T \cup \{z\}$ is an irredundant set of H.

Case 2: $u \in T_1$. Then T_1 is an irredundant set of G. If T_1 is maximal, then $k = ir(G) \leqslant |T_1| < k$, a contradiction. Therefore T_1 is not a maximal irredundant set of G. Therefore there exists $x \in V(G) - T_1$ such that $T_1 \cup \{x\}$ is irredundant in G. Since $u \in T_1$, we get that $x \neq u$. Therefore $T_1 \cup \{x\} \cup \{v\}$ is an irredundant set in H. That is $T \cup \{x\}$ is irredundant in H, a contradiction to the maximality of T. Therefore $ir(H) > k$. That is $ir(H) \geqslant k+1$. But $S \cup \{v\}$ for any ir-set S of G is a maximal irredundant set of H. Therefore $ir(H) \leqslant |S \cup \{v\}| = k+1$. Therefore $ir(H) = k+1$. Therefore $S \cup \{v\}$ is an ir-set of H for any ir-set S of G. Therefore every ir-good vertex in G as well as v is ir-good in H. Moreover for any ir-set S of G, $S \cup \{u\}$ is irredundant in H since u has a private neighbour v in H. Therefore $S \cup \{u\}$ is an ir-set of H, which implies u is also ir-good in H. Therefore H is ir-excellent. G is an induced subgraph of H. Further, $ir(H) = ir(G)+1$. □

Conjecture. There does not exist any graph G which is both γ-excellent and ir-excellent and $ir(G) < \gamma(G)$.

COROLLARY 2.2. *If G_1, G_2 are ir-excellent, then G_1+G_2 is ir-excellent if and only if $ir(G_1) = ir(G_2)$.*

3. Definition and Properties of just ir-excellent graphs

In this section, we introduce the concept of just ir-excellent graphs and study its properties.

DEFINITION 3.1. A graph G is said to be just ir-excellent graph, if every vertex of G belongs to exactly one ir-set of G.

REMARK 3.1. If G is just ir-excellent then G admits a partition where each element of the partition is an ir-set of G.

EXAMPLE 3.1.
C_{3n}, K_n, $H_{5,10}$.

REMARK 3.2. Every just ir-excellent graph is ir-excellent graph.

REMARK 3.3. If $\gamma(G) = 2$, then $ir(G) = 2$.

PROOF. Suppose $ir(G) = 1$. Then G has a full degree vertex. Hence $\gamma(G) = 1$, a contradiction. Therefore $ir(G) \geqslant 2$. But $ir(G) \leqslant \gamma(G) = 2$. Therefore $ir(G) = 2$. The converse is not true, since in $A.L$ graph $\gamma(G) = 3$ and $ir(G) = 2$. □

PROPOSITION 3.1. *It has been proved in [**3**] that if G is a graph containing no induced subgraph isomorphic to $K_{1,3}$, or $A.L$ graph, then $ir(G) = \gamma(G) = i(G)$. Since C_n and P_n does not contain $K_{1,3}$ or $A.L$ graph as an induced subgraph, $ir(C_n) = \gamma(C_n) = i(C_n)$ and $ir(P_n) = \gamma(P_n) = i(P_n)$.*

OBSERVATION 3.1. *C_n is γ-excellent if and only if $n \equiv 0(mod\ 3)$. Therefore C_n is ir-excellent if and only if $n \equiv 0(mod\ 3)$.*

PROPOSITION 3.2. *Every just ir-excellent graph $G(\neq \overline{K_n})$,is connected.*

PROOF. Let G be a disconnected graph, $G \neq \overline{K_n}$. Let G_1 be a component of G. If $|V(G_1)| = 1$, then G is not just ir-excellent. Hence $|V(G_1)| \geqslant 2$.

Claim: G_1 is just ir-excellent.

Let S be an ir-set of G. Let $S_1 = S \cap V(G_1)$. Clearly S_1 is non-empty. Since S is an ir-set, S_1 is an irredundant set of G_1 and clearly it is a maximal irredundant set of G_1.

Suppose $|S_1| > ir(G)$. Let S' be an ir-set of G_1. Then $S' \cup (S - S_1)$ is an irredundant set of G of cardinality greater than $|S|$, a contradiction (since S is an ir-set of G). Therefore S_1 is an ir-set of G_1. Since G is just ir-excellent, G_1 is also just ir-excellent. Since G_1 is connected, $ir(G_1) \leqslant \gamma(G_1) \leqslant \frac{n}{2}$. As G_1 is just ir-excellent, G_1 has at least two ir-sets, say T_1 and T_2. Let D_1 be an ir-set of $G - G_1$. Then $D_1 \neq \phi$ and $T_1 \cup D_1$, $T_2 \cup D_2$ are ir-sets of G with non-empty intersection, a contradiction, since G is just ir-excellent. Therefore G is connected. □

Proposition 3.3. *Let $G \neq \overline{K_n}$ is just ir-excellent. Then for any ir-set D of G, $|pn[u, D]| \geqslant 2$ for all $u \in D$.*

Proof. **Case A:** Since $G \neq \overline{K_n}$, order of G is greater than or equal to 2.

Since D is an ir-set of G, $|pn[u, D]| \geqslant 1$ for all $u \in D$. Suppose $|pn[u, D]| = 1$.

Case (i): $|pn[u, D]| = 1$. Let $pn(u, D) = \{v\}$ where $v \in V - D$. Let $D_1 = (D - \{u\} \cup \{v\}$. Then v being not adjacent to any vertex of $D - \{u\}$, $v \in pn[v, (D - \{u\}) \cup \{v\}]$. Also, if $x \in D - \{u\}$, then $pn[x, D] = pn[x, (D - \{u\}) \cup \{v\}]$, since v is not adjacent with x. Therefore D_1 is an irredundant set of G of cardinality $ir(G)$.

Suppose D_1 is not a maximal irredundant set of G. Then there exists a maximal irredundant set say D_2 of G such that $D_1 \subsetneq D_2$. Let $w \in D_2 - D_1$.

Subcase (i): $w = u$. In this case $D_1 \subsetneq D_2$ and $v \in D_2$. Since u and v are adjacent and D_2 is irredundant, $w = u$ has a private neighbour say x with respect to D_2 outside D_2. Clearly $x \notin D$. Therefore x and v are two private neighbours of u with respect to D belonging to $V - D$, a contradiction since $|pn(u, D)| = 1$.

Subcase (ii): $w \neq u$. Clearly $w \neq v$. Since u is adjacent with $v \in D_2$, u cannot be a private neighbour of w with respect to D_2. Therefore w is a private neighbour of u with respect to D. Hence $|pn(u, D)| \geqslant 2$, a contradiction.

Subcase (iii): Suppose u is not a private neighbour of w with respect to D_2. Then either w is an isolate of D_2 or there exists $y \in V - D_2$ such that $y \in pn(w, D_2)$. Let w be an isolate of D_2. Consider $D' = D \cup \{w\}$. If w is not adjacent with u then w is an isolate of D' and hence D' is an irredundant set containing D, a contradiction to maximality of D. If w is adjacent with u, then w being not adjacent with any vertex of $D - \{u\}$, is a private neighbour of u with respect to D in $V - D$. That is u has two private neighbours v, w with respect to D in $V - D$, a contradiction since $|pn(u, D)| = 1$. Suppose there exists $y \in V - D_2$ such that $y \in pn(w, D_2)$. Let w be a private neighbour of some $x \in D$ with respect to D. If $x = u$, then u has two private neighbours with respect to D, a contradiction. If $x \neq u$, then as x, $w \in D_2$, x has a private neighbour say z outside D_2 with respect to D_2. Then z is a private neighbour of x with respect to D. Hence $D \cup \{w\}$ is an irredundant set of G, containing D, a contradiction to the maximality of D.

Case (ii): u is an isolate of D. Since u is not an isolate of G (if u is an isolate of G, then u belongs to every irredundant set contradicting just ir-excellent), there exists $v \in V - D$ such that u and v are adjacent. Since $pn[u, D] = \{u\}$, v is not a private neighbour of u with respect to D. Therefore v is adjacent to some vertex say $w \neq u \in D$. Consider $D_1 = (D - \{u\}) \cup \{v\}$. Since u is a private neighbour of v with respect to D_1 and since every vertex of $D - \{u\}$ has a private neighbour not equal to v with respect to D, D_1 is an irredundant set of G strictly containing D_1. Let $w \in D_2 - D_1$. Suppose $w = u$. Since u is adjacent with v, w in D_2 is not an isolate of D_2. $w = u$ has a private neighbour in $V - D$ with respect to D. Therefore $|pn[u, D]| \geqslant 2$, a contradiction.

Suppose $w \neq u$.

Subcase (i): w is an isolate of D_2. Then w is not adjacent with any vertex of $(D - \{u\}) \cup \{v\}$. (If w is adjacent with u then w is a private neighbour of u in $V - D$ with respect to D a contradiction). Therefore w is not adjacent with u. w is an isolate of $D \cup \{w\}$. Hence $D \cup \{w\}$ is an irredundant set containing D, a contradiction to the maximality of D.

Subcase (ii): w is not an isolate of D_2. Then w has a private neighbour say z in $V - D_2$. If $z = u$, then z is not adjacent with any vertex in $D_2 - w$. But u is adjacent with v in D_2, a contradiction. Therefore $z \neq u$. Consider $D \cup \{w\}$. If w is not a private neighbour of any vertex of D with respect to D, then $D \cup \{w\}$ is irredundant. If w is a private neighbour of some $x \in D$ with respect to D, then $x \neq u$ (since $pn[u, D] = 1$). As x and w are adjacent in D_2, x has a private neighbour say y in $V - D_2$ with respect to D_2. That is x has a private neighbour y in $V - D$ with respect to D. Therefore $D \cup \{w\}$ is an irredundant set of G is a contradiction to the maximality of D. Therefore $D_1 = (D - \{u\}) \cup \{v\}$ is a maximal irredundant set of G. $|D| = 1$ implies $ir(G) = 1$. As G is just excellent and $ir(G) = 1$, $G = K_n$, a contradiction. Therefore $|D| \geqslant 2$. Hence $\phi \neq D - \{u\}$, is contained in two ir-sets namely D and D_1, a contradiction to just excellence. Therefore $|pn[u, D]| \geqslant 2$ for all $u \in D$.

Case B: $G = K_n$, $n \geqslant 2$. Here $ir(G) = 1$ and every verex constitutes an ir-set of G. Let D be any ir-set of G. Then $D = \{u\}$ for some $u \in V(G)$. $|pn[u, D]| = n \geqslant 2$. □

Remark 3.4. Let G be the graph obtained from $K_{n,n}$ by removing a 1-factor. Then G is just ir-excellent.

Proof. If V_1 and V_2 are the partite sets and if $V_1 = \{u_1, u_2, \cdots, u_n\}$, $V_2 = \{v_1, v_2, \cdots, v_n\}$ and u_i and v_i are not adjacent ($1 \leqslant i \leqslant n$), then the ir-sets are

$\{u_1, v_1\}, \{u_2, v_2\}, \cdots, \{u_n, v_n\}$. □

Theorem 3.1. *Let G be a graph of order n. Then G is ir-excellent if and only if the following conditions hold.*

(i) $ir(G)$ divides n.

(ii) G has exactly $\frac{n}{ir(G)}$ distinct ir-sets.

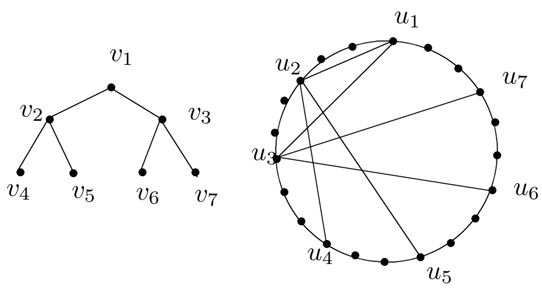
Proof. (i) Let G be just ir-excellent. Then G can be partitioned into t sets each of which is an ir-set. Therefore $t\ ir(G) = n$. Therefore $ir(G)$ divides n.

(ii): $V(G) = S_1 \cup S_2 \cup, \cdots, \cup S_m$ where each S_i is an ir-set of G and these sets are pairwise disjoint. Therefore there are m distinct ir-sets of G where $m = \frac{n}{ir(G)}$. Suppose there exists a ir-set T different from $S_1, S_2, \cdots, S_m$. Since $S_1 \cup S_2 \cup, \cdots, \cup S_m = V(G) \supseteq T$, every element $x \in T$ belongs to some S_i, $1 \leqslant i \leqslant m$. Therefore x belongs to two ir-sets of G, a contradiction.
Conversely, suppose the three conditions hold. Let $m = \frac{n}{ir(G)}$. By (iii) G has exactly m distinct ir-sets. Suppose $V = S_1 \cup S_2 \cup, \cdots, \cup S_m$ is a decomposition of $V(G)$ where each S_i is a maximal irredundant set, $1 \leqslant i \leqslant m$. Then $n = \sum_{i=1}^{m} |S_i| \geqslant m\ ir(G)$. But $n = m\ ir(G)$. Therefore each S_i is an ir-set of G. Since G has exactly m distinct ir-sets, $S_1, S_2, \cdots, S_m$ are the distinct ir-sets of G and hence $V = S_1 \cup S_2 \cup, \cdots, \cup S_m$ is a partition into disjoint ir-sets of G. Therefore each vertex v belongs to exactly one S_i, for some i, $1 \leqslant i \leqslant m$. Therefore G is just ir-excellent. $\square$

Theorem 3.2. *Every graph is an induced subgraph of a just ir-excellent graph.*

Proof. Let G be a given graph. If G is just ir-excellent, then there is nothing to prove. Assume that G is not just ir-excellent. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$. Consider the cycle C_{3n}. It is just ir-excellent. Let S_1, S_2, S_3 be the distinct ir-sets of C_{3n}. Label the vertices of S_1 by $u_1, u_2, u_3, \cdots, u_n$. Now in C_{3n} we add edges $u_i u_j$ if and only if $v_i v_j$ is an edge in G. Let the resulting graph be H. Then the induced subgraph $\langle S_1 \rangle$ in H is isomorphic to G. By theorem 4.12 in [**6**], H is just ir-excellent and $ir(H) = n$. Every ir-set is a γ set. Thus the given graph G is an induced subgraph of a just ir-excellent graph H. $\square$

Example 3.2.



G H

G is an induced subgraph of H which is a just ir-excellent graph.

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