MILDLY g-ω-CLOSED SETS

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ABSTRACT. In this paper, another generalized class of τ called mildly g-ω-closed sets is introduced and the notion of mildly g-ω-open sets in topological spaces is introduced and studied. The relationships of mildly g-ω-closed sets with various other sets are investigated.

1. Introduction

The first step of generalizing closed sets (briefly, g-closed sets) was done by Levine in 1970 [7]. He defined a subset S of a topological space (X, τ) to be g-closed if its closure is contained in every open superset of S. As the weak form of g-closed sets, the notion of weakly g-closed sets was introduced and studied by Sundaram and Nagaveni [12]. Sundaram and Pushpalatha [13] introduced and studied the notion of strongly g-closed sets, which are weaker than the class of closed sets and stronger than the class of g-closed sets. Park and Park [10] introduced and studied the notion of mildly g-closed sets, which is properly placed between the class of strongly g-closed sets and the class of weakly g-closed sets. Moreover, the relations with other notions directly or indirectly connected with the notion of g-closed sets were investigated by them.

The notions of ω-open and ω-closed sets in topological spaces introduced by Hdeib [5] have been studied in recent years by a good number of researchers like Noiri et al [9], Al-Omari and Noorani [1, 2] and Khalid Y. Al-Zoubi [6].

The main aim of this paper is to introduce another generalized class called mildly g-ω-open sets in topological spaces. Moreover, this generalized class of τ generalize g-ω-open sets and mildly g-ω-open sets. The relationships of mildly g-ω-closed sets with various other sets are discussed.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $(\mathbb{R} - \mathbb{Q})$, $(\mathbb{R} - \mathbb{Q})_-$ and $(\mathbb{R} - \mathbb{Q})_+$) denotes the set of real numbers (resp. the set of natural numbers, the set of rational numbers, the set of irrational numbers, the set of negative irrational numbers and the set of positive irrational numbers).

In this paper, (X, τ) represents a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a subset G of a topological space (X, τ) will be denoted by $cl(G)$ and $int(G)$, respectively.

DEFINITION 2.1. *A subset G of a topological space (X, τ) is said to be*

(1) *g-closed* [**7**] *if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is open in X;*
(2) *g-open* [**7**] *if $X \backslash G$ is g-closed;*
(3) *weakly g-closed* [**12**] *if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is open in X;*
(4) *strongly g-closed* [**13**] *if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is g-open in X.*

DEFINITION 2.2. *A subset G of a topological space (X, τ) is said to be*

(1) *regular closed* [**11**] *if $G = cl(int(G))$.*
(2) *preopen* [**8**] *if $G \subseteq int(cl(G))$.*
(3) *preclosed* [**8**] *if $X \backslash G$ is preopen (or) $cl(int(G)) \subseteq G$.*

REMARK 2.1. [**11**] *In any topological space, every regular closed set is closed but not conversely.*

DEFINITION 2.3. [**15**] *In a topological space (X, τ), a point p in X is called a condensation point of a subset H if for each open set U containing p, $U \cap H$ is uncountable.*

DEFINITION 2.4. [**5**] *A subset H of a topological space (X, τ) is called ω-closed if it contains all its condensation points.*

The complement of an ω-closed set is called ω-open.

It is well known that a subset W of a topological space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

LEMMA 2.1. [**5**] *Let H be a subset of a topological space (X, τ). Then*

(1) *H is ω-closed in X if and only if $H = cl_\omega(H)$.*
(2) *$cl_\omega(X \backslash H) = X \backslash int_\omega(H)$.*
(3) *$cl_\omega(H)$ is ω-closed in X.*
(4) *$x \in cl_\omega(H)$ if and only if $H \cap G \neq \phi$ for each ω-open set G containing x.*
(5) *$cl_\omega(H) \subset cl(H)$.*
(6) *$int(H) \subset int_\omega(H)$.*

LEMMA 2.2. [**6**] *If A is an ω-open subset of a topological space (X, τ), then $A - C$ is ω-open for every countable subset C of X.*

DEFINITION 2.5. [**14**] *A subset G of a topological space (X, τ) is said to be g-ω-closed if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is ω-open in (X, τ).*

DEFINITION 2.6. *A subset H of a topological space (X, τ) is called generalized ω-closed (briefly, $g\omega$-closed) [**6**] if $cl_\omega(H) \subseteq U$ whenever $H \subseteq U$ and U is open in X.*

The complement of a $g\omega$-closed set is called $g\omega$-open.

REMARK 2.2. [**2**] *For a subset of a topological space (X, τ), the following relations hold:*

$$\begin{array}{ccc} \textbf{\textit{open}} & \longrightarrow & \textbf{\textit{g-open}} \\ \downarrow & & \downarrow \\ \textbf{\textit{ω-open}} & \longrightarrow & \textbf{\textit{$g\omega$-open}} \end{array}$$

None of the above implications is reversible.

DEFINITION 2.7. [**3**] *A subset H of a topological space (X, τ) is said to be ω-closed if $cl(G) \subseteq H$ for every countable subset $G \subseteq H$.*

The complement of an ω-closed set is called ω-open.

The family of all ω-open subsets of a topological space (X, τ) forms a topology for it.

3. ω_{71}-closed and ω_{82}-closed sets

In section 2 of this paper, we have taken up two different topological subsets in same name called ω-closed. To understand them in right sense in section 3 alone, we call one subset as ω_{71}-closed and other as ω_{82}-closed keeping view of the years in which they were introduced.

Recall that $(\mathbb{R}, \tau_{coc})$ is a co-countable topological space in which a subset A is open $\Leftrightarrow$ its complement A^c is countable.

PROPOSITION 3.1. *In a topological space (X, τ), every closed set is ω_{71}-closed.*

PROOF. Let A be closed in (X, τ). Let B be any countable subset such that $B \subseteq A$. Then $cl(B) \subseteq cl(A) = A$ and thus A is ω_{71}-closed. $\qquad \square$

REMARK 3.1. *The converse of Proposition 3.1 is not true in general as shown in the following example.*

EXAMPLE 3.1. *In $(\mathbb{R}, \tau_{coc})$, consider $A = \mathbb{R} - \mathbb{Q} =$ the set of irrational numbers. If B is any countable subset such that $B \subseteq A = \mathbb{R} - \mathbb{Q}$, then $cl(B) = B \subseteq A$. Thus A is ω_{71}-closed. But A is not closed in $(\mathbb{R}, \tau_{coc})$ since $A = \mathbb{R} - \mathbb{Q}$ is not countable.*

PROPOSITION 3.2. *In a topological space (X, τ), every closed set is ω_{82}-closed.*

PROOF. Let A be closed in (X, τ). Then $cl_\omega(A) \subseteq cl(A) = A$ and so $cl_\omega(A) = A$. Therefore A is ω_{82}-closed. $\qquad \square$

REMARK 3.2. *The converse of Proposition 3.2 is not true in general as shown in the following example.*

EXAMPLE 3.2. *In $\mathbb{R}$ with the usual topology τ_u, $A = \mathbb{Q}$ is ω_{82}-closed since $\mathbb{Q}$ is countable. But $A = \mathbb{Q}$ is not closed for $cl(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{Q} = A$.*

REMARK 3.3. *In a topological space* (X, τ), *ω_{71}-closed and ω_{82}-closed sets are independent concepts.*

EXAMPLE 3.3. *In $\mathbb{R}$ with the usual topology τ_u, $A = \mathbb{Q} =$ the set of rational numbers is ω_{82}-closed since A is countable. But A is not ω_{71}-closed for $\mathbb{Q} \subseteq \mathbb{Q}$ and $A = \mathbb{Q}$ is countable whereas $cl(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{Q} = A$.*

EXAMPLE 3.4. *In $(\mathbb{R}, \tau_{coc})$, $A = \mathbb{R} - \mathbb{Q}$ is ω_{71}-closed by Example 3.1. But A is not ω_{82}-closed for any $x \in \mathbb{Q}$ is a condensation point for $A = \mathbb{R} - \mathbb{Q}$ and $x \notin \mathbb{R} - \mathbb{Q}$.*

COROLLARY 3.1. *In a topological space (X, τ),*

(1) *every open set is ω_{82}-open but not conversely.*
(2) *every open set is ω_{71}-open but not conversely* [**4**]*.*

REMARK 3.4. *For every open set G, we have $int_\omega(G) = int(G)$ but the converse is not true as shown in the following example.*

EXAMPLE 3.5. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$, for $G = \mathbb{N}$, $int_\omega(G) = int(G)$ but G is not open.*

THEOREM 3.1. *Let G be a subset of a topological space (X, τ). The following are equivalent.*

(1) *G is open,*
(2) *G is ω_{82}-open (ω_{71}-open) and $int_\omega(G) = int(G)$.*

PROOF. $(1) \Rightarrow (2)$: Proof follows from Corollary 3.1 and Remark 3.4.
$(2) \Rightarrow (1)$: Let G be an ω_{82}-open (ω_{71}-open) and $int_\omega(G) = int(G)$. Then $G = int_\omega(G) = int(G)$. Thus, G is open. $\square$

4. Other generalized classes of τ

DEFINITION 4.1. *In a topological space (X, τ), a subset G of X is said to be*

(1) *weakly g-ω-closed if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is ω-open in X;*
(2) *mildly g-ω-closed if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is $g\omega$-open in X;*
(3) *strongly g-ω-closed if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is $g\omega$-open in X.*

EXAMPLE 4.1. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$,*

(1) *For $G = \mathbb{R} - \mathbb{Q}$, if H is any ω-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $cl(int(G)) = cl(\phi) = \phi \subseteq H$ and hence G is weakly g-ω-closed in X.*
(2) *$K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being ω-open whereas $cl(int(\mathbb{Q})) = cl(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{Q}$ which implies $K = \mathbb{Q}$ is not weakly g-ω-closed in X.*
(3) *For $G = \mathbb{R} - \mathbb{Q}$, if H is any $g\omega$-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $cl(int(G)) = cl(\phi) = \phi \subseteq H$ and hence G is mildly g-ω-closed in X.*
(4) *$K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being $g\omega$-open whereas $cl(int(\mathbb{Q})) = cl(\mathbb{Q}) = \mathbb{R} \not\subseteq \mathbb{Q}$ which implies $K = \mathbb{Q}$ is not mildly g-ω-closed in X.*

(5) *For $G = \mathbb{R} - \mathbb{Q}$, if H is any $g\omega$-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $cl(G) = cl(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q} = G \subseteq H$ and hence G is strongly g-ω-closed in X.*

(6) *$K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being $g\omega$-open whereas $cl(K) = cl(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{Q} = K$ which implies $K = \mathbb{Q}$ is not strongly g-ω-closed in X.*

DEFINITION 4.2. *A subset G in a topological space (X, τ) is said to be mildly g-ω-open (resp. strongly g-ω-open, weakly g-ω-open) if $X \backslash G$ is mildly g-ω-closed (resp. strongly g-ω-closed, weakly g-ω-closed).*

THEOREM 4.1. *In a topological space (X, τ), a subset G of X is mildly g-ω-closed $\Leftrightarrow cl(int(G)) \subseteq G$.*

PROOF. $\Rightarrow$ If $cl(int(G)) \nsubseteq G$, there exists $x \in X$ such that $x \in cl(int(G)) - G$. Then $x \in cl(int(G)) - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open being ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open. But $cl(int(G)) \nsubseteq X - \{x\}$ since $x \in cl(int(G))$. This implies that G is not mildly g-ω-closed which proves the necessary part.

$\Leftarrow$ Let $cl(int(G)) \subseteq G$ and H be any $g\omega$-open subset such that $G \subseteq H$. Then $cl(int(G)) \subseteq G \subseteq H$. This implies that G is mildly g-ω-closed which proves the sufficiency part. $\square$

THEOREM 4.2. *In a topological space (X, τ), a subset G of X is weakly g-ω-closed $\Leftrightarrow cl(int(G)) \subseteq G$.*

PROOF. $\Rightarrow$ If $cl(int(G)) \nsubseteq G$, there exists $x \in X$ such that $x \in cl(int(G)) - G$. Then $x \in cl(int(G)) - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. But $cl(int(G)) \nsubseteq X - \{x\}$ since $x \in cl(int(G))$. This implies that G is not weakly g-ω-closed which proves the necessary part.

$\Leftarrow$ Let $cl(int(G)) \subseteq G$ and H be any ω-open set such that $G \subseteq H$. Then $cl(int(G)) \subseteq G \subseteq H$. This implies that G is weakly g-ω-closed which proves the sufficiency part. $\square$

THEOREM 4.3. *In a topological space (X, τ), a subset G of X is mildly g-ω-closed $\Leftrightarrow G$ is weakly g-ω-closed.*

PROOF. Proof follows by Theorem 4.1 and Theorem 4.2. $\square$

THEOREM 4.4. *In a topological space (X, τ), a subset G of X is strongly g-ω-closed $\Leftrightarrow cl(G) \subseteq G$.*

PROOF. $\Rightarrow$ If $cl(G) \nsubseteq G$, there exists $x \in X$ such that $x \in cl(G) - G$. Then $x \in cl(G) - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open being ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is $g\omega$-open. But $cl(G) \nsubseteq X - \{x\}$ since $x \in cl(G)$. This implies that G is not strongly g-ω-closed which proves the necessary part.

$\Leftarrow$ Let $cl(G) \subseteq G$ and H be any $g\omega$-open set such that $G \subseteq H$. Then $cl(G) \subseteq G \subseteq H$. This implies that G is strongly g-ω-closed which proves the sufficient part. $\square$

THEOREM 4.5. *In a topological space (X, τ), a subset G of X is g-ω-closed $\Leftrightarrow cl(G) \subseteq G$.*

PROOF. $\Rightarrow$ If $cl(G) \nsubseteq G$, there exists $x \in X$ such that $x \in cl(G) - G$. Then $x \in cl(G) - G \subseteq X - G$ and so $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. Thus $G \subseteq X - \{x\}$ where $X - \{x\}$ is ω-open. But $cl(G) \nsubseteq X - \{x\}$ since $x \in cl(G)$. This implies that G is not g-ω-closed which proves the necessary part.

$\Leftarrow$ Let $cl(G) \subseteq G$ and H be any ω-open set such that $G \subseteq H$. Then $cl(G) \subseteq G \subseteq H$. This implies that G is g-ω-closed which proves the sufficient part. $\square$

THEOREM 4.6. *In a topological space (X, τ), a subset G of X is strongly g-ω-closed $\Leftrightarrow G$ is g-ω-closed.*

PROOF. Proof follows from Theorem 4.4 and Theorem 4.5. $\square$

PROPOSITION 4.1. *In a topological space (X, τ),*

  (1) *Every g-ω-closed set is weakly g-ω-closed.*
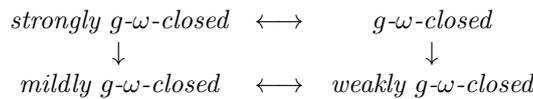 (2) *Every strongly g-ω-closed set is mildly g-ω-closed.*

PROOF. Obvious. $\square$

REMARK 4.1. *The converses of Proposition 4.1 are not true in general as shown in the following Example.*

EXAMPLE 4.2. *Consider $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \{1\}\}$. Let $G = \mathbb{R} - \mathbb{Q}$.*

  (1) *Let H be any ω-open subset of $\mathbb{R}$ such that $G = \mathbb{R} - \mathbb{Q} \subseteq H$. Now $cl(int(G)) = cl(\phi) = \phi \subseteq H$. Hence $G = \mathbb{R} - \mathbb{Q}$ is weakly g-ω-closed. Also $G \subseteq G$ and G is ω-open. But $cl(G) = \mathbb{R}\backslash\{1\} \nsubseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not g-ω-closed.*
  (2) *Let H be any $g\omega$-open subset of $\mathbb{R}$ such that $G = \mathbb{R} - \mathbb{Q} \subseteq H$. Now $cl(int(G)) = cl(\phi) = \phi \subseteq H$. Hence $G = \mathbb{R} - \mathbb{Q}$ is mildly g-ω-closed. But $G \subseteq G$, G being $g\omega$-open whereas $cl(G) = \mathbb{R}\backslash\{1\} \nsubseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not strongly g-ω-closed.*

REMARK 4.2. *In a topological space (X, τ), the following relations hold for a subset G of X.*

$$\begin{array}{ccc} strongly\ g\text{-}\omega\text{-}closed & \longleftrightarrow & g\text{-}\omega\text{-}closed \\ \downarrow & & \downarrow \\ mildly\ g\text{-}\omega\text{-}closed & \longleftrightarrow & weakly\ g\text{-}\omega\text{-}closed \end{array}$$

Where $A \longleftrightarrow B$ means A implies and is implied by B and $A \to B$ means A implies B but not conversely.

THEOREM 4.7. *In a topological space (X, τ), for a subset G of X, the following properties are equivalent.*

  (1) *G is mildly g-ω-closed;*
  (2) *$cl(int(G))\backslash G = \phi$;*
  (3) *$cl(int(G)) \subseteq G$;*
  (4) *G is preclosed.*

PROOF. (1) $\Leftrightarrow$ (2) G is mildly g-ω-closed $\Leftrightarrow cl(int(G)) \subseteq G$ by Theorem 4.1 $\Leftrightarrow cl(int(G))\backslash G = \phi$.

(2) $\Leftrightarrow$ (3) $cl(int(G))\backslash G = \phi \Leftrightarrow cl(int(G)) \subseteq G$.

(3) $\Leftrightarrow$ (4) $cl(int(G)) \subseteq G \Leftrightarrow G$ is preclosed by (3) of Definition 2.2. $\square$

THEOREM 4.8. *In a topological space (X, τ), if G is mildly g-ω-closed, then $G \cup (X - cl(int(G)))$ is mildly g-ω-closed.*

PROOF. Since G is mildly g-ω-closed, $cl(int(G)) \subseteq G$ by Theorem 4.1. Then $X - G \subseteq X - cl(int(G))$ and $G \cup (X - G) \subseteq G \cup (X - cl(int(G)))$. Thus $X \subseteq G \cup (X - cl(int(G)))$ and so $G \cup (X - cl(int(G))) = X$. Hence $G \cup (X - cl(int(G)))$ is mildly g-ω-closed. $\square$

THEOREM 4.9. *In a topological space (X, τ), the following properties are equivalent:*

(1) *G is a closed set and an open set,*

(2) *G is a regular closed set and an open set,*

(3) *G is a mildly g-ω-closed set and an open set.*

PROOF. (1) $\Rightarrow$ (2): Since G is closed and open, $G = cl(G)$ and $G = int(G)$ implies $G = cl(int(G))$ and $G = int(G)$. Hence G is regular closed and open.

(2) $\Rightarrow$ (3): Since G is regular closed and open, $G = cl(int(G))$ and $G = int(G)$. Since $G = cl(int(G))$, $cl(int(G)) \subseteq G$. By Theorem 4.1, G is mildly g-ω-closed and open.

(3) $\Rightarrow$ (1): Since G is mildly g-ω-closed, $cl(int(G)) \subseteq G$ by Theorem 4.1. Again G is open implies $cl(G) \subseteq G$. Thus G is closed and open. $\square$

PROPOSITION 4.2. *In a topological space (X, τ), every closed set is mildly g-ω-closed.*

PROOF. Let H be any $g\omega$-open subset of X such that $G \subseteq H$. Then $cl(int(G)) \subseteq cl(G) = G \subseteq H$ and hence G is mildly g-ω-closed. This shows that G is closed $\Rightarrow$ mildly g-ω-closed. $\square$

REMARK 4.3. *The converse of Proposition 4.2 is not true in general as shown in the following example.*

EXAMPLE 4.3. *In Example 4.2(2), $G = \mathbb{R} - \mathbb{Q}$ is mildly g-ω-closed but not closed for $cl(G) = \mathbb{R}\backslash\{1\} \nsubseteq G$.*

5. Further properties

THEOREM 5.1. *In a topological space (X, τ), if G is mildly g-ω-closed and H is a subset such that $G \subseteq H \subseteq cl(int(G))$, then H is mildly g-ω-closed.*

PROOF. Since G is mildly g-ω-closed, $cl(int(G)) \subseteq G$ by (3) of Theorem 4.7. Thus by assumption, $G \subseteq H \subseteq cl(int(G)) \subseteq G$. Then $G = H$ and so H is mildly g-ω-closed. $\square$

COROLLARY 5.1. *In a topological space (X, τ), if G is a mildly g-ω-closed set and an open set, then $cl(G)$ is mildly g-ω-closed.*

PROOF. Since G is open in X, $G \subseteq cl(G) = cl(int(G))$. G is mildly g-ω-closed implies $cl(G)$ is mildly g-ω-closed by Theorem 5.1. $\square$

THEOREM 5.2. *In a topological space (X, τ), a nowhere dense subset is mildly g-ω-closed.*

PROOF. If G is a nowhere dense subset in X then $int(cl(G)) = \phi$. Since $int(G) \subseteq int(cl(G))$, $int(G) = \phi$. Hence $cl(int(G)) = cl(\phi) = \phi \subseteq G$. Thus, G is mildly g-ω-closed in (X, τ) by Theorem 4.1. $\square$

REMARK 5.1. *The converse of Theorem 5.2 is not true in general as shown in the following Example.*

EXAMPLE 5.1. *Let $\mathbb{R}$ and τ be as in Example 4.1. For $G = \mathbb{Q} - \{1\}$, if H is any $g\omega$-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $cl(int(G)) = cl(\phi) = \phi \subseteq G$. Hence $G = \mathbb{Q} - \{1\}$ is mildly g-ω-closed. On the other hand, $int(cl(G)) = int(\mathbb{R}) = \mathbb{R} \neq \phi$ and thus $G = \mathbb{Q} - \{1\}$ is not nowhere dense in X.*

REMARK 5.2. *In a topological space (X, τ), the intersection of two mildly g-ω-closed subsets is mildly g-ω-closed.*

PROOF. Let A and B be mildly g-ω-closed subsets in (X, τ). Then $cl(int(A)) \subseteq A$ and $cl(int(B)) \subseteq B$ by Theorem 4.1. Also $cl[int(A \cap B)] = cl[int(A) \cap int(B)] \subseteq cl(int(A)) \cap cl(int(B)) \subseteq A \cap B$. This implies that $A \cap B$ is mildly g-ω-closed by Theorem 4.1. $\square$

REMARK 5.3. *In a topological space (X, τ), the union of two mildly g-ω-closed subsets need not be mildly g-ω-closed.*

EXAMPLE 5.2. *In Example 3.5, for $A = (\mathbb{R} - \mathbb{Q})_+ =$ the set of positive irrationals and $B = (\mathbb{R} - \mathbb{Q})_- =$ the set of negative irrationals, $int(A) = \phi$ and $int(B) = \phi$ respectively. So $cl(int(A)) = cl(\phi) = \phi \subseteq A$ and thus A is mildly g-ω-closed by Theorem 4.1. Similarly B is also mildly g-ω-closed. But $int(A \cup B) = int(\mathbb{R} - \mathbb{Q}) = \mathbb{R} - \mathbb{Q}$. So $cl[int(A \cup B)] = cl(\mathbb{R} - \mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{R} - \mathbb{Q} = A \cup B$. Hence $A \cup B$ is not mildly g-ω-closed.*

THEOREM 5.3. *In a topological space (X, τ), a subset G is mildly g-ω-open if and only if $G \subseteq int(cl(G))$.*

PROOF. G is mildly g-ω-open $\Leftrightarrow$ $X \backslash G$ is mildly g-ω-closed $\Leftrightarrow$ $X \backslash G$ is preclosed by (4) of Theorem 4.7 $\Leftrightarrow$ G is preopen $\Leftrightarrow$ $G \subseteq int(cl(G))$. $\square$

THEOREM 5.4. *In a topological space (X, τ), if the subset G is mildly g-ω-closed, then $cl(int(G)) \backslash G$ is mildly g-ω-open in (X, τ).*

PROOF. Since G is mildly g-ω-closed, $cl(int(G)) \backslash G = \phi$ by (2) of Theorem 4.7. Thus $cl(int(G)) \backslash G$ is mildly g-ω-open in (X, τ). $\square$

THEOREM 5.5. *In a topological space (X, τ), if G is mildly g-ω-open, then $int(cl(G)) \cup (X - G) = X$.*

PROOF. Since G is mildly g-ω-open, $G \subseteq int(cl(G))$ by Theorem 5.3. So $(X - G) \cup G \subseteq (X - G) \cup int(cl(G))$ which implies $X = (X - G) \cup int(cl(G))$. $\square$

THEOREM 5.6. *In a topological space (X, τ), if G is mildly g-ω-open and $int(cl(G)) \subseteq H \subseteq G$, then H is mildly g-ω-open.*

PROOF. Since G is mildly g-ω-open, $G \subseteq int(cl(G))$ by Theorem 5.3. By assumption $int(cl(G)) \subseteq H \subseteq G$. This implies $G \subseteq int(cl(G)) \subseteq H \subseteq G$. Thus $G = H$ and so H is mildly g-ω-open. $\square$

COROLLARY 5.2. *In a topological space (X, τ), if G is a mildly g-ω-open set and a closed set, then $int(G)$ is mildly g-ω-open.*

PROOF. If G is a mildly g-ω-open set and a closed set in (X, τ), then $int(cl(G)) = int(G) \subseteq int(G) \subseteq G$. Thus, by Theorem 5.6, $int(G)$ is mildly g-ω-open in (X, τ). $\square$

DEFINITION 5.1. *A subset H of a topological space (X, τ) is called a G-set if $H = M \cup N$ where M is $g\omega$-closed and N is preopen.*

PROPOSITION 5.1. *Every preopen (resp. $g\omega$-closed) set is a G-set.*

REMARK 5.4. *The separate converses of Proposition 5.1 are not true in general as shown in the following Example.*

EXAMPLE 5.3.   (1) *Let $\mathbb{R}$ and τ be as in Example 4.1 and $G = \mathbb{R} - \mathbb{Q}$. Since G is closed, it is $g\omega$-closed and hence a G-set. But $int(cl(G)) = int(G) = \phi \not\supseteq G$. Hence $G = \mathbb{R} - \mathbb{Q}$ is not preopen.*
  (2) *Let $\mathbb{R}$ and τ be as in Example 3.5. For $G = \mathbb{R} - \mathbb{Q}$, $int(cl(G)) = int(\mathbb{R}) = \mathbb{R} \supseteq G$. Thus G is preopen and hence a G-set. But G is open and $G \subseteq G$, whereas $cl_\omega(G) = \mathbb{R} \not\subseteq G$. Hence G is not $g\omega$-closed.*

REMARK 5.5. *The following example shows that the concepts of preopenness and $g\omega$-closedness are independent of each other.*

EXAMPLE 5.4. *In Example 5.3(1), $G = \mathbb{R} - \mathbb{Q}$ is $g\omega$-closed but not preopen. In Example 5.3(2), $G = \mathbb{R} - \mathbb{Q}$ is preopen but not $g\omega$-closed.*

THEOREM 5.7. *Let (X, τ) be a topological space and $G \subseteq X$. Then G is mildly g-ω-open if and only if $F \subseteq int(cl(G))$ whenever F is $g\omega$-closed and $F \subseteq G$.*

PROOF. Suppose G is mildly g-ω-open. If F is $g\omega$-closed and $F \subseteq G$, then $X - G \subseteq X - F$ and so $cl(int(X - G)) \subseteq X - F$. Therefore $F \subseteq X - cl(int(X - G)) = int(cl(G))$.

Conversely, the condition holds. Let U be an $g\omega$-open set such that $X - G \subseteq U$. Then $X - U \subseteq G$ and so $X - U \subseteq int(cl(G))$. Therefore $cl(int(X - G)) \subseteq U$. Thus $X - G$ is mildly g-ω-closed and so G is mildly g-ω-open. $\square$

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