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COMMON BEST PROXIMITY POINTS IN COMPLEX VALUED METRIC SPACES

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Abstract. In this paper, we obtain the existence and the uniqueness of common best proximity point theorems for non-self mappings between two subsets of a complex valued metric space satisfying certain contractive conditions. Our results supported by some examples.

1. Introduction and Preliminaries

Fixed point theory focuses on solving the equation $Tx = x$, where T is a self-mapping defined on a subset of a metric space or other suitable space. If it is assumed that, T is not a self-mapping then the equation $Tx = x$ is likely to have no solution. Consequently, the significant aim is determining an element x that is in close proximity to Tx in some sense. Eventually, the target is finding an element x in a metric space, that satisfy in the following condition, $d(x, Tx) = d(A, B)$ and $d(x, Sx) = d(A, B)$ which d is a metric function and $d(A, B) := \inf\{d(x, y) : x \in A,\ y \in B\}$. Now, if $T, S : A \to B$ are two non-self mappings, then the equations $Sx = x$ and $Tx = x$ are likely to have no solution, the solution known as a common fixed point of the mappings S and T (see, [**1, 7, 9, 12, 8, 15**]). So, the purpose is finding an element x in A such that $d(x, Sx) = d(A, B)$ and $d(x, Tx) = d(A, B)$ which x is called the common best proximity point of mappings S and T in a metric space (see, [**2, 13, 14**]). In 2011, Azam et al. [**3**] introduced the notion of complex valued metric space, which is a generalization of the classical metric space and established the existence of common fixed point theorems for mappings satisfying contraction condition (see [**3**], Theorem 4). The purpose of this article is generalizing some well-known results about common best proximity points that

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were established in the classic metric space (see, [**2, 13**]), in the complex valued metric space by some new definitions and presenting a type of contractive condition and developing a common best proximity point theorem for non-self mappings which satisfy in this contractive condition, in the complex valued metric space.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \text{ if and only if } Re(z_1) \leqslant Re(z_2),\ Im(z_1) \leqslant Im(z_2).$$

It follows that $z_1 \preceq z_2$ if and only if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied where we denote $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \preceq z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|,$$
$$z_1 \preceq z_2, z_2 \precnsim z_3 \Longrightarrow z_1 \prec z_3.$$

DEFINITION 1.1. [**3**] *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:*

(a) $0 \preceq d(x, y)$*, for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$*;*
(b) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(c) $d(x, z) \preceq d(x, y) + d(y, z)$*, for all* $x, y, z \in X$*.*

Then d is called a complex valued metric on X , and (X, d) is called a complex valued metric space.

EXAMPLE 1.1. Let $X = \mathbb{C}$. Define the mapping $d : X \times X \to \mathbb{C}$ for all $x, y \in X$, by

$$d(x, y) = i|x - y|.$$

Clearly, the pair (X, d) is a complex valued metric space.

DEFINITION 1.2. [**3**] *Let (X, d) be a complex valued metric space.*

(a) *A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.*
(b) *A point $x \in X$ is called a limit point of a subset $A \subseteq X$ whenever for every $0 \prec r \in C$, $B(x, r) \bigcap (A \setminus \{x\}) \neq \emptyset$.*
(c) *A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.*
(d) *A subset $A \subseteq X$ is called closed whenever each limit point of A belongs to A.*
(e) *The family $F = \{B(x, r) : x \in X,\ 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on X.*

DEFINITION 1.3. [**4**] *Let A be a subset of $\mathbb{C}$. If there exists $u \in \mathbb{C}$ such that $z \preceq u$ for all $z \in A$, then A is bounded above and u is an upper bound. Similarly,* *if there exists* $l \in \mathbb{C}$ *such that* $l \preceq z$, *for all* $z \in A$, *then* A *is bounded below and* l *is a lower bound.*

Definition 1.4. [**4**] *For a* $A \subseteq \mathbb{C}$ *which is bounded above if there exists an upper bound* s *of* A *such that, for every upper bound* u *of* A, $s \preceq u$, *then the upper bound* s *is called* $\sup A$. *Similarly, for a subset* $A \subseteq \mathbb{C}$ *which is bounded below if there exists a lower bound* t *of* A *such that for every lower bound* l *of* A, $l \preceq t$, *then the lower bound* t *is called* $\inf A$.

Suppose that $A \subseteq \mathbb{C}$ is bounded above. Then there exists $q = u + iv \in \mathbb{C}$ such that $z = x + iy \preceq q = u + iv$, for all $z \in A$. It follows that $x \preceq u$ and $y \preceq v$, for all $z = x + iy \in A$; that is, $S = \{x : z = x + iy \in A\}$ and $T = \{y : z = x + iy \in A\}$ are two sets of real numbers which are bounded above. Hence both $\sup S$ and $\inf T$ exist. Let $\bar{x} = \sup S$ and $\bar{y} = \sup T$. Then $\bar{z} = \bar{x} + i\bar{y}$ is $\sup A$.

Similarly, if $A \subseteq \mathbb{C}$ is bounded below, then $z^* = x^* + iy^*$ is $\inf A$, where $x^* = \inf S = \inf\{x : z = x + iy \in A\}$ and $y^* = \inf T = \inf\{y : x + iy \in A\}$.

Any subset $A \subseteq \mathbb{C}$ which is bounded above has supremum. Equivalently, any subset $A \subseteq \mathbb{C}$ which is bounded below has infimum.

Definition 1.5. [**3**] *Let* (X, d) *be a complex valued metric space. Let* $\{x_n\}$ *be a sequence in* X *and* $x \in X$.

- (i) *If for every* $c \in \mathbb{C}$, *with* $0 \prec c$ *there is* $n_0 \in N$ *such that* $d(x_n, x) \prec c$, *for all* $n > n_0$, *then* $\{x_n\}$ *is said to be convergent,* $\{x_n\}$ *converges to* x, x *is the limit point of* $\{x_n\}$. *We denote this by* $lim_n x_n = x$ *or* $x_n \to x$ *as* $n \longrightarrow \infty$.
- (ii) *If for every* $c \in \mathbb{C}$, *with* $0 \prec c$ *there is* $N \in \mathbb{N}$ *such that for all* $n > N$, $d(x_n, x_{n+m}) \prec c$, *where* $m \in \mathbb{N}$, *then* $\{x_n\}$ *is said to be Cauchy sequence.*
- (iii) *If every Cauchy sequence is convergent in* (X, d), *then* (X, d) *is called a complete complex valued metric space.*

Lemma 1.1 ([**3**], Lemma 3). *Let* (X, d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *is a Cauchy sequence if and only if* $|d(x_n, x_{n+m})| \to 0$ *as* $n \to \infty$.

Lemma 1.2 ([**3**], Lemma 2). *Let* (X, d) *be a complex valued metric space and let* $\{x_n\}$ *be a sequence in* X. *Then* $\{x_n\}$ *converges to* x *if and only if* $|d(x_n, x)| \to 0$ *as* $n \to \infty$.

Given nonempty subsets A and B of complex valued metric space (X, d). Then $\{d(x, y) :\ x \in A,\ y \in B\} \subseteq \mathbb{C}$ is always bounded below by $z_0 = 0 + i0$ and hence $\inf\{d(x, y) :\ x \in A,\ y \in B\}$ exists. Here we define

$$d(A, B) = inf\{d(x, y) : x \in A \; and \; y \in B\},$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \; for \; some \; y \in B\},$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \; for \; some \; x \in A\}.$$

From the above definition, it is clear that for every $x \in A_0$ there exists $y \in B_0$ such that $d(x, y) = d(A, B)$ and conversely, for every $y \in B_0$ there exists $x \in A_0$ such that $d(x, y) = d(A, B)$.

DEFINITION 1.6. *Given non-self mapping $S : A \to B$ and $T : A \to B$, an element $x \in X$ is called a common best proximity point of the mappings if they satisfy the condition that*

$$d(x, Sx) = d(x, Tx) = d(A, B).$$

DEFINITION 1.7. *Let (A, B) be a pair of nonempty subsets of a complex valued metric space (X, d) with $A_0 \neq \emptyset$. Then that pair (A, B) is said to have the weak P-property if and only if*

$$\begin{cases} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{cases} \Longrightarrow d(x_1, x_2) \preceq d(y_1, y_2), \tag{1.1}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

DEFINITION 1.8. *The mappings $S : A \to B$ and $T : A \to B$ are said to be commute proximally if they satisfy the condition that*

$$[d(u, Sx) = d(v, Tx) = d(A, B)] \Rightarrow Sv = Tu.$$

DEFINITION 1.9. *Let S and T be two non-empty subsets of a complex valued metric space (X, d). Non-self mappings $S, T : A \longrightarrow B$ are said to satisfy a L-contractive condition if there exist non-negative numbers α_i where $i = 1, ..., 4$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$, then for each $x, y \in A$,*

$$\begin{aligned} d(Sx, Sy) \preceq \alpha_1 d(Tx, Ty) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Ty, Sy) \\ + \alpha_4[d(Ty, Sx) + d(Sy, Tx)]. \end{aligned}$$

DEFINITION 1.10. *A mapping $T : A \to B$ is said to dominate a mapping $S : A \to B$ proximally if there exists a non-negative real number $\alpha < 1$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A,*

$$\begin{aligned} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2) \\ \Rightarrow d(u_1, u_2) \preceq \alpha d(v_1, v_2) \end{aligned}$$

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DEFINITION 1.11. *A mapping $T : A \to B$ is said to weakly dominate a mapping $S : A \to B$ proximally if there exists a non-negative real number $\alpha < 1$ such that for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A,*

$$\begin{aligned} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2) \\ \Rightarrow d(u_1, u_2) \preceq \alpha\, \omega_{u_1,u_2,v_1,v_2}. \end{aligned}$$

where $\omega_{u_1,u_2,v_1,v_2} = Re\ \omega_{u_1,u_2,v_1,v_2} + i\ Im\ \omega_{u_1,u_2,v_1,v_2}$ and
$Re\ \omega_{u_1,u_2,v_1,v_2} = \max\{Re\ d(v_1, v_2), Re\ d(v_1, u_1), Re\ d(v_2, u_2), \frac{Re\ d(v_1,u_2)+Re\ d(v_2,u_1)}{2}\}$,
$Im\ \omega_{u_1,u_2,v_1,v_2} =$
$\max\{Im\ d(v_1, v_2), Im\ d(v_1, u_1), Im\ d(v_2, u_2), \frac{Im\ d(v_1,u_2)+Im\ d(v_2,u_1)}{2}\}$.

If T dominates S then T weakly dominates S. But the converse is not true.

EXAMPLE 1.2. Let us consider the complex valued metric space (X, d) where $X = \mathbb{C}$ and let $d : X \times X \longrightarrow \mathbb{C}$ be given as

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Let A and B be two subsets of X given by

$$A = \{z \in \mathbb{C} : \ Re(z) = -1, \ 0 \leqslant Im(z) \leqslant 1\},$$

$$B = \{z \in \mathbb{C} : \ Re(z) = 1, \ 0 \leqslant Im(z) \leqslant 1\}.$$

So we have that $A_0 = A$, $B_0 = B$ and $d(A, B) = 2 + 0i$. Let $T, S : A \longrightarrow B$ be defined as

$$Tz = -x + iy \ \ for\ each\ z = x + iy \in A$$

and

$$Sz = \begin{cases} 1 + i\frac{1}{4} & 0 \leqslant y < 1 \\ 1 + i\frac{1}{3} & y = 1 \end{cases}$$

for each $z = x + iy \in A$. If we suppose that $v_1 = x_1 = -1 + \frac{12}{13}i$, $v_2 = x_2 = -1 + i$, $u_1 = -1 + \frac{1}{4}i$, $u_2 = -1 + \frac{1}{3}i$, it implies that

$$d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2).$$

Clearly, $0 + \frac{1}{12}i = d(u_1, u_2) \not\precsim \alpha d(v_1, v_2) = \alpha(0 + \frac{1}{13}i)$ for each non-negative real number $\alpha < 1$. But obviously, we have that for $\alpha = \frac{1}{8}$, T weakly dominates S proximally.

2. Common Best Proximity Point by Weakly Dominate Proximally Property

THEOREM 2.1. *Let (X, d) be a complete complex valued metric space, A and B be two non-empty subsets of X. Assume that A_0 and B_0 are nonempty and A_0 is closed. Let $S : A \to B$ and $T : A \to B$ be two non-self mappings that satisfy the following conditions:*

(a) *T weakly dominates S proximally*
(b) *S and T commute proximally*
(c) *S and T are continuous*
(d) *$S(A_0) \subseteq B_0$*
(e) *$S(A_0) \subseteq T(A_0)$*

Then there exists a unique element $x \in A$ such that

$$d(x, Tx) = d(A, B) \ \ and \ \ d(x, Sx) = d(A, B).$$

PROOF. Let x_0 be a fixed element in A_0. Since $S(A_0) \subseteq T(A_0)$, then there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Then by continuing this process we can choose $x_n \in A_0$ such that there exists $x_{n+1} \in A_0$ satisfying

$$Sx_n = Tx_{n+1} \quad for\ each\ n \in N$$

since $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$d(Sx_n, u_n) = d(A, B) \quad for\ each\ n \in N. \tag{2.1}$$

By choosing x_n and u_n it follows that

$$d(Sx_n, u_n) = d(Sx_{n+1}, u_{n+1}) \tag{2.2}$$

and

$$d(A, B) = d(Tx_n, u_{n-1}) = d(Tx_{n+1}, u_n).$$

Since T weakly dominates S proximally then we have

$$d(u_n, u_{n+1}) \preceq \alpha\, \omega_{u_n, u_{n+1}, u_{n-1}, u_n},$$

where $\alpha < 1$ and

$$\begin{aligned} Re\, \omega_{u_n,u_{n+1},u_{n-1},u_n} &= \alpha \max\{Re\ d(u_{n-1}, u_n), Re\ d(u_{n-1}, u_n), \\ &Re\ d(u_n, u_{n+1}), \frac{Re\ d(\ u_{n-1}, u_{n+1}) + Re\ d(u_n, u_n)}{2}\}. \end{aligned}$$

and

$$\begin{aligned} Im\, \omega_{u_n,u_{n+1},u_{n-1},u_n} &= \alpha \max\{Im\ d(u_{n-1}, u_n), Im\ d(u_{n-1}, u_n), \\ &Im\ d(u_n, u_{n+1}), \frac{Im\ d(\ u_{n-1}, u_{n+1}) + Im\ d(u_n, u_n)}{2}\}. \end{aligned}$$

We focus on $Re\ \ d(u_n, u_{n+1})$ and conclude for $Im\ \ d(u_n, u_{n+1})$ and finally for $d(u_n, u_{n+1})$,

$$\begin{aligned} Re\ d(u_n, u_{n+1}) &\leqslant \alpha\ \max\{Re\ d(u_{n-1}, u_n), \frac{Re\ d(u_{n-1}, u_{n+1})}{2}\} \\ &\leqslant \alpha\ \max\{Re\ d(u_{n-1}, u_n), \frac{Re\ d(u_{n-1}, u_n) + Re\ d(u_n, u_{n+1})}{2}\}. \end{aligned}$$

We will prove that $\{u_n\}$ is a Cauchy sequence. We distinguish two cases.

Case I. Suppose that

$$Re\ d(u_n, u_{n+1}) \leqslant \alpha\ Re\ d(u_{n-1}, u_n),$$

so we get that

$$Re\ d(u_n, u_{n+1}) \leqslant \alpha^n\ Re\ d(u_0, u_1),$$

Therefore for any $m > n$ we have

$$\begin{aligned} Re\ d(u_n, u_m) &\leqslant Re\ d(u_n, u_{n+1}) + Re\ d(u_{n+1}, u_{n+2}) + \ \dots\ + Re\ d(u_{m-1}, u_m) \\ &\leqslant \alpha^n\ Re\ d(u_0, u_1) + \alpha^{n+1}\ Re\ d(u_0, u_1) + \ \dots\ + \alpha^{m-1}\ Re\ d(u_0, u_1) \\ &\leqslant (\tfrac{\alpha^n}{1-\alpha})\ Re\ d(u_0, u_1) \to 0 \quad as\ \ m, n \to \infty. \end{aligned}$$

Case II. Assume that

$$\begin{aligned} Re\ d(u_n, u_{n+1}) &\leqslant \alpha\ \frac{Re\ d(u_{n-1}, u_n) + Re\ d(u_n, u_{n+1})}{2} \\ &\leqslant \frac{\alpha/2}{1-\alpha/2}\ Re\ d(u_{n-1}, u_n). \end{aligned}$$

Put $h = \frac{\alpha/2}{1-\alpha/2} < 1$, so we have that

$$Re\ d(u_n, u_{n+1}) \leqslant h^n\ Re\ d(u_0, u_1).$$

It follows that for any $m > n$,

$$Re\ d(u_n, u_m) \leqslant (\frac{h^n}{1-h})\ Re\ d(u_0, u_1) \to 0 \quad as\ \ m, n \to \infty.$$

Similarly we can conclude that for any $m > n$,

$$Im\ d(u_n, u_m) \leqslant (\frac{\alpha^n}{1-\alpha})\ Im\ d(u_0, u_1) \to 0 \quad as\ \ m, n \to \infty,$$

or

$$Im\ d(u_n, u_m) \leqslant (\frac{h^n}{1-h})\ Im\ d(u_0, u_1) \to 0 \quad as\ \ m, n \to \infty.$$

This implies that for any $m > n$,

$$d(u_n, u_m) \to 0 \quad as\ \ m, n \to \infty.$$

Then $\{u_n\}$ is a Cauchy sequence and since X is complete and A_0 is closed, there exists $u \in A_0$ such that $u_n \to u$. By hypothesis, mappings S and T are commuting proximally and by (2.2) we have that

$$Tu_n = Su_{n-1}, \quad for\ every\ \ n \in N.$$

Since T and S are continuous it implies that

$$Tu = \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Su_{n-1} = Su.$$

As $Su \in S(A_0) \subseteq B_0$, there exists an $x \in A_0$ such that

$$d(x, Su) = d(A, B) = d(x, Tu). \tag{2.3}$$

Since S and T commute proximally, $Sx = Tx$. Also, $Sx \in S(A_0) \subseteq B_0$, there exists a $z \in A_0$ such that

$$d(z, Sx) = d(A, B) = d(z, Tx). \tag{2.4}$$

Since T weakly dominates S then from (2.3) and (2.4) we can conclude that

$$d(x, z) \preceq \alpha\ \omega_{x,z,x,z} = \alpha\ (Re\ d(x, z) + iIm\ d(x, z)) = \alpha\ d(x, z).$$

It follows that $x = z$, therefore we have that

$$d(x, Sx) = d(A, B) = d(x, Tx). \tag{2.5}$$

We now show that S and T have unique common best proximity point. For this, assume that x^* in A is a second common best proximity point of S and T, then

$$d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*). \tag{2.6}$$

Since T weakly dominate S proximally then from (2.5) and (2.6), we have

$$d(x, x^*) \preceq \alpha d(x, x^*).$$

Consequently, $x = x^*$ and S and T have a unique common best proximity point. □

EXAMPLE 2.1. Let us consider the complex valued metric space (X, d) where $X = \mathbb{C}$ and let $d : X \times X \longrightarrow \mathbb{C}$ be given as

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Let A and B be two subsets of X given by

$$A = \{z \in \mathbb{C} : \ Re(z) = -1, \ 0 \leqslant Im(z) \leqslant 1\} \\ \cup \{z \in \mathbb{C} : \ Re(z) = 1, \ 0 \leqslant Im(z) \leqslant 1\},$$

$$B = \{z \in \mathbb{C} : \ Re(z) = -2, \ 0 \leqslant Im(z) \leqslant 1\} \\ \cup \{z \in \mathbb{C} : \ Re(z) = 2, \ 0 \leqslant Im(z) \leqslant 1\}.$$

Then A and B are closed and bounded subsets of X such that

$$d(A, B) = 1, \quad A_0 = A, \quad B_0 = B.$$

Let $T, S : A \longrightarrow B$ be defined as

$$Tz = 2|x| + iy \ \ for \ each \ \ z = x + iy \in A$$

and

$$Sz = 2|x| + i\frac{y}{2} \ \ for \ \ each \ z = x + iy \in A.$$

Therefore T and S satisfy the properties mentioned in Theorem 2.1. Hence the conditions of Theorem 2.1 are satisfied and $1 + 0i$ is the unique common best proximity point of S and T.

By Theorem 2.1 we obtain the following results in the fixed point theorem.

COROLLARY 2.1. *Let (X,d) be a complex valued metric space. Let $T : X \to X$ be a continuous mapping and S be any self-mapping on X that commutes with T. Further let S and T satisfy $S(X) \subseteq T(X)$ and there exists a constant $\alpha \in [0, 1)$ such that for every $x, y \in X$*

$$d(Sx, Sy) \preceq \alpha \ \omega_{Sx,Sy,Tx,Ty}.$$

where

$Re \ \omega_{Sx,Sy,Tx,Ty}$
$= \max\{Re \ d(Tx, Ty), Re \ d(Tx, Sx), Re \ d(Ty, Sy), \frac{Re \ d(Tx,Sy)+Re \ d(Ty,Sx)}{2}\},$
and
$Im \ \omega_{Sx,Sy,Tx,Ty}$
$= \max\{Im \ d(Tx, Ty), Im \ d(Tx, Sx), Im \ d(Ty, Sy), \frac{Im \ d(Tx,Sy)+Im \ d(Ty,Sx)}{2}\}.$

Then S and T have a unique common fixed point.

If T is assumed to be identity mapping in Corollary 2.1, then we have the following result.

COROLLARY 2.2. *Let (X,d) be a complex valued metric space. Let S be a self-mapping on X and there exists a constant $\alpha \in [0, 1)$ such that for every $x, y \in X$*

$$d(Sx, Sy) \preceq \alpha \ \omega_{Sx,Sy,x,y}.$$

where $Re\ \omega_{Sx,Sy,x,y}$
$= \max\{Re\ d(x,y), Re\ d(x,Sx), Re\ d(y,Sy), \frac{Re\ d(x,Sy)+Re\ d(y,Sx)}{2}\}$,
and
$Im\ \omega_{Sx,Sy,x,y}$
$= \max\{Im\ d(x,y), Im\ d(x,Sx), Im\ d(y,Sy), \frac{Im\ d(x,Sy)+Im\ d(y,Sx)}{2}\}$.

Then S has a fixed point.

3. Common Best Proximity Point for L-contractive Condition Mappings

THEOREM 3.1. *Let (X,d) be a complex valued metric space, A and B be two non-empty closed subsets of X and the pair (A,B) satisfies the weak P-property. Let A_0 and B_0 are non-empty. Assume also that $S,T : A \longrightarrow B$ are two non-self mappings satisfying the following conditions:*
(a) *S and T commute proximally;*
(b) *S and T are continuous;*
(c) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$;*
(d) *S and T satisfy L-contractive condition.*
Then, there exists a unique point $x \in A$ such that

$$d(x,Tx) = d(A,B) = d(x,Sx).$$

PROOF. Let x_0 be a fixed element in A_0. Since $S(A_0) \subseteq T(A_0)$, then there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Then by continuing this process we can choose $x_n \in A_0$ such that there exists $x_{n+1} \in A_0$ satisfying

$$Sx_n = Tx_{n+1} \ \ for\ each\ n \in \mathbb{N}. \tag{3.1}$$

Since $S(A_0) \subseteq B_0$ there exists an element $u_n \in A_0$ such that

$$d(Sx_n,u_n) = d(A,B) \ \ for\ each\ n \in N. \tag{3.2}$$

Further, it follows from the choice x_n and u_n that

$$d(Sx_n,u_n) = d(A,B) = d(Sx_{n+1},u_{n+1}),$$

By using the weak P-property and L-contractive condition, we have

$$\begin{aligned} d(u_n,u_{n+1}) &\preceq d(Sx_n,Sx_{n+1}) \\ &\preceq \alpha_1 d(Tx_n,Tx_{n+1}) + \alpha_2 d(Tx_n,Sx_n) + \alpha_3 d(Tx_{n+1},Sx_{n+1}) \\ &\quad +\alpha_4[d(Tx_{n+1},Sx_n) + d(Sx_{n+1},Tx_n)] \\ &\preceq \alpha_1 d(Sx_{n-1},Sx_n) + \alpha_2 d(Sx_{n-1},Sx_n) + \alpha_3 d(Sx_n,Sx_{n+1}) \\ &\quad +\alpha_4 d(Sx_{n-1},Sx_n) + \alpha_4 d(Sx_n,Sx_{n+1}). \end{aligned}$$

Consequently, it implies that

$$d(u_n,u_{n+1}) \preceq hd(Sx_{n-1},Sx_n) \preceq \ \ldots \preceq h^n d(Sx_0,Sx_1),$$

where $h = \frac{\alpha_1+\alpha_2+\alpha_4}{1-(\alpha_3+\alpha_4)} < 1$. Therefore, $\{u_n\}$ is a Cauchy sequence and since (X, d) is a complete complex valued metric space and A is closed, then there exists $u \in A$ such that $u_n \to u$ as $n \to \infty$. Also, we have that

$$d(Sx_n, u_n) = d(A, B) = d(Tx_n, u_{n-1}),$$

Since S and T commute proximally we get that

$$Tu_n = Su_{n-1}.$$

Thus, it follows that $Tu = Su$, because S and T are continuous. Since $\{Sx_n\}$ is also a Cauchy sequence, X is complete and B is closed we can easily prove that $Su \in S(A_0) \subseteq B_0$. Therefore, there exists $x \in A_0$ such that

$$d(x, Su) = d(A, B) = d(x, Tu). \tag{3.3}$$

Therefore, $Tx = Sx$, because S and T commute proximally. Since $Sx \in S(A_0) \subseteq B_0$, there exists $z \in A_0$, it implies that

$$d(z, Sx) = d(A, B) = d(z, Tx). \tag{3.4}$$

By L-contractive condition, we get that

$$\begin{aligned} d(Su, Sx) &\preceq \alpha_1 d(Tu, Tx) + \alpha_2 d(Su, Tu) + \alpha_3 d(Sx, Tx) \\ &\quad + \alpha_4[d(Su, Tx) + d(Sx, Tu)] \\ &= (\alpha_1 + 2\alpha_4) d(Su, Sx). \end{aligned} \tag{3.5}$$

Therefore, $Su = Sx$. From (3.3) and (3.4) we have

$$d(x, Su) = d(A, B) = d(z, Sx),$$

the weak P-property of the pair (A,B) implies

$$d(x, z) \preceq d(Sx, Su) = 0.$$

So $x = z$ and

$$d(x, Sx) = d(A, B) = d(x, Tx). \tag{3.6}$$

Suppose that x^* is another common best proximity point of the mappings S and T so that

$$d(x^*, Sx^*) = d(A, B) = d(x^*, Tx^*). \tag{3.7}$$

Since S and T commute proximally, then $Sx = Tx$ and $Sx^* = Tx^*$. So we have

$$\begin{aligned} d(Sx, Sx^*) &\preceq \alpha_1 d(Tx, Tx^*) + \alpha_2 d(Tx, Sx) + \alpha_3 d(Tx^*, Sx^*) \\ &\quad + \alpha_4[d(Tx^*, Sx) + d(Tx, Sx^*)] \\ &= (\alpha_1 + 2\alpha_4) d(Sx, Sx^*), \end{aligned}$$

Which implies that $Sx = Sx^*$. Since the pair (A, B) satisfies weak P-property, from (3.6) and (3.7) we have that

$$d(x, x^*) \preceq d(Sx, Sx^*).$$

Eventually, we have that $x = x^*$. Hence S and T have a unique common best proximity point. □

Example 3.1. Let (X,d) be a complex valued metric space defined as in Example 2.1 and A, B be two subsets of X given by

$$A = \{z \in \mathbb{C} : \ Re(z) = 0, \ 0 \leqslant Im(z) \leqslant 1\},$$
$$B = \{z \in \mathbb{C} : \ Re(z) = 1, \ 0 \leqslant Im(z) \leqslant 1\}.$$

Let $T, S : A \longrightarrow B$ be defined as

$$T(0+iy) = 1 + iy \ \ for\ each \ \ 0 \leqslant y \leqslant 1$$

and

$$S(0+iy) = 1 + i\frac{y}{4} \ \ for\ \ each\ 0 \leqslant y \leqslant 1.$$

Then (A,B) is a pair of nonempty closed and bounded subsets of X such that $A_0 = A$, $B_0 = B$ and $d(A,B) = 1 + 0i$. It is verified that the (A,B) satisfies the weak P-property. Also T and S satisfy the properties mentioned in Theorem 3.1. Hence the conditions of Theorem 3.1 are satisfied and it is seen that $0 = 0 + i0$ is the unique common best proximity point of S and T.

If we suppose that S and T are self-mappings, then Theorem 3.1 implies the following common fixed point theorem, that generalizes and complements the results of [**5**], [**6**], [**10**], [**11**] and others in complex valued metric spaces.

Corollary 3.1. *Let (X,d) be a complete complex valued metric space. Assume that $S,T : X \longrightarrow X$ are two self mappings satisfying the following conditions:*

(a) *there exist non-negative numbers α_i where $i = 1,...,4$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$, such that for each $x, y \in A$,*

$$d(Sx,Sy) \preceq \alpha_1 d(Tx,Ty) + \alpha_2 d(Tx,Sx) + \alpha_3 d(Ty,Sy)$$
$$+\alpha_4[d(Ty,Sx) + d(Sy,Tx)].$$

(b) *S and T commute;*
(c) *T is continuous;*
(d) *$S(X) \subseteq T(X)$;*

Then S and T have a unique common fixed point.

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