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ODD HARMONIOUS LABELING OF PLUS GRAPHS

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Abstract. A graph $G(p, q)$ is said to be odd harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. In this paper we prove that the plus graph Pl_n , open star of plus graph $S(t.Pl_n)$, path union of plus graph Pl_n, joining of C_m and plus graph Pl_n with a path, one point union of path of plus graph $P_n^t(t.n.Pl_n)$ are odd harmonious graphs.

1. Introduction

Throughout this paper by a graph we mean a finite, simple and undirected one. For standard terminology and notation we follow Harary [**5**]. A graph $G = (V, E)$ with p vertices and q edges is called a (p, q) − graph. The graph labeling is an assignment of integers to the set of vertices or edges or both, subject to certain conditions have often been motivated by practical problems. An extensive survey of various graph labeling problems is available in [**3**]. Labeled graphs serves as useful mathematical models for many applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. Graham and Sloane [**4**] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes. A graph G is said to be harmonious if there exists an injection $f : V(G) \to Z_q$ such that the induced function $f^* : E(G) \to Z_q$ defined by

$$f^*(uv) = (f(u) + f(v)) \pmod q$$

is a bijection and f is called harmonious labeling of G. The concept of odd harmonious labeling was due to Liang and Bai [**6**]. A graph G is said to be odd

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harmonious if there exists an injection $f : V(G) \to \{0, 1, 2, \cdots, 2q-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \cdots, 2q-1\}$ defined by

$$f^*(uv) = f(u) + f(v)$$

is a bijection. If $f(V(G)) = \{0, 1, 2, ...q\}$ then f is called as strongly odd harmonious labeling and G is called as strongly odd harmonious graph.The odd harmoniousness of graph is useful for the solution of undetermined equations. Several results have been published on odd harmonious labeling.

In [**6**], the following results have been proved: 1. If G is an odd harmonious graph, then G is a bipartite graph. Hence any graph that contains an odd cycle is not an odd harmonious.

2.If a $(p, q)-$ graph G is odd harmonious, then $2\sqrt{q} \leqslant p \leqslant (2q-1)$.

3.If G is an odd harmonious Eulerian graph with q edges, then $q \equiv 0, 2(mod\ 4)$.

We [**7, 8**] proved that the shadow and splitting of graph $K_{2,n}$, C_n for $n \equiv 0(mod\ 4)$ and the graph $H_{n,n}$ are odd harmonious. Also, we proved that any two even cycles sharing a common vertex and a common edge are also odd harmonious graph. For further results, the interested reader can refer to [**3, 6, 9, 10**].

We use the following definitions in the subsequent section.

Definition 1.1. [**1**] *Let G be a graph and $G_1, G_2, ..., G_n, n \geqslant 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1}, $(1 \leqslant i \leqslant n-1)$ is called path union of G.*

Definition 1.2. [**12**] *Let G be a graph on n vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G is called a star of G denoted by G^*.*

Definition 1.3. [**1**] *Take $P_n, P_n, ..., P_{n-2}, P_{n-4}, ..., P_4, P_2$ paths on*

$$n, n, ..., n-2, n-4, ...4, 2$$

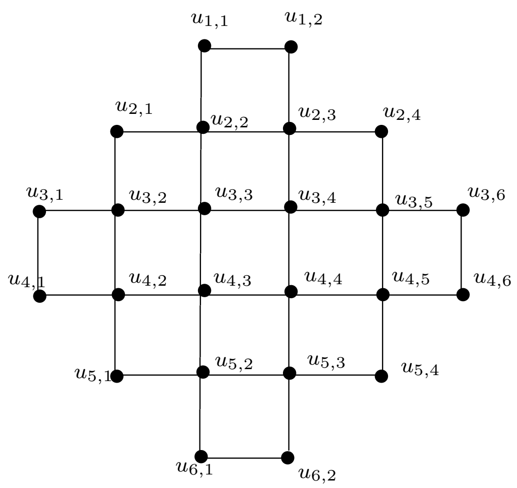
vertices and arrange them centrally horizontal, where $n \equiv 0(mod\ 2)$, $n \neq 2$. A graph obtained by joining vertical vertices of given successive paths is called a double step grid graph of size n denoted by DS_{tn}.

Definition 1.4. [**1**] *Take $P_2, P_4, ..., P_{n-2}, P_n, P_n, P_{n-2}, ..., P_4, P_2$ paths on*

$$2, 4, ..., n-2, n, n, n-2, ..., 4, 2$$

vertices and arrange them centrally horizontal, where $n \equiv 0(mod\ 2)$, $n \neq 2$. A graph obtained by vertical vertices of given successive paths is known as a plus graph of size n denoted by Pl_n. Obviously, $|V(G)| = \dfrac{n^2}{2} + n$ and $|E(G)| = n^2$.

The example for the plus graph Pl_6 is shown below (Figure 0):



2. Main Results

In this section we prove that the plus graph Pl_n , open star of plus graph $S(t.Pl_n)$, path union of plus graph Pl_n, joining of C_m and plus graph Pl_n with a path of arbitrary length, one point union of path of plus graph $P_n^t(t.n.Pl_n)$ are odd harmonious graphs.

THEOREM 2.1. *A plus graph Pl_n is odd harmonious, where $n \equiv 0(mod\ 2)$, $n \neq 2$.*

PROOF. Let $G = Pl_n$ be any graph of size n, $n \equiv 0(mod\ 2)$, $n \neq 2$. We denote each vertices of first row as $u_{1,j}$ $(1 \leqslant j \leqslant 2)$ and 2^{nd} row as $u_{2,j}$ $(1 \leqslant j \leqslant 4)$. Similarly $\left(\frac{n}{2} - 1\right)^{th}$ row as $u_{\frac{n}{2}-1,j}$ $(1 \leqslant j \leqslant n-2)$ and $\left(\frac{n}{2}\right)^{th}$ row as $u_{n,j}$ $(1 \leqslant j \leqslant n)$ and $\left(\frac{n}{2} + 1\right)^{th}$ row as $u_{\frac{n}{2}+1,j}$ $(1 \leqslant j \leqslant n)$. Similarly $(n-1)^{th}$ row as $u_{n-1,j}$ $(1 \leqslant j \leqslant 4)$ and n^{th} row as $u_{n,j}$ $(1 \leqslant j \leqslant 2)$. Note that the number of vertices in $G = Pl_n$ is $|V(G)| = \frac{n^2}{2} + n$ and $|E(G)| = n^2$.

We define a labeling $f : V(G) \to \{0, 1, 2, \cdots, 2n^2 - 1\}$ as follows:

$$f(u_{i,j}) = nj - 2i + j - 1\ ,\quad j = 1, 3, 5, \cdots, n-1 \text{ and } i = \frac{j+1}{2}, \frac{j+3}{2}, \cdots, \frac{n}{2}.$$

$$f(u_{i,j}) = n(j-2) + n - 2i + j + 1,\quad j = 2, 4, \cdots, n \text{ and } i = \frac{j}{2}, \frac{j+2}{2}, \cdots, \frac{n}{2}.$$

$$f(u_{i,j}) = nj + 2in - n^2 - 2n + j - 2\ ,\quad i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n \text{ and } j = 2, 4, \cdots, n - [2i - n - 2].$$

$$f(u_{i,j}) = nj + 2in - n^2 - 3n + j,\quad i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n \text{ and } j = 1, 3, 5, \cdots, n - [2i - n - 1].$$

The induced edge labels are

$$f^*(u_{i,j}u_{i,j+1}) = 2nj - 4i + 2j + 1, \quad j = 1,2,3,\cdots,n-1 \text{ and } i = 1,2,...,\frac{n}{2}.$$

$$f^*(u_{i,j}u_{i,j+1}) = 2nj + 4in - 2n^2 - 4n + 2j - 1, \ \ j = 1,2,3,\cdots,n \text{ and } i = \frac{n}{2}+1, \frac{n}{2}+2,\cdots,n.$$

$$f^*(u_{i,j}u_{i+1,j+1}) = 2nj - 4i + 2j - 1, \ i = 1,2,\cdots,\frac{n-2}{2} \text{ and } j = 1,2,\cdots,n-2.$$

$$f^*(u_{i,j}u_{i+1,j}) = 2nj + 2in - n^2 - n - 2i + 2j - 1, \ i = \frac{n}{2} \text{ and } j = 1,2,\cdots,n.$$

$$f^*(u_{i,j}u_{i+1,j-1}) = 2nj + 4in - 2n^2 - 4n + 2j - 3, \ i = \frac{n}{2}+1, \frac{n}{2}+2,\cdots,n-1 \text{ and } j = 2,3,\cdots,n-1.$$

In the view of above defined labeling pattern, the plus graph Pl_n is odd harmonious, where $n \equiv 0(mod\ 2)$, $n \neq 2$. □

The odd harmonious labeling of Pl_6 is shown in Figure 1.

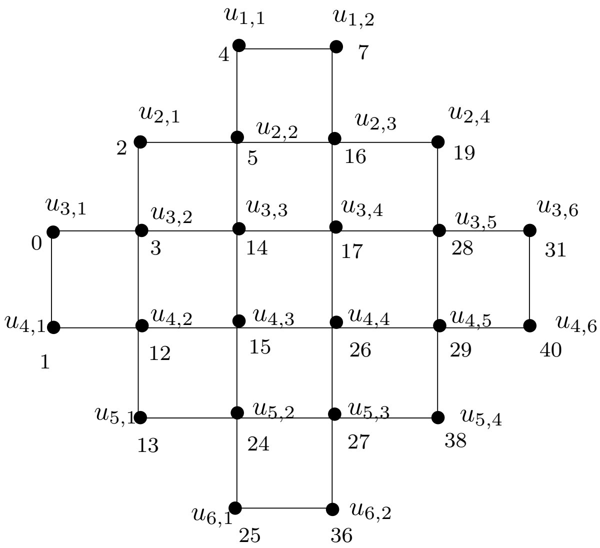


FIGURE 1: *Odd harmonious labeling of Pl_6.*

THEOREM 2.2. *Path union of finite copies of the plus graph Pl_n is odd harmonious, where $n \equiv 0(mod\ 2)$, $n \neq 2$.*

PROOF. Let G be a path union of r copies of the plus graph pl_n, where $n \equiv 0(mod\ 2)$, $n \neq 2$. Let $u_{k,i,j}$ ($\forall i = 1,2,\cdots,\frac{n}{2}$ and $\forall j = 1,2,\cdots,2i$ and $\forall i = \frac{n}{2}+1,\cdots,n$ and $\forall j = 1,2,\cdots,2n-2i+2$) be the vertices of the k^{th}copy of Pl_n, $\forall k = 1,2,\cdots,r$, where the vertices of k^{th} copy of Pl_n is $p = \frac{n^2}{2} + n$ and the edges of the k^{th} copy of Pl_n is $q = n^2$. Join the vertices $u_{k,\frac{n}{2}+1,n}$ to $u_{k+1,\frac{n}{2}+1,1}$ for $k = 1, 2, \cdots, r-1$ by an edge.

Note that in the graph G, the vertices $|V(G)| = \left(\dfrac{n^2}{2} + n\right) r$ and $|E(G)| = rn^2 + r - 1$.

We define the labeling

$$f : V(G) \to \{0, 1, 2, \cdots, 2[rn^2 + r - 1] - 1\}$$

as follows:

$$f(u_{k,i,j}) = n(j-1) + n - 2i + j - 1 + n(n+1)(k-1), \ \ j = 1, 3, 5, \cdots, n-1 \text{ and } i = \frac{j+1}{2}, \frac{j+3}{2}, \cdots, \frac{n}{2}, \ \ k = 1, 2, \cdots, r.$$

$$f(u_{k,i,j}) = n(j-2) + n - 2i + j + 1 + [n(n-1) + 2](k-1), \ \ j = 2, 4, 6, \cdots, n \text{ and } i = \frac{j}{2}, \frac{j+2}{2}, \cdots, \frac{n}{2}, \ \ k = 1, 2, \cdots, r.$$

$$f(u_{k,i,j}) = nj + 2in - n^2 - 3n + j + [n(n-1) + 2](k-1), \ \ j = 1, 3, 5, \cdots, n-1 \text{ and } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n, \ k = 1, 2, \cdots, r.$$

$$f(u_{k,i,j}) = nj + 2in - n^2 - 2n + j - 2 + n(n+1)(k-1), \ \ j = 2, 4, 6, \cdots, n \text{ and } i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n, \ k = 1, 2, \cdots, r.$$

The induced edge labels are

$$f^*(u_{k,i,j} \ u_{k,i,j+1}) = 2nj + 2n^2k - 4i - 2n^2 + 2j + 2k - 1, \ \ j = 1, 2, 3, \cdots, n-1 \text{ and } i = 1, 2, \cdots, \frac{n}{2}, \ \ k = 1, 2, \cdots, r.$$

$$f^*(u_{k,i,j} \ u_{k,i,j+1}) = 2nj + 2n^2k + 4in - 4n^2 - 4n + 2j + 2k - 3, \ \ j = 1, 2, 3, \cdots, n-1 \text{ and } i = \frac{n}{2} + 1, \cdots, n, \ \ k = 1, 2, \cdots, r.$$

$$f^*(u_{k,i,j} \ u_{k,i+1,j}) = 2nj + 2n^2k + 2in - 3n^2 - n - 2i + 2j + 2k - 3, \ \ j = 1, 2, \cdots, n \text{ and } i = \frac{n}{2}, \ \ k = 1, 2, \cdots, r.$$

$$f^*(u_{k,i,j} \ u_{k,i+1,j-1}) = 2nj + 2n^2k + 4in - 4n^2 - 4n + 2j + 2k - 5, \ \ j = 2, 3, \cdots, n-1 \text{ and } i = \frac{n}{2} + 1, \cdots, n-1, \ \ k = 1, 2, \cdots, r.$$

$$f^*(u_{k,i,j} \ u_{k,i+1,j+1}) = 2nj + 2n^2k - 4i - 2n^2 + 2j + 2k - 3, \ \ j = 1, 2, 3, \cdots, n-2 \text{ and } i = 1, 2, \cdots, \frac{n-2}{2}, \ \ k = 1, 2, \cdots, r.$$

$$f^*(u_{k,i,n} \ u_{k+1,i,1}) = 4in + 2n^2k - 2n^2 - 4n + 2k - 1, \ \ j = 2, 3, \cdots, n-1 \text{ and } i = \frac{n}{2} + 1, \ k = 1, 2, \cdots, r-1.$$

In the view of above defined labeling pattern, path union of finite copies of the plus graph Pl_n is odd harmonious, where $n \equiv 0(mod\ 2)$, $n \neq 2$. □

The odd harmonious labeling of path union of 3 copies of the plus graph Pl_4 is shown in Figure 2.

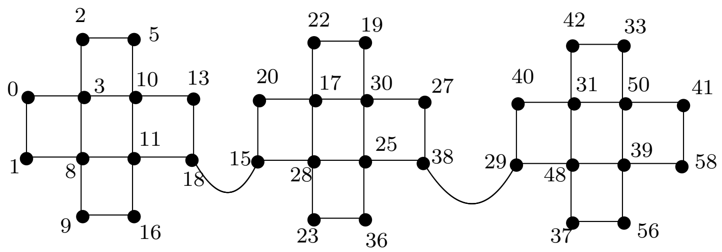


FIGURE 2: *Odd harmonious lableing path union of 3 copies of* Pl_4

THEOREM 2.3. *Open star of plus graph* $S(t.Pl_n)$ *is odd harmonious, where* $n \equiv 0(mod\ 2)$, $n \neq 2$ *and if* t *is odd.*

PROOF. Let $G = S(t.Pl_n)$ be a graph obtained by replacing all vertices of $K_{1,t}$ except the apex vertex of $K_{1,t}$ by the graph Pl_n. Let u_0 be the apex vertex of $K_{1,t}$. That is it is the central vertex of the graph G. Let $u_{k,i,j}$ ($\forall i = 1, 2, \cdots, \frac{n}{2}$ and $\forall j = 1, 2, \cdots, 2i$ and $\forall i = \frac{n}{2} + 1, \cdots, n$ and $\forall j = 1, 2, \cdots, 2n - 2i + 2$) be the vertices of the k^{th}copy of Pl_n in G, $\forall k = 1, 2, \cdots, t$, where the vertices of k^{th} copy of Pl_n is $p = \frac{n^2}{2} + n$ and the edges of the k^{th} copy of Pl_n is $q = n^2$. Join the vertices $u_{k,\frac{n}{2}+1,1}$ with the vertex u_0 by an edge, $\forall k = 1, 2, \cdots, t$.

In graph G, $|V(G)| = t(\frac{n^2}{2} + n) + 1$ and $|E(G)| = t(n^2 + 1)$.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2t(n^2 + 1) - 1\}$ as follows:

$f(u_0) = 0$.

$f(u_{k,i,j}) = jnt + (j + 3)t - 2it - 2 - 4(k - 1)$, $i = \frac{j+1}{2}, \frac{j+3}{2}, \cdots, \frac{n}{2}$, $j = 1, 3, \cdots, n - 1$, $k = 1, 2, \cdots, t$.

$f(u_{k,i,j}) = (j - 1)nt + jt - 2it + 1 + 2(k - 1)$, $i = \frac{j}{2}, \frac{j+2}{2}, \cdots, \frac{n}{2}$, $j = 2, 4, \cdots, n$, $k = 1, 2, \cdots, t$.

$f(u_{k,i,j}) = jnt + 2int - n^2t - 3nt + (j - 1)t + 1 + 2(k - 1)$, $i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n$, $j = 1, 3, \cdots, n - [2i - n - 1]$, $k = 1, 2, \cdots, t$.

$f(u_{k,i,j}) = jnt + 2int - n^2t - 2nt + (j + 2)t - 2 - 4(k - 1)$, $i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n$, $j = 2, 4, \cdots, n - [2i - n - 2]$, $k = 1, 2, \cdots, t$.

The induced edge labels are

$f^*(u_{k,i,j}\ u_{k,i,j+1}) = 2jnt + 2jt + 4t - 4it - 1 - 2(k - 1)$, $i = 1, 2, \cdots, \frac{n}{2}$, $j = 1, 2, \cdots, n - 1$, $k = 1, 2, \cdots, t$.

$f^*(u_{k,i,j}\ u_{k,i,j+1}) = 2jnt + 4int - 2n^2t - 4nt + 2jt + 2t - 1 - 2(k - 1)$, $i = \frac{n}{2} + 1, \frac{n}{2} + 2, \cdots, n$, $j = 1, 2, \cdots, n - 1$, $k = 1, 2, \cdots, t$.

$f^*(u_{k,i,j}\ u_{k,i+1,j+1}) = 2jnt + 2jt + 2t - 4it - 1 - 2(k-1),\ i = 1, 2, \cdots, \dfrac{n-2}{2},\ j = 1, 2, \cdots, n-2,\ k = 1, 2, \cdots, t.$

$f^*(u_{k,i,j}\ u_{k,i+1,j-1}) = 2jnt + 4int - 2n^2t - 4nt + 2jt - 1 - 2(k-1),\ i = \dfrac{n}{2} + 1, \cdots, n-1,\ j = 2, 3, \cdots, n-1,\ k = 1, 2, \cdots, t.$

$f^*(u_{k,i,j}\ u_{k,i+1,j}) = 2jnt + 2int - n^2t - nt + 2jt - 2it + 2t - 1 - 2(k-1),\ i = \dfrac{n}{2},\ j = 1, 2, \cdots, n,\ k = 1, 2, \cdots, t.$

$f^*(u_{k,\frac{n}{2}+1,1}\ u_0) = 1 + 2(k-1),\ k = 1, 2, \cdots, t.$

In the view of above defined labeling pattern, open star of plus graph $S(t.Pl_n)$ is odd harmonious. □

The odd harmonious labeling of $S(5.Pl_4)$ is shown in Figure 3.

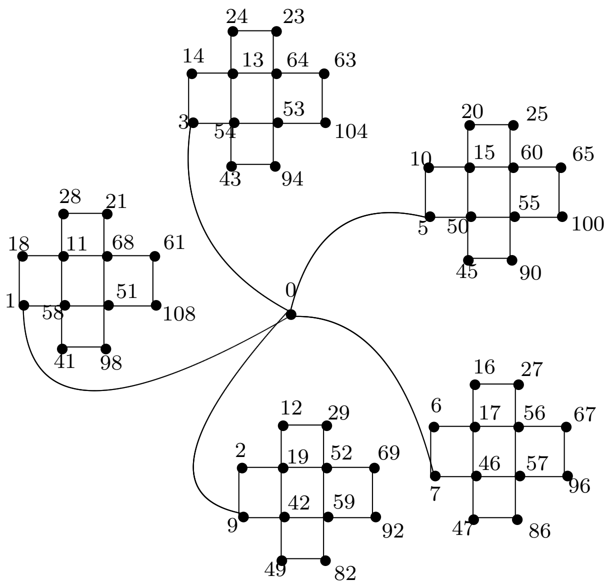


FIGURE 3: *Odd harmonious labeling of* $S(5.Pl_4)$.

THEOREM 2.4. *A graph obtained by joining* C_m, $m \equiv 0(mod\ 4)$ *and a plus graph* Pl_n,$n \equiv 0(mod\ 2)$, $n \neq 2$ *with a path of arbitrary length t is odd harmonious.*

PROOF. Let G be a graph obtained by joining a cycle C_m, $m \equiv 0(mod\ 4)$ and a plus graph Pl_n with P_t, a path of length t on $t+1$ vertices. Let $u_1, u_2, \cdots, u_m$ be the vertices of the cycle C_m and $v_1 = u_m, v_2, \cdots, v_{t+1}$ be the vertices of path P_t of t length and $w_{i,j}$, ($\forall i = 1, 2, \cdots, \dfrac{n}{2}$ and $\forall j = 1, 2, \cdots, 2i$ and $\forall i = \dfrac{n}{2} + 1, \cdots, n$ and $\forall j = 1, 2, \cdots, 2n-2i+2$) be the vertices of the plus graph Pl_n, where $v_{t+1} = w_{\frac{n}{2}+1,1}$ if t is odd and $v_{t+1} = w_{\frac{n}{2},1}$ if t is even.

In G , $|V(G)| = m + (t-1) + \dfrac{n^2}{2} + n$ and $|E(G)| = m + t + n^2$.

We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2(m+t+n^2)-1\}$ as follows:
$f(u_i) = i - 1,\ 1 \leqslant i \leqslant \dfrac{m}{2}$.

$$f(u_i) = \begin{cases} i+1 & \text{if } i \text{ is odd} \\ i-1 & \text{if } i \text{ is even,} \end{cases} \quad \frac{m}{2} + 1 \leqslant i \leqslant m.$$

$$f(v_i) = \begin{cases} m+i-2 & \text{if } i \text{ is odd} \\ m+i & \text{if } i \text{ is even,} \end{cases} \quad 1 \leqslant i \leqslant t+1.$$

If t is even

$f(w_{i,j}) = nj - 2i + j - 1 + m + t - 1,\ j = 1, 3, \cdots, n-1$ and $i = \dfrac{j+1}{2}, \dfrac{j+3}{2}, \cdots, \dfrac{n}{2}$.

$f(w_{i,j}) = n(j-2) + n - 2i + j + 1 + m + t + 1,\ j = 2, 4, \cdots, n$ and $i = \dfrac{j}{2}, \dfrac{j+2}{2}, \cdots, \dfrac{n}{2}$.

$f(w_{i,j}) = nj + 2in - n^2 - 2n + j - 2 + m + t - 1,\ i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, \cdots, n,\ j = 2, 4, \cdots, n - [2i - n - 2]$.

$f(w_{i,j}) = nj + 2in - n^2 - 3n + j + m + t + 1,\ i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, \cdots, n,\ j = 1, 3, \cdots, n - [2i - n - 1]$.

If t is odd

$f(w_{i,j}) = nj - 2i + j - 1 + m + t,\ j = 1, 3, \cdots, n-1$ and $i = \dfrac{j+1}{2}, \dfrac{j+3}{2}, \cdots, \dfrac{n}{2}$.

$f(w_{i,j}) = n(j-2) + n - 2i + j + 1 + m + t,\ j = 2, 4, \cdots, n$ and $i = \dfrac{j}{2}, \dfrac{j+2}{2}, \cdots, \dfrac{n}{2}$.

$f(w_{i,j}) = nj + 2in - n^2 - 2n + j - 2 + m + t,\ i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, \cdots, n,\ j = 2, 4, \cdots, n - [2i - n - 2]$.

$f(w_{i,j}) = nj + 2in - n^2 - 3n + j + m + t,\ i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, \cdots, n,\ j = 1, 3, \cdots, n - [2i - n - 1]$.

The induced edge labels are

$f^*(u_i\ u_{i+1}) = 2i - 1,\ \ 1 \leqslant i \leqslant \dfrac{m}{2} - 1$.

$f^*(u_{\frac{m}{2}} u_{\frac{m}{2}+1}) = m + 1$.
$f^*(u_i\ u_{i+1}) = 2i + 1,\ \ \dfrac{m}{2} + 1 \leqslant i \leqslant m - 1$.
$f^*(u_m\ u_1) = m - 1$.

$f^*(v_i\ v_{i+1}) = 2m + 2i - 1,\ \ 1 \leqslant i \leqslant t$.

$f^*(v_t\ w_{\frac{n}{2},1}) = 2m + 2t - 1$, if t is even.

$f^*(v_t\ w_{\frac{n}{2}+1,1}) = 2m + 2t - 1$, if t is odd.

$f^*(w_{i,j}\ w_{i,j+1}) = 2nj - 4i + 2j + 2m + 2t + 1,\ i = 1, 2, \cdots, \frac{n}{2}$ and $j = 1, 2, 3, \cdots, n-1$.

$f^*(w_{i,j}\ w_{i,j+1}) = 2nj + 4in + 2j - 2n^2 - 4n + 2m + 2t - 1,\ i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, \cdots, n$ and $j = 1, 2, 3, \cdots, n-1$.

$f^*(w_{i,j}\ w_{i+1,j+1}) = 2nj - 4i + 2j + 2m + 2t - 1,\ i = 1, 2, \cdots, \dfrac{n-2}{2}$ and $j = 1, 2, 3, \cdots, n-2$.
$f^*(w_{i,j}\ w_{i+1,j}) = 2nj + 2in - 2i + 2j - n^2 - n + 2m + 2t - 1,\ i = \dfrac{n}{2}$ and $j = 1, 2, 3, \cdots, n$.
$f^*(w_{i,j}\ w_{i+1,j-1}) = 2nj + 4in + 2j - 2n^2 - 4n + 2m + 2t - 3,\ i = \dfrac{n}{2} + 1, \dfrac{n}{2} + 2, \cdots, n-1$ and $j = 2, 3, \cdots, n-1$.
In the view of above defined labeling pattern, the cycle C_m, $m \equiv 0 (mod\ 4)$ and a plus graph Pl_n with a path of arbitrary length t is odd harmonious. □

The odd harmonious labeling of cycle C_8 and a plus graph Pl_4 with a path of arbitrary length $t = 5$ (odd) is shown in Figure 4.

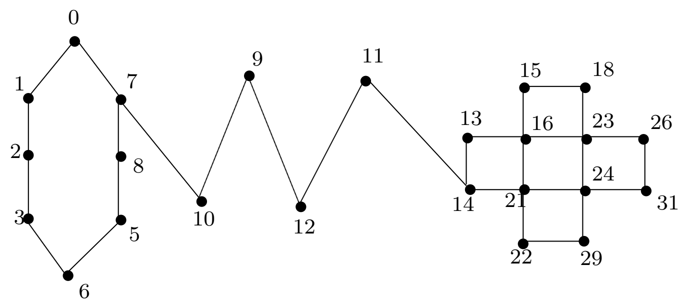


FIGURE 4: *Odd harmonious lableing of cycle C_8 and a plus graph pl_4* with a path of length $t = 5$.

The odd harmonious labeling of cycle C_8 and a plus graph Pl_6 with a path of arbitrary length $t = 6$ (even) is shown in Figure 5.

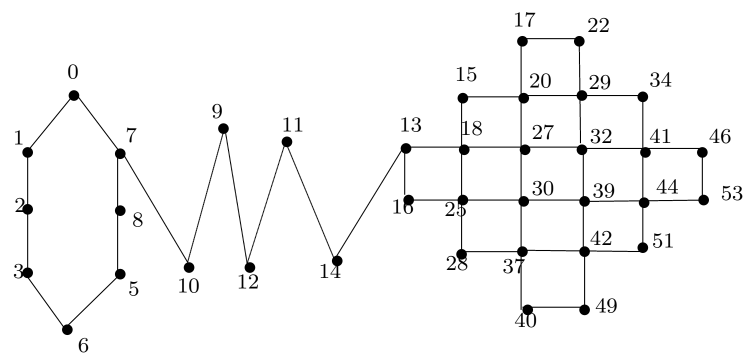


FIGURE 5: *Odd harmonious lableing of cycle C_8 and a plus graph pl_4 with a path of length $t = 6$.*

Theorem 2.5. *One point union for path of plus graph $P_n^t(t.n.Pl_m)$, $m \equiv 0(mod\ 2)$, $m \neq 2$ is odd harmonious if t is odd.*

Proof. Let $G = P_n^t(t.n.Pl_m)$ be a graph obtained by replacing all vertices of P_n^t except the central vertex by the graph Pl_m. That means G is the graph obtained by replacing all vertices of $K_{1,t}$ except the apex vertex by the path union of n copies of the graph Pl_m. Let u_0 be the central vertex for the graph G with t branches. Let $u_{s,k,i,j}$ $(\forall i = 1, 2, \cdots, \frac{m}{2}, \ \forall j = 1, 2, \cdots, 2i, \ \forall i = \frac{m}{2} + 1, \cdots, m, \ \forall j = 1, 2, \cdots, 2m - 2i + 2)$ be the vertices of k^{th} copy of path union of n copies of Pl_m lies in the s^{th} branch of the graph G, $\forall s = 1, 2, \cdots, t$. Join the vertices of $u_{s,1,\frac{m}{2}+1,1}$ with u_0 by an edge. Also join the vertices $u_{s,k,\frac{m}{2}+1,m}$ to $u_{s,k+1,\frac{m}{2}+1,1}$ for $k = 1, 2, \cdots, n-1$, $s = 1, 2, \cdots, t$ by an edge.

This graph G with $|V(G)| = tn(\frac{m^2}{2} + m) + 1$ and $|E(G)| = tn(m^2 + 1)$.
We define the labeling $f : V(G) \to \{0, 1, 2, \cdots, 2tn(m^2 + 1) - 1\}$ as follows:
$f(u_0) = 0$.
$f(u_{s,k,i,j}) = jmt + (j + 3)t - 2it - 2 - 4(s - 1) + (k - 1)tm(m + 1)$, $j = 1, 3, \cdots, m - 1$, $i = \frac{j+1}{2}, \frac{j+3}{2}, \cdots, \frac{m}{2}$, $k = 1, 2, \cdots, n$, $s = 1, 2, \cdots, t$.

$f(u_{s,k,i,j}) = (j-1)mt+jt-2it+1+2(s-1)+(k-1)[tm(m+1)-2t(m-1)]$, $j = 2, 4, \cdots, m$, $i = \frac{j}{2}, \frac{j+2}{2}, \cdots, \frac{m}{2}$, $k = 1, 2, \cdots, n$, $s = 1, 2, \cdots, t$.

$f(u_{s,k,i,j}) = jmt+2imt-m^2t-3mt+(j-1)t+1+2(s-1)+(k-1)[tm(m+1)-2t(m-1)]$,

$i = \frac{m}{2} + 1, \frac{m}{2} + 2, \cdots, m$, $j = 1, 3, \cdots, m - [2i - m - 1]$, $k = 1, 2, \cdots, n$, $s = 1, 2, \cdots, t$.

$f(u_{s,k,i,j}) = jmt + 2imt - m^2t - 2mt + (j + 2)t - 2 - 4(s - 1) + (k - 1)tm(m + 1)$, $i = \frac{m}{2} + 1, \frac{m}{2} + 2, \cdots, m$, $j = 2, 4, \cdots, m - [2i - m - 2]$, $k = 1, 2, \cdots, n$, $s = 1, 2, \cdots, t$.
The induced edge labels are
$f^*(u_{s,k,i,j}\ u_{s,k,i,j+1}) = 2jmt + (j + 3)t + jt - 4it + t - 2(s - 1) + (k - 1)[2tm(m + 1) - 2t(m - 1)] - 1$, $i = 1, 2, \cdots, \frac{m}{2}$, $j = 1, 2, \cdots, m - 1$, $s = 1, 2, \cdots, t$, $k = 1, 2, \cdots, n$.

$f^*(u_{s,k,i,j}\ u_{s,k,i,j+1}) = 2jmt + 4imt - 2m^2t - 4mt + 2jt + 2t - 1 - 2(s - 1) + (k - 1)[2tm(m + 1) - 2t(m - 1)] - 1$, $i = \frac{m}{2} + 1, \cdots, m$, $j = 1, 2, \cdots, m - 1$, $s = 1, 2, \cdots, t$, $k = 1, 2, \cdots, n$.

$f^*(u_{s,k,i,j}\ u_{s,k,i+1,j+1}) = 2jmt+2jt+2t-4it-1-2(s-1)+(k-1)[2tm(m+1)-2t(m-1)]-1$,

$i = 1, 2, \cdots, \frac{m-2}{2}$, $j = 1, 2, \cdots, m - 2$, $s = 1, 2, \cdots, t$, $k = 1, 2, \cdots, n$.

$f^*(u_{s,k,i,j}\ u_{s,k,i+1,j}) = 2jmt + 2imt - m^2t - mt + 2jt - 2it + 2t - 1 - 2(s - 1) + (k - 1)[2tm(m + 1) - 2t(m - 1)] - 1$, $i = \frac{m}{2}$, $j = 1, 2, \cdots, m$, $s = 1, 2, \cdots, t$, $k = 1, 2, \cdots, n$.

$f^*(u_{s,k,i,j}\ u_{s,k,i+1,j-1}) = 2jmt + 4imt - 2m^2t - 4mt + 2jt - 1 - 2(s - 1) + (k - 1)[2tm(m +$

$1)-2t(m-1)]-1,\ i=\dfrac{m}{2}+1,\cdots,m-1,\ j=1,2,\cdots,m-1,\ s=1,2,\cdots,t,\ k=1,2,\cdots,n.$

$f^*(u_0\ u_{s,1,\frac{m}{2}+1,1})=1+2(s-1),\ s=1,2,\cdots,t.$

$f^*(u_{s,k,\frac{m}{2}+1,m}\ u_{s,k+1,\frac{m}{2}+1,1})=tm(m+1)+2t-1-2(s-1)+tm(m+1)(2k-1)-2kt(m-1),$

$s=1,2,\cdots,t,\ k=1,2,\cdots,n.$

In the view of above defined labeling pattern, $P_n^t(t.n.Pl_m)$ is odd harmonious.

□

The odd harmonious labeling of $P_3^3(3.3.Pl_4)$ *is shown in Figure 6.*

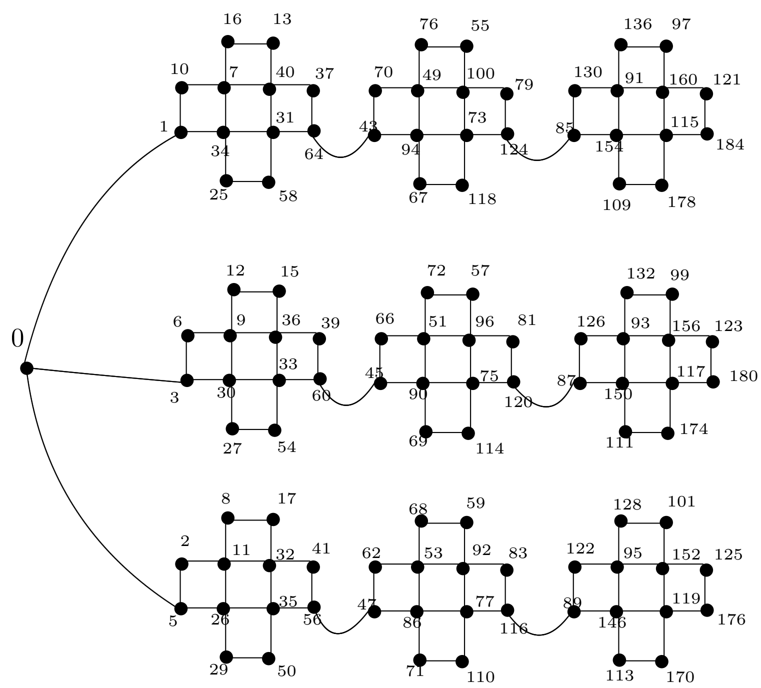


FIGURE 6: *Odd harmonious labeling of* $P_3^3(3.3.Pl_4)$

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