BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
7(2017), 527-538
DOI:10.7251/BIMVI1703527M

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

GENERALIZED FUZZY RIGHT h-IDEALS OF HEMIRINGS REDEFINED BY FUZZY SUMS AND FUZZY PRODUCTS

G. Mohanraj and E. Prabu

Abstract. In this paper, we redefine the concepts of (λ, μ)-fuzzy right [left] ideals of hemirings by using the notions of fuzzy sum and fuzzy product. And also the notions of (λ, μ)-fuzzy right [left] h-ideals of hemirings are redefined by fuzzy sum, fuzzy closure and fuzzy product. Further, using the notions of fuzzy h-sum and fuzzy h-product, we characterize (λ, μ)-fuzzy right [left] h-ideals. In particular, we investigate (λ, μ)-fuzzy right [left] h-ideals by using fuzzy h-sum and fuzzy h-intrinsic product.

1. Introduction

The notion of a fuzzy set, which was firstly proposed by Lotfi Aliasker Zadeh [**20**], provides a natural framework for generalizing many of the concepts of mathematics. Rosenfeld [**14**] combined fuzzy sets and groups in a fruitful way by defining fuzzy groups. Since then, the fuzzy set theory have been applied to many branch of mathematics and engineering. The notion of h-ideals of hemirings was initiated by Torre [**6**]. Young Bae Jun [**5**] introduced the concepts of fuzzy h-ideals of hemirings. In [**2**], Bhakat and Das introduced the concept of redefined fuzzy subrings and ideals. M. Shabir and W.A. Dudek explored the concepts of fuzzy h-ideals and (α, β)-fuzzy ideals of hemirings in [**15, 16**]. Yao initiated the notions of (λ, μ)-fuzzy groups [**18**] and (λ, μ)-fuzzy subrings [**19**]. Moharaj et al. introduced (λ, μ)-fuzzy ideals and (λ, μ)-fuzzy prime ideals of semirings [**4, 3**] and generalized fuzzy weakly interior ideals of ordered semigroups [**13**]. Recently, G.Mohanraj and E.Prabu investigated the concepts of redefined generalized L-h-bi-ideals of

2010 *Mathematics Subject Classification.* 03E72, 08A72, 16Y99, 16D25.

Key words and phrases. Fuzzy h-sum, Fuzzy h-product, Fuzzy h-intrinsic product, Fuzzy h-closure, (λ, μ)-fuzzy right h-ideals.

hemirings [**11**] and characterizations of generalized fuzzy h-bi-ideals of hemirings [**12**].

In this paper, the concepts of (λ, μ)-fuzzy right [left] h-ideals are redefined by the notions of fuzzy sum, fuzzy closure and fuzzy product. Moreover, using the notions of fuzzy h-sum and fuzzy h-product, the concept of (λ, μ)-fuzzy right [left] h-ideals is redefined. In particular using the notions of fuzzy h-sum and fuzzy h-intrinsic product, the concept of (λ, μ)-fuzzy right [left] h-ideals are characterized.

2. Preliminaries

An algebraic structure $(R, +, \cdot)$ in which $(R, +)$ and $(R, \cdot)$ are semigroups that satisfy both distributive laws is called a hemiring if " $+$ " is commutative and there is an absorbing element $0 \in R$ such that $0+x = x = x+0$ and $0 \cdot x = 0 = x \cdot 0$ for all $x \in R$.

A subset $A \neq \emptyset$ of a hemiring R which is closed under addition is called a right [left] ideal of R if $ar \in A$ $[ra \in A]$ for all $a \in A,\ r \in R$. A right [left] ideal A of R is called a right [left] h-ideal of R if $p, q \in A$ and $y + p + z = q + z$ imply $y \in A$ for $y, z \in R$. Further the h-closure of a subset A of R denoted by $\overline{A}$ is defined as $\overline{A} = \{y \in R | y + p + z = q + z \text{ for some } p, q \in A, z \in R\}$.

An element "1" is called an unity of a hemiring R, if $1 \cdot a = a \cdot 1 = a$ for all $a \in R$. Recall that a mapping $f : R \to [0, 1]$ is called a fuzzy set of a hemiring R. The level set of a fuzzy set f denoted by f_t of R is defined as $f_t = \{a \in R | f(a) \geqslant t\}$ for all $t \in [0, 1]$. The fuzzy set "1" is defined as $1 = 1(x) = \chi_R(x)$ for every $x \in R$. The intersection of fuzzy sets f and g of R, denoted by $f \cap g$ and is defined as $(f \cap g)(x) = min\{f(x), g(x)\}$ for all $x \in R$. For fuzzy sets f and g of R, we denote $f \subseteq g$ if $g(x) \geqslant f(x)$ for all $x \in R$.

Definition 2.1. [**16**] *The fuzzy sum $f + g$ and the fuzzy h-sum $f +_h g$ of the fuzzy sets f and g are defined respectively as follows*

$$(f + g)(x) = \bigvee_{x=y+z} [f(y) \wedge g(z)]$$

for $x, y, z \in R$.

$$(f +_h g)(x) = \bigvee_{x+a_1+b_1+z=a_2+b_2+z} [f(a_1) \wedge f(a_2) \wedge g(b_1) \wedge g(b_2)]$$

for $x, a_1, b_1, a_2, b_2, z \in R$.

Definition 2.2. [**12**] *(i)The fuzzy product $f \cdot g$ of the fuzzy sets f and g of R is defined as*

$$(f \cdot g)(x) = \begin{cases} \bigvee_{x=yz} [f(y) \wedge g(z)] & \text{if } x = yz \\ 0 & \text{if } x \text{ cannot be expressible as } x = yz. \end{cases}$$

(ii)The fuzzy h-product $f \circ g$ of the fuzzy sets f and g of R is defined as

$$(f \circ g)(x) = \begin{cases} \bigvee_{x+a_1b_1+z=a_2b_2+z} [f(a_1) \wedge f(a_2) \wedge g(b_1) \wedge g(b_2)] \\ 0 \text{ if } x \text{ cannot be expressible as } x + a_1b_1 + z = a_2b_2 + z. \end{cases}$$

DEFINITION 2.3. [**16**] *For a fuzzy sets f and g of R, the h-intrinsic product $f \odot g$ of f and g of R is defined as*

$$(f \odot g)(x) = \begin{cases} \bigvee\limits_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \left[\bigwedge\limits_{i=1}^{m} [f(a_i) \wedge g(b_i)] \wedge \bigwedge\limits_{j=1}^{n} [f(a'_j) \wedge g(b'_j)] \right] \\ 0 \text{ if } x \text{ cannot be expressible as } x + \sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a'_j b'_j + z. \end{cases}$$

DEFINITION 2.4. [**12**] *The fuzzy h-closure $\overline{f}$ of fuzzy set f of R is defined by*

$$(\overline{f})(x) = \bigvee_{x+a+z=b+z} [f(a) \wedge f(b)]$$

for $x, a, b, z \in R$.

DEFINITION 2.5. [**13**] *Let f and g be two fuzzy sets of R. We write $f \subseteq_\mu^\lambda g$, if $g(x) \vee \lambda \geqslant f(x) \wedge \mu$ for all $x \in R$ and $0 \leqslant \lambda < \mu \leqslant 1$.*

REMARK 2.6. *If $\lambda = 0$ and $\mu = 1$, then $f \subseteq_\mu^\lambda g$ coincides with $f \subseteq g$.*

THEOREM 2.7. [**15**] *The fuzzy set f of R is an $(\in, \in \vee q_k)$-fuzzy right [left] ideal of R if and only if*
1. $f(x+y) \geqslant f(x) \wedge f(y) \wedge \frac{1-k}{2}$,
2. $f(xy) \geqslant f(x) \wedge \frac{1-k}{2}$ $[f(xy) \geqslant f(y) \wedge \frac{1-k}{2}]$ for all $x, y \in R$.

3. Redefined $(\lambda, \mu)-$fuzzy right [left] h-ideals

Throughout this paper, we represents R as a hemiring and $0 \leqslant \lambda < \mu \leqslant 1$.

DEFINITION 3.1. *The fuzzy set f is called a $(\lambda, \mu)-$fuzzy right [left] ideal of R if for all $x, y \in R$*
F1a. $f(x+y) \vee \lambda \geqslant f(x) \wedge f(y) \wedge \mu$,
F1b. $f(xy) \vee \lambda \geqslant f(x) \wedge \mu$ $[f(xy) \vee \lambda \geqslant f(y) \wedge \mu]$.

REMARK 3.2. *1. By taking $\lambda = 0$ and $\mu = 1$ in Definition 3.1, f is known as a fuzzy right [left] ideal of a hemiring R.*
2. By taking $\lambda = 0$ and $\mu = \frac{1-k}{2}$ in Definition 3.1, f coincides with $(\in, \in \vee q_k)$-fuzzy right [left] ideal of a hemiring R.
3. Therefore $(\lambda, \mu)-$fuzzy right [left] ideal of R is a generalization of fuzzy right [left] ideal and $(\in, \in \vee q_k)$-fuzzy right [left] ideal of R.

Now, using fuzzy sum and fuzzy product, $(\lambda, \mu)-$fuzzy right [left] ideals of R are redefined.

THEOREM 3.3. *The fuzzy set f of R is a $(\lambda, \mu)-$fuzzy right [left] ideal of R if and only if*
F2a) $f + f \subseteq_\mu^\lambda f$
F2b) $f \cdot 1 \subseteq_\mu^\lambda f$ $[1 \cdot f \subseteq_\mu^\lambda f]$

PROOF. Let f be a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R and for every $x \in R$. If $x = y+z$, then $f(x)\vee\lambda = f(y+z)\vee\lambda \geqslant f(y)\wedge f(z)\wedge\mu$. Thus,

$$\begin{aligned} f(x)\vee\lambda &\geqslant \bigvee_{x=y+z} f(y)\wedge f(z)\wedge\mu \\ &= \left[\bigvee_{x=y+z} f(y)\wedge f(z)\right]\wedge\mu \\ &= (f+f)(x)\wedge\mu \end{aligned}$$

Therefore $f+f \subseteq^{\lambda}_{\mu} f$.
If x cannot be expressible as $x = yz$, then $0 = (f\cdot 1)(x) = (f\cdot 1)(x)\wedge\mu \leqslant f(x)\vee\lambda$. If $x = yz$, then $f(x)\vee\lambda = f(yz)\vee\lambda \geqslant f(y)\wedge\mu$. Thus,

$$\begin{aligned} f(x)\vee\lambda &\geqslant \bigvee_{x=yz} f(y)\wedge\mu \\ &= \left[\bigvee_{x=yz} f(y)\wedge 1(z)\right]\wedge\mu \\ &= (f\cdot 1)(x)\wedge\mu \end{aligned}$$

Therefore $f\cdot 1 \subseteq^{\lambda}_{\mu} f$. Similarly, we prove that $f+f \subseteq^{\lambda}_{\mu} f$ and $1\cdot f \subseteq^{\lambda}_{\mu} f$ if f is a $(\lambda,\mu)-$fuzzy left ideal of R.

Conversely, for $x,y \in R$, $f(x+y)\vee\lambda \geqslant (f+f)(x+y)\wedge\mu \geqslant f(x)\wedge f(y)\wedge\mu$ and $f(xy)\vee\lambda \geqslant (f\cdot 1)(xy)\wedge\mu \geqslant f(x)\wedge\mu$. Therefore f is a $(\lambda,\mu)-$fuzzy right ideal of R. Similarly, we prove that f is a $(\lambda,\mu)-$fuzzy left ideal of R if $f+f \subseteq^{\lambda}_{\mu} f$ and $1\cdot f \subseteq^{\lambda}_{\mu} f$. □

Now, the concept of $(\lambda,\mu)-$fuzzy right [left] h-ideals of R are redefined by using the notions of fuzzy sum, fuzzy product and fuzzy closure.

DEFINITION 3.4. *The (λ,μ)-fuzzy right [left] ideal f of R is called a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R if*
$(F1c)$ for any $x,a,b,z \in R$, $x+a+z = b+z$ implies $f(x)\vee\lambda \geqslant f(a)\wedge f(b)\wedge\mu$.

THEOREM 3.5. *The $(\lambda,\mu)-$fuzzy right [left] ideal f of R is a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R if and only if*
F2c) $\overline{f} \subseteq^{\lambda}_{\mu} f$

PROOF. Let f be a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R and for all $x \in R$. By Theorem 3.3, $f+f \subseteq^{\lambda}_{\mu} f$ and $f\cdot 1 \subseteq^{\lambda}_{\mu} f$ $[1\cdot f \subseteq^{\lambda}_{\mu} f]$
If $x+a+z = b+z$, then $f(x)\vee\lambda \geqslant f(a)\wedge f(b)\wedge\mu$. Thus,

$$\begin{aligned} f(x)\vee\lambda &\geqslant \bigvee_{x+a+z=b+z} f(a)\wedge f(b)\wedge\mu \\ &= \left[\bigvee_{x+a+z=b+z} f(a)\wedge f(b)\right]\wedge\mu \\ &= \overline{f}(x)\wedge\mu \end{aligned}$$

Therefore $\overline{f} \subseteq_{\mu}^{\lambda} f$.

Conversely, by Theorem 3.3, f is a $(\lambda, \mu)-$fuzzy right [left] ideal of R. Now $x + a + z = b + z$ implies $f(x) \vee \lambda \geqslant \overline{f}(x) \wedge \mu \geqslant f(a) \wedge f(b) \wedge \mu$. Therefore f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. □

THEOREM 3.6. *A fuzzy set f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R if and only if*
F2a) $f + f \subseteq_{\mu}^{\lambda} f$
F2b) $f \cdot 1 \subseteq_{\mu}^{\lambda} f$ *[*$1 \cdot f \subseteq_{\mu}^{\lambda} f$*]*
F2c) $\overline{f} \subseteq_{\mu}^{\lambda} f$

PROOF. By Theorem 3.3 and 3.5, the proof follows. □

Now, we redefine $(\lambda, \mu)-$fuzzy right [left] h-ideal of R by using fuzzy h-sum and fuzzy product.

LEMMA 3.7. [**12**] *If f is a fuzzy set of R such that $f + f \subseteq_{\mu}^{\lambda} f$ and $\overline{f} \subseteq_{\mu}^{\lambda} f$, then $f +_h f \subseteq_{\mu}^{\lambda} f$.*

LEMMA 3.8. [**12**] *If f is a fuzzy set of R such that $f(0) \vee \lambda \geqslant f(x) \wedge \mu$ for all $x \in R$ and $f +_h f \subseteq_{\mu}^{\lambda} f$, then $\overline{f} \subseteq_{\mu}^{\lambda} f$ and $f + f \subseteq_{\mu}^{\lambda} f$.*

LEMMA 3.9. *If $f \cdot 1 \subseteq_{\mu}^{\lambda} f$ or $1 \cdot f \subseteq_{\mu}^{\lambda} f$ for a fuzzy set f of R, then $f + f \subseteq_{\mu}^{\lambda} f +_h f$.*

PROOF. Let f be a fuzzy set of R such that $f \cdot 1 \subseteq_{\mu}^{\lambda} f$ or $1 \cdot f \subseteq_{\mu}^{\lambda} f$. Then for all $x \in R$, $f(0) \vee \lambda \geqslant (f \cdot 1)(0) \wedge \mu \geqslant f(x) \wedge 1(0) \wedge \mu = f(x) \wedge \mu$. Similarly, $1 \cdot f \subseteq_{\mu}^{\lambda} f$ implies $f(0) \vee \lambda \geqslant f(x) \wedge \mu$ for all $x \in R$. Now,

$$\begin{aligned}
(f +_h f)(x) \vee \lambda &= \left(\bigvee_{x+a_1+b_1+z=a_2+b_2+z} f(a_1) \wedge f(a_2) \wedge f(b_1) \wedge f(b_2) \right) \vee \lambda \\
&\geqslant \left(\bigvee_{x+0+0+z=a+b+z} f(0) \wedge f(a) \wedge f(0) \wedge f(b) \right) \vee \lambda \\
&= \bigvee_{x=a+b} (f(0) \vee \lambda) \wedge (f(a) \vee \lambda) \wedge (f(0) \vee \lambda) \wedge (f(b) \vee \lambda) \\
&\geqslant \bigvee_{x=a+b} f(a) \wedge \mu \wedge f(b) \wedge \mu \\
&= \left(\bigvee_{x=a+b} f(a) \wedge f(b) \right) \wedge \mu \\
&= (f + f)(x) \wedge \mu
\end{aligned}$$

Therefore $f + f \subseteq_{\mu}^{\lambda} f +_h f$. □

REMARK 3.10. *Let f be a $(\lambda, \mu)-$fuzzy left or right ideal of R, then by Lemma 3.9, $f + f \subseteq_{\mu}^{\lambda} f +_h f$.*

LEMMA 3.11. *If* $f +_h f \subseteq^{\lambda}_{\mu} f$ *and* $f \cdot 1 \subseteq^{\lambda}_{\mu} f$ *or* $f +_h f \subseteq^{\lambda}_{\mu} f$ *and* $1 \cdot f \subseteq^{\lambda}_{\mu} f$ *for a fuzzy set* f *of* R, *then* $\overline{f} \subseteq^{\lambda}_{\mu} f$.

PROOF. Let f be a fuzzy set of R such that $f +_h f \subseteq^{\lambda}_{\mu} f$ and $f \cdot 1 \subseteq^{\lambda}_{\mu} f$ or $f +_h f \subseteq^{\lambda}_{\mu} f$ and $1 \cdot f \subseteq^{\lambda}_{\mu} f$. Then for all $x \in R$, $f(0) \vee \lambda \geqslant f(x \cdot 0) \wedge \mu \geqslant f(x) \wedge \mu$. Now,

$$\begin{aligned}
(f(x) \vee \lambda) \vee \lambda &\geqslant \Big((f +_h f)(x) \wedge \mu\Big) \vee \lambda \\
&\geqslant \left[\left(\bigvee_{x+a+0+z=0+b+z} f(a) \wedge f(0) \wedge f(0) \wedge f(b)\right) \vee \lambda\right] \wedge \mu \\
&= \left(\bigvee_{x+a+z=b+z} (f(a) \vee \lambda) \wedge (f(0) \vee \lambda) \wedge (f(0) \vee \lambda)\right. \\
&\qquad \left. \wedge\, (f(b) \vee \lambda)\right) \wedge \mu \\
&\geqslant \left(\bigvee_{x+a+z=b+z} (f(a) \wedge \mu) \wedge f(a) \wedge (f(b) \wedge \mu) \wedge f(b)\right) \wedge \mu \\
&\geqslant \left(\bigvee_{x+a+z=b+z} f(a) \wedge f(b) \wedge \mu\right) \wedge \mu \\
&= \left(\bigvee_{x+a+z=b+z} f(a) \wedge f(b)\right) \wedge \mu \\
&= \overline{f}(x) \wedge \mu
\end{aligned}$$

Therefore $\overline{f} \subseteq^{\lambda}_{\mu} f$. □

THEOREM 3.12. *The fuzzy set* f *of* R *is a* $(\lambda, \mu)-$*fuzzy right [left] h-ideal of* R *if and only if*
F3a) $f +_h f \subseteq^{\lambda}_{\mu} f$
F2b) $f \cdot 1 \subseteq^{\lambda}_{\mu} f$ *[*$1 \cdot f \subseteq^{\lambda}_{\mu} f$*]*

PROOF. Now, f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R then by Theorem 3.6 and Lemma 3.7, we have $F3a$ and $F2b$.
Conversely, by Lemma 3.9, $f + f \subseteq^{\lambda}_{\mu} f +_h f$ implies $f + f \subseteq^{\lambda}_{\mu} f$ and by Lemma 3.11, we have $\overline{f} \subseteq^{\lambda}_{\mu} f$. Then by Theorem 3.6, f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. □

4. Fuzzy h-product and $(\lambda, \mu)-$fuzzy right [left] h-ideals

In this section, using fuzzy h-sum and fuzzy h-product, the concept of (λ, μ)-fuzzy right [left] h-ideal of R is characterized.

LEMMA 4.1. *If f is a (λ,μ)-fuzzy right [left] ideals of R, then $f\cdot 1 \subseteq_\mu^\lambda f\circ 1$ [$1\cdot f \subseteq_\mu^\lambda 1\circ f$].*

PROOF. By Theorem 3.6, we have $f\cdot 1 \subseteq_\mu^\lambda f$ [$1\cdot f \subseteq_\mu^\lambda f$]. For each $x\in R$,

$$f(0)\vee\lambda \geqslant (f\cdot 1)(x0)\wedge\mu \geqslant f(x)\wedge\mu.$$

If f is a (λ,μ)-fuzzy left ideal of R, then for each $x\in R$,

$$f(0)\vee\lambda \geqslant (1\cdot f)(0x)\wedge\mu \geqslant f(x)\wedge\mu.$$

Now $x=ab$ implies $x+0b+z=ab+z$. Then,

$$\begin{aligned}(f\circ 1)(x)\vee\lambda &\geqslant [f(0)\wedge f(a)\wedge 1(b)\wedge 1(b)]\vee\lambda\\ &\geqslant (f(a)\wedge\mu)\wedge(f(a)\vee\lambda)\wedge(1(b)\wedge\mu)\wedge(1(b)\vee\lambda)\\ &\geqslant f(a)\wedge 1(b)\wedge\mu\end{aligned}$$

Thus,

$$\begin{aligned}(f\circ 1)(x)\vee\lambda &\geqslant \bigvee_{x=ab} f(a)\wedge 1(b)\wedge\mu\\ &= \left[\bigvee_{x=ab} f(a)\wedge 1(b)\right]\wedge\mu\\ &= (f\cdot 1)(x)\wedge\mu\end{aligned}$$

Therefore $f\cdot 1\subseteq_\mu^\lambda f\circ 1$. Similarly we prove that $1\cdot f\subseteq_\mu^\lambda 1\circ f$ if f is a (λ,μ)-fuzzy left ideal of R. □

THEOREM 4.2. *A fuzzy set f of R is a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R if and only if*
F2a) $f+f\subseteq_\mu^\lambda f$
F4b) $f\circ 1\subseteq_\mu^\lambda f$ [$1\circ f\subseteq_\mu^\lambda f$]
F2c) $\overline{f}\subseteq_\mu^\lambda f$

PROOF. Let f be a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R. By Theorem 3.6, we have $F2a$, $F2b$ and $F2c$. If $x+a_1b_1+z=a_2b_2+z$, then by $F2c$ and $F2b$,

$$\begin{aligned}f(x)\vee\lambda &\geqslant (\overline{f}(x)\wedge\mu)\vee\lambda\\ &\geqslant (f(a_1b_1)\wedge f(a_2b_2)\wedge\mu)\vee\lambda\\ &= (f(a_1b_1)\vee\lambda)\wedge(f(a_2b_2)\vee\lambda)\wedge\mu\\ &\geqslant ((f\cdot 1)(a_1b_1)\wedge\mu)\wedge((f\cdot 1)(a_2b_2)\wedge\mu)\wedge\mu\\ &\geqslant f(a_1)\wedge f(a_2)\wedge 1(b_1)\wedge 1(b_2)\wedge\mu\end{aligned}$$

Now $x + a_1b_1 + z = a_2b_2 + z$ implies $f(x) \vee \lambda \geqslant f(a_1) \wedge f(a_2) \wedge 1(b_1) \wedge 1(b_2) \wedge \mu$. Thus,

$$\begin{aligned} f(x) \vee \lambda &\geqslant \bigvee_{x+a_1b_1+z=a_2b_2+z} f(a_1) \wedge f(a_2) \wedge 1(b_1) \wedge 1(b_2) \wedge \mu \\ &= \left[\bigvee_{x+a_1b_1+z=a_2b_2+z} f(a_1) \wedge f(a_2) \wedge 1(b_1) \wedge 1(b_2)\right] \wedge \mu \\ &= (f \circ 1)(x) \wedge \mu \end{aligned}$$

Therefore $f \circ 1 \subseteq_\mu^\lambda f$. Similarly, we prove that $f + f \subseteq_\mu^\lambda f$, $1 \circ f \subseteq_\mu^\lambda f$ and $\overline{f} \subseteq_\mu^\lambda f$ if f is a $(\lambda, \mu)-$fuzzy left h-ideal of R.

Conversely, by Lemma 4.1, $f \cdot 1 \subseteq_\mu^\lambda f \circ 1 \subseteq_\mu^\lambda f$ implies $f \cdot 1 \subseteq_\mu^\lambda f$ and $1 \cdot f \subseteq_\mu^\lambda 1 \circ f \subseteq_\mu^\lambda f$ implies $1 \cdot f \subseteq_\mu^\lambda f$. Therefore by Theorem 3.6, f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. □

THEOREM 4.3. *If R has an unity, then the fuzzy set f is a (λ, μ)-fuzzy right [left] h-ideal of R if and only if*
F2a) $f + f \subseteq_\mu^\lambda f$
F4b) $f \circ 1 \subseteq_\mu^\lambda f$ [$1 \circ f \subseteq_\mu^\lambda f$]

PROOF. Let f be a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. By Theorem 4.2, we have $f + f \subseteq_\mu^\lambda f$, $f \circ 1 \subseteq_\mu^\lambda f$ [$1 \circ f \subseteq_\mu^\lambda f$].

Conversely, $x + a + z = b + z$ implies $x + a \cdot 1 + z = b \cdot 1 + z$. Now,

$$\begin{aligned} f(x) \vee \lambda &\geqslant (f \circ 1)(x) \wedge \mu \\ &= \left[\bigvee_{x+a_1b_1+z=a_2b_2+z} f(a_1) \wedge f(a_2) \wedge 1(b_1) \wedge 1(b_2)\right] \wedge \mu \\ &\geqslant \left[\bigvee_{x+a\cdot 1+z=b\cdot 1+z} f(a) \wedge f(b) \wedge 1 \wedge 1\right] \wedge \mu \\ &= \left[\bigvee_{x+a+z=b+z} f(a) \wedge f(b)\right] \wedge \mu \\ &= \overline{f}(x) \wedge \mu \end{aligned}$$

Thus $\overline{f} \subseteq_\mu^\lambda f$ and by Theorem 4.2, f is a (λ, μ)-fuzzy right [left] h-ideal of R. □

THEOREM 4.4. *A fuzzy set f of R is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R if and only if*
F3a) $f +_h f \subseteq_\mu^\lambda f$
F4b) $f \circ 1 \subseteq_\mu^\lambda f$ [$1 \circ f \subseteq_\mu^\lambda f$]

PROOF. Let f be a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. Then by Theorem 4.2 and by Lemma 3.7, we have F3a and F4b.

Conversely, by Lemma 4.1, 3.9, 3.11 and by Theorem 4.2, f is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. □

5. Fuzzy h-intrinsic product and (λ, μ)-fuzzy right [left] h-ideal

In this section, using the notion of fuzzy h-sum and fuzzy h-intrinsic product, we establish a necessary and sufficient condition for a fuzzy set to be a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R.

Lemma 5.1. *If f and g are fuzzy sets of R, then $f \circ g \subseteq f \odot g$.*

Proof. Let f and g be fuzzy sets of R. Now, for all $x \in R$,

$$\begin{aligned}
(f \odot g)(x) &= \bigvee_{x+\sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'} + z} \left[\left(\bigwedge_{i=1}^{m} (f(a_i) \wedge g(b_i)) \right) \wedge \right. \\
&\qquad \left. \left(\bigwedge_{j=1}^{n} (f(a_j^{'}) \wedge g(b_j^{'})) \right) \right] \\
&\geqslant \bigvee_{x+a_1 b_1 + z = a_1^{'} b_1^{'} + z} f(a_1) \wedge g(b_1) \wedge f(a_1^{'}) \wedge g(b_1^{'}) \\
&= \bigvee_{x+a_1 b_1 + z = a_1^{'} b_1^{'} + z} f(a_1) \wedge f(a_1^{'}) \wedge g(b_1) \wedge g(b_1^{'}) \\
&= (f \circ g)(x)
\end{aligned}$$

Therefore $f \circ g \subseteq f \odot g$. □

Theorem 5.2. *The fuzzy set f of R is a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R if and only if*
F2a) $f + f \subseteq_{\mu}^{\lambda} f$
F5b) $f \odot 1 \subseteq_{\mu}^{\lambda} f$ [$1 \odot f \subseteq_{\mu}^{\lambda} f$]
F2c) $\overline{f} \subseteq_{\mu}^{\lambda} f$

Proof. Let f be a $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. By Theorem 3.6, we have $F2a$, $F2c$ and $F2b$. Now, $x + \sum\limits_{i=1}^{m} a_i b_i + z = \sum\limits_{j=1}^{n} a_j^{'} b_j^{'} + z$, then by F2a, F2b and F2c,

$$\begin{aligned}
f(x) \vee \lambda &= (f(x) \vee \lambda) \vee \lambda \\
&\geqslant (\overline{f}(x) \wedge \mu) \vee \lambda \\
&\geqslant (f(\sum_{i=1}^{m} a_i b_i) \wedge f(\sum_{j=1}^{n} a_j^{'} b_j^{'}) \wedge \mu) \vee \lambda \\
&= (f(\sum_{i=1}^{m} a_i b_i) \vee \lambda) \wedge (f(\sum_{j=1}^{n} a_j^{'} b_j^{'}) \vee \lambda) \wedge \mu
\end{aligned}$$

$$\geqslant ((f+f+...+f)(\sum_{i=1}^{m} a_i b_i) \wedge \mu)$$
$$\wedge ((f+f+...+f)(\sum_{j=1}^{n} a_j^{'} b_j^{'}) \wedge \mu) \wedge \mu$$
$$\geqslant \left(\bigwedge_{i=1}^{m}(f(a_i b_i)\right) \wedge \left(\bigwedge_{j=1}^{n} f(a_j^{'} b_j^{'})\right) \wedge \mu \tag{5.1}$$

Now by Equation 5.1,

$$\begin{aligned} f(x) \vee \lambda &= (f(x) \vee \lambda) \vee \lambda \\ &\geqslant \left(\bigwedge_{i=1}^{m}(f(a_i b_i) \vee \lambda\right) \wedge \left(\bigwedge_{j=1}^{n}(f(a_j^{'} b_j^{'}) \vee \lambda\right) \wedge \mu \end{aligned}$$

Then by $F2b$,

$$\begin{aligned} f(x) \vee \lambda &\geqslant \left(\bigwedge_{i=1}^{m}(f \cdot 1)(a_i b_i) \wedge \mu\right) \wedge \left(\bigwedge_{j=1}^{n}(f \cdot 1)(a_j^{'} b_j^{'}) \wedge \mu\right) \wedge \mu \\ &= \left(\bigwedge_{i=1}^{m} f(a_i) \wedge 1(b_i)\right) \wedge \left(\bigwedge_{j=1}^{n} f(a_j^{'}) \wedge 1(b_j^{'})\right) \wedge \mu \end{aligned}$$

Thus,

$$\begin{aligned} f(x) \vee \lambda &\geqslant \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j^{'} b_j^{'} + z} \left(\bigwedge_{i=1}^{m}(f(a_i) \wedge 1(b_i))\right) \wedge \\ &\qquad \left(\bigwedge_{j=1}^{n}(f(a_j^{'}) \wedge 1(b_j^{'}))\right) \wedge \mu \\ &= \left[\bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j^{'} b_j^{'} + z} \left(\bigwedge_{i=1}^{m}(f(a_i) \wedge 1(b_i))\right) \wedge \right. \\ &\qquad \left. \left(\bigwedge_{j=1}^{n}(f(a_j^{'}) \wedge 1(b_j^{'}))\right)\right] \wedge \mu \\ &= (f \odot 1)(x) \wedge \mu \end{aligned}$$

Therefore $f \odot 1 \subseteq_{\mu}^{\lambda} f$. Similarly we prove that $f + f \subseteq_{\mu}^{\lambda} f$, $1 \odot f \subseteq_{\mu}^{\lambda} f$ and $\overline{f} \subseteq_{\mu}^{\lambda} f$ if f is (λ, μ)-fuzzy left h-ideal of R.

Conversely, by Lemma 5.1, $f \circ 1 \subseteq f \odot 1 \subseteq_{\mu}^{\lambda} f$ implies $f \circ 1 \subseteq_{\mu}^{\lambda} f$. Similarly $1 \circ f \subseteq 1 \odot f \subseteq_{\mu}^{\lambda} f$ implies $1 \circ f \subseteq_{\mu}^{\lambda} f$. By Theorem 4.2, f is $(\lambda, \mu)-$fuzzy right [left] h-ideal of R. □

THEOREM 5.3. *If R has an unity, then the fuzzy set f is a (λ,μ)-fuzzy right [left] h-ideal of R if and only if*
F2a) $f+f\subseteq_\mu^\lambda f$
F5b) $f\odot 1\subseteq_\mu^\lambda f$ [$1\odot f\subseteq_\mu^\lambda f$]

PROOF. Let f be a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R. By Theorem 5.2, we have $f+f\subseteq_\mu^\lambda f$ and $f\odot 1\subseteq_\mu^\lambda f$ [$1\odot f\subseteq_\mu^\lambda f$].

Conversely, $x+\sum\limits_{i=1}^{m}a_ib_i+z=\sum\limits_{j=1}^{n}a_j^{'}b_j^{'}+z$ implies

$$\begin{aligned} f(x)\vee\lambda &\geqslant (f\odot 1)(x)\wedge\mu \\ &\geqslant \left[\bigvee_{x+a\cdot 1+z=b\cdot 1+z} f(a)\wedge f(b)\right]\wedge\mu \\ &= \left[\bigvee_{x+a+z=b+z} f(a)\wedge f(b)\right]\wedge\mu \\ &= \overline{f}(x)\wedge\mu \end{aligned}$$

Thus $\overline{f}\subseteq_\mu^\lambda f$ and by Theorem 5.2, f is a (λ,μ)-fuzzy right [left] h-ideal of R. □

THEOREM 5.4. *A fuzzy set f of R is a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R if and only if*
F3a) $f+_h f\subseteq_\mu^\lambda f$
F5b) $f\odot 1\subseteq_\mu^\lambda f$ [$1\odot f\subseteq_\mu^\lambda f$]

PROOF. Now, f is a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R and by Theorem 5.2 and by Lemma 3.7, we have $F3a$ and $F5b$.

Conversely, by Lemma 5.1 and by Theorem 4.4, f is a $(\lambda,\mu)-$fuzzy right [left] h-ideal of R. □

THEOREM 5.5. *If f and g are (λ,μ)-fuzzy right h-ideal and (λ,μ)-fuzzy left h-ideal of R respectively, then $f\odot g\subseteq_\mu^\lambda f\cap g$.*

PROOF. Let f and g be (λ,μ)-fuzzy right h-ideal and (λ,μ)-fuzzy left h-ideal of R respectively.

$$\begin{aligned} \text{Then, } f\odot g &\subseteq f\odot 1 \\ &\subseteq_\mu^\lambda f \end{aligned}$$

$$\begin{aligned} f\odot g &\subseteq 1\odot g \\ &\subseteq_\mu^\lambda g \end{aligned}$$

Therefore $f\odot g\subseteq_\mu^\lambda f\cap g$. □

COROLLARY 5.6. *If f and g are (λ,μ)-fuzzy h-ideals of a hemiring R, then $f\odot g\subseteq_\mu^\lambda f\cap g$.*

PROOF. Straightforward □

Corollary 5.7. [**16**] *If f and g are fuzzy h-ideals of a hemiring R, then $f \odot g \subseteq f \cap g$.*

Proof. By taking $\lambda = 0$ and $\mu = 1$ in Corollary 5.6, we get the result. $\square$

References

[1] S. K. Bhakat and P. Das. $(\in, \in \vee q)$-fuzzy group, *Fuzzy Sets and Systems*, **80**(1996), 359-368.

[2] S. K. Bhakat and P. Das. Fuzzy subrings and ideals redefined, *Fuzzy Sets and Systems*, **81**(1996), 383-393.

[3] P. Dheena and G. Mohanraj. (λ, μ)-fuzzy ideals in Semirings, *Advances in Fuzzy Mathematics*, **6**(2)(2011), 183-192.

[4] P. Dheena and G. Mohanraj. On (λ, μ)-fuzzy Prime ideals of semirings, *J. Fuzzy Math.*, **20**(4)(2012), 889-898.

[5] Y. B. Jun, M. A. *Öztürk* and S.Z. Song. On fuzzy h-ideals in hemirings, *Information Science*, **162**(3-4)(2004), 211-226.

[6] D. R. La Torre. On h-ideals and k-ideals in hemirings, *Publ. Math. Debrecen*, **12**(1965), 219-226.

[7] X. Ma and J. Zhan. Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings, *Information Sciences*, **179**(9)(2009), 1249-1268.

[8] G. Mohanraj and E. Prabu. Weakly fuzzy prime ideals of hemiring, *International Conference on Mathematical Sciences (ICMS2014), Sathyabama University, Chennai, Conference, Tamil Nadu, July 17-19 2014*, (Vol. 1, pp. 480-483), Elsevier, ISBN-978-93-5107-261-4

[9] G. Mohanraj and E. Prabu. Generalized Fuzzy Right h-Ideals of Hemirings, *Inter. J. Fuzzy Math. Archive*, **7**(2)(2015), 147-155.

[10] G. Mohanraj and E. Prabu. Redefined T-fuzzy right h-ideals of hemirings, *Global Journal of Pure and Applied Mathematics*, Print ISSN 0973-1768, Volume 12, Number 4, (2016), pp.35-38. Link: http://ucepktmaths.wixsite.com/amase-2016/gjpam

[11] G. Mohanraj and E. Prabu. Redefined generalized L-h-bi-ideals of hemirings, (to appear in *Fuzzy Information and Engineering*).

[12] G. Mohanraj and E. Prabu. Characterizations of generalized fuzzy h-bi-ideals of hemirings by fuzzy sums and products. (to appear in *Journal of Fuzzy Mathematics*).

[13] G. Mohanraj, D. Krishnasamy and R. Hema. On generalized fuzzy weakly interior ideals of ordered semigroups, *Ann. Fuzzy Math. Infor.*, **8**(5)(2014), 803-814.

[14] A. Rosenfeld. Fuzzy groups, *J. Math. Anal. Appl.*, **35**(3)(1971), 512-517.

[15] Y. B. Jun, W. A. Dudek, M. Shabir and M. S. Kang. General types of (α, β)-fuzzy ideals of hemirings, *Honam Math. J.*, **32**(3)(2010), 413-439.

[16] W.A. Dudek, M. Shabir and R. Anjum. Characterizations of hemirings by their h-ideals, *Comput. Math. Appl.*, **59**(9)(2010), 3167-3179.

[17] X. Ma, Y. Yin and J. Zhan. Characterizations of h-intra- and h-quasi-hemiregular hemirings, *Comput. Math. Appl.*, **63**(4)(2012), 783-793.

[18] B. Yao. (λ, μ)-fuzzy normal subgroups and (λ, μ)-fuzzy quotients subgroup, *J. Fuzzy Math.*, **13**(3)(2005), 695-705.

[19] B. Yao. (λ, μ)-fuzzy subrings and (λ, μ)-fuzzy ideals, *J. Fuzzy Math.*, **15**(4)(2007), 981-987.

[20] L. A. Zadeh. Fuzzy sets, *Information and Control*, **8**(3)(1965), 338-353.

[21] J. Zhan and W. A. Dudek. Fuzzy h-ideals of hemirings, *Information Science*, **177**(3)(2007), 876-886.

Received by editors 02.02.2017; Revised version 24.03.2017; Available online 24.04.2017.

Department of Mathematics, Annamalai University, Annamalainagar-608002, India
E-mail address: gmohanraaj@gmail.com

Department of Mathematics, Annamalai University, Annamalainagar-608002, India
E-mail address: prabumathsrep@gmail.com