BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 7(3)(2017), 573-580
DOI: 10.7251/BIMVI1703573S

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BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

FORCING TOTAL DETOUR MONOPHONIC SETS IN A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a *total detour monophonic set* of a graph G is a detour monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total detour monophonic set of G is the *total detour monophonic number* of G and is denoted by $dm_t(G)$. A subset T of a minimum total detour monophonic set S of G is a *forcing total detour monophonic subset* for S if S is the unique minimum total detour monophonic set containing T. A forcing total detour monophonic subset for S of minimum cardinality is a *minimum forcing total detour monophonic subset* of S. The *forcing total detour monophonic number* $f_{tdm}(S)$ in G is the cardinality of a minimum forcing total detour monophonic subset of S. The *forcing total detour monophonic number* of G is $f_{tdm}(G) = min\{f_{tdm}(S)\}$, where the minimum is taken over all minimum total detour monophonic sets S in G. We determine bounds for it and find the forcing total detour monophonic number of certain classes of graphs. It is shown that for every pair a, b of positive integers with $0 \leqslant a < b$ and $b > 2a+1$, there exists a connected graph G such that $f_{tdm}(G) = a$ and $dm_t(G) = b$.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [**2**]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic* [**1**]. The

2010 *Mathematics Subject Classification.* 05C12.
Key words and phrases. total detour monophonic set, total detour monophonic number, forcing total detour monophonic set, forcing total detour monophonic number.
The third author research work is supported by NBHM Project No. NBHM/R.P.29/2015/Fresh/157.

neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A longest $x-y$ monophonic path is called an $x-y$ *detour monophonic path*. A set S of vertices of G is a *detour monophonic set* if each vertex v of G lies on an $x-y$ detour monophonic path for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour monophonic set of cardinality $dm(G)$ is called dm-set. The detour monophonic number of a graph was introduced in [**4**] and further studied in [**3**].

A *total detour monophonic set* of a graph G is a detour monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total detour monophonic set of G is the *total detour monophonic number* of G and is denoted by $dm_t(G)$. The total detour monophonic number of a graph was introduced and studied in [**5**].

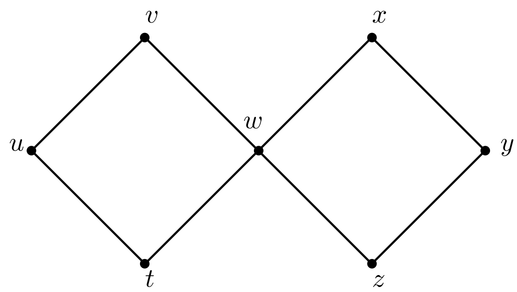

Figure 1.1: G

For the graph G given in Figure 1.1, $S_1 = \{u, v, x, y\}$, $S_2 = \{u, v, y, z\}$, $S_3 = \{t, u, x, y\}$ and $S_4 = \{t, u, y, z\}$ are the minimum total detour monophonic sets of G and so $dm_t(G) = 4$.

A connected graph G may contain more than one minimum total detour monophonic sets. For example, the graph G given in Figure.1.1 contains four minimum total detour monophonic sets. For each minimum total detour monophonic set S in G there is always some subset T of S that uniquely determines S as the minimum total detour monophonic set containing T. Such sets are called "forcing total detour monophonic subsets "and we discuss these sets in this paper.

The following theorems will be used in the sequel.

THEOREM 1.1. [**5**] Each extreme vertex and each support vertex of a connected graph G belongs to every total detour monophonic set of G. If the set S of all extreme vertices and support vertices form a total detour monophonic set, then it is the unique minimum total detour monophonic set of G.

THEOREM 1.2. [**5**] For the complete graph $K_p (p \geqslant 2)$, $dm_t(K_p) = p$.

THEOREM 1.3. [**5**] For any non-trivial tree T, the set of all endvertices and support vertices of T is the unique minimum total detour monophonic set of G.

THEOREM 1.4. [**5**] For any connected graph G, $dm_t(G) = 2$ if and only if $G = K_2$.

THEOREM 1.5. [**4**] No cutvertex of a connected graph G belongs to any minimum detour monophonic set of G.

Throught this paper G denotes a connected graph with at least two vertices.

2. Forcing Total Detour Monophonic Sets

DEFINITION 2.1. *Let G be a connected graph and let S be a minimum total detour monophonic set of G. A subset T of a minimum total detour monophonic set S of G is a forcing total detour monophonic subset for S if S is the unique minimum total detour monophonic set containing T. A forcing total detour monophonic subset for S of minimum cardinality is a minimum forcing total detour monophonic subset of S. The forcing total detour monophonic number $f_{tdm}(S)$ in G is the cardinality of a minimum forcing total detour monophonic subset of S. The forcing total detour monophonic number of G is $f_{tdm}(G) = min\{f_{tdm}(S)\}$, where the minimum is taken over all minimum total detour monophonic sets S in G.*

EXAMPLE 2.1. *For the graph G given in Figure 1.1, $S_1 = \{u, v, x, y\}$, $S_2 = \{u, v, y, z\}$, $S_3 = \{t, u, x, y\}$ and $S_4 = \{t, u, y, z\}$ are the minimum total detour monophonic sets of G. It is clear that $f_{tdm}(S_1) = 2$, $f_{tdm}(S_2) = 2$, $f_{tdm}(S_3) = 2$ and $f_{tdm}(S_4) = 2$ so that $f_{tdm}(G) = 2$. For the graph G given in Figure 2.1, $S' = \{v_1, v_2, v_3\}$ and $S'' = \{v_1, v_3, v_4\}$ are the minimum total detour monophonic sets of G. Clearly, $f_{tdm}(S') = 1$ and so $f_{tdm}(G) = 1$.*

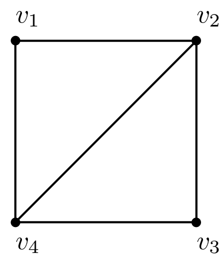

Figure 2.1: G

The next theorem follows immediately from the definitions of the total detour monophonic number and forcing total detour monophonic number of a graph G.

THEOREM 2.1. *For a connected graph G, $0 \leqslant f_{tdm}(G) \leqslant dm_t(G) \leqslant p$.*

REMARK 2.1. *The bounds in Theorem 2.1 are sharp. By Theorem 1.2, for the complete graph $K_p (p \geqslant 2)$, $dm_t(K_p) = p$, also $V(K_p)$ is the unique total detour monophonic set of K_p and so $f_{tdm}(K_p) = 0$. The inequalities in Theorem 2.1 are strict. For the graph G given in Figure 2.1, $dm_t(G) = 3$ and $f_{tdm}(G) = 1$. Thus $0 < f_{tdm}(G) < dm_t(G) < p$.*

The following theorem is an easy consequence of the definitions of the toal detour monophonic number and forcing total detour monophonic number. In fact, the theorem characterizes graphs G for which the lower bound in Theorem 2.1 is attained and also graphs G for which $f_{tdm}(G) = 1$ and $f_{tdm}(G) = dm_t(G)$.

THEOREM 2.2. *Let G be a connected graph . Then*
(i) $f_{tdm}(G) = 0$ *if and only if G has a unique minimum total detour monophonic set.*
(ii) $f_{tdm}(G) = 1$ *if and only if G has at least two minimum total detour monophonic sets, one of which is a unique minimum total detour monophonic set containing one of its elements, and*
(iii) $f_{tdm}(G) = dm_t(G)$ *if and only if no minimum total detour monophonic set of G is the unique minimum total detour monophonic set containing any of its proper subsets.*

DEFINITION 2.2. *A vertex v of a connected graph G is said to be a total detour monophonic vertex of G if v belongs to every minimum total detour monophonic set of G.*

We observe that if G has a unique minimum total detour monophonic set S, then every vertex in S is a total detour monophonic vertex of G. Also, if x is an extreme vertex of G or a support vertex of G, then x is a total detour monophonic vertex of G. For the graph G given in Figure 2.1, v_1 and v_3 are the total detour monophonic vertices of G.

The following theorem and corollary follows immediately from the definitions of total detour monophonic vertex and forcing total detour monophonic subset of G.

THEOREM 2.3. *Let G be a connected graph and let $\mho_{dm}$ be the set of relative complements of the minimum forcing total detour monophonic subsets in their respective minimum total detour monophonic sets in G. Then $\bigcap_{F \in \mho_{dm}} F$ is the set of total detour monophonic vertices of G.*

COROLLARY 2.1. *Let G be a connected graph and let S be a minimum total detour monophonic set of G. Then no total detour monophonic vertex of G belongs to any minimum forcing total detour monophonic subset of S.*

THEOREM 2.4. *Let G be a connected graph and let M be the set of all total detour monophonic vertices of G. Then $f_{tdm}(G) \leqslant dm_t(G) - |M|$.*

PROOF. Let S be any minimum total detour monophonic set of G. Then $dm_t(G) = |S|$, $M \subseteq S$ and S is the unique minimum total detour monophonic set containing $S - M$. Thus $f_{tdm}(G) \leqslant |S - M| = |S| - |M| = dm_t(G) - |M|$. □

COROLLARY 2.2. *If G is a connected graph with l extreme vertices and k support vertices, then $f_{tdm}(G) \leqslant dm_t(G) - (l + k)$.*

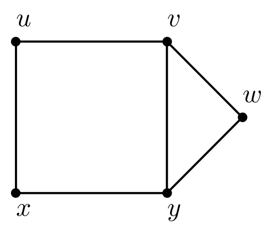

Figure 2.2: G

REMARK 2.2. *The bound in Theorem 2.4 is sharp. For the graph G given in Figure 2.1, $dm_t(G) = 3$ and $f_{tdm}(G) = 1$. Also, $M = \{v_1, v_3\}$ is the set of all total detour monophonic vertices of G and so $f_{tdm}(G) = dm_t(G) - |M|$. Also the inequality in Theorem 2.4 can be strict. For the graph G given in Figure 2.2, $S_1 = \{u, v, w\}$ and $S_2 = \{x, y, w\}$ are the minimum total detour monophonic sets of G and so that $dm_t(G) = 3$. Since S_1 is the unique minimum total detour monophonic set contains the subset $\{u\}$ so that $f_{tdm}(S_1) = 1$ and S_2 is the unique minimum total detour monophonic set contains the subset $\{x\}$ so that $f_{tdm}(S_2) = 1$. Hence, we have $f_{tdm}(G) = 1$. Also, the vertex w is the unique total detour monophonic vertex of G, we have $f_{tdm}(G) < dm_t(G) - |M|$.*

THEOREM 2.5. *Let G be a connected graph and let S be a minimum total detour monophonic set of G. Then no cutvertex of G (which is not a support vertex) belongs to any minimum forcing total detour monophonic subset of S.*

PROOF. Let v be a cutvertex of G which is not a support vertex. By Theorems 1.1 and 1.5, v does not belong to any minimum total detour monophonic set of G. Since any minimum forcing total detour monophonic subset of S is a subset of S, then it is clear. □

The next result follows from Theorem 1.4.

THEOREM 2.6. *If G is a connected graph with $dm_t(G) = 2$, then $f_{tdm}(G) = 0$.*

Now, we proceed to determine the forcing total detour monophonic number of certain classes of graphs.

THEOREM 2.7. *For any cycle $C_n (n \geqslant 4)$, $f_{tdm}(C_n) = 3$.*

PROOF. Let $C_n : v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1$ be a cycle of order n. It is easily observed that any three consecutive vertices of C_n is a minimum total detour monophonic set of C_n. Then clearly no minimum total detour monophonic set of C_n is the unique minimum total detour monophonic set containing any of its proper subsets. Hence by Theorem 2.2(iii), $f_{tdm}(C_n) = 3$. □

THEOREM 2.8. *For any complete graph $G = K_p (p \geqslant 2)$ or any non-trivial tree $G = T$, $f_{tdm}(G) = 0$.*

Proof. For $G = K_p$, it follows from Theorem 1.2 that the set of all vertices of G is the unique minimum total detour monophonic set of G. Now, it follows from Theorem 2.2 (i) that $f_{tdm}(G) = 0$. If G is a non-trivial tree, then by Theorem 1.3, the set of all endvertices and support vertices of G is the unique minimum total detour monophonic set of G and so by Theorem 2.2 (i), $f_{tdm}(G) = 0$. □

Theorem 2.9. *For the complete bipartite graph* $G = K_{m,n}(m, n \geqslant 2)$,

$$f_{tdm}(G) = \begin{cases} 1 & \text{if } 2 = m < n \\ 3 & \text{if } 2 = m = n \\ 4 & \text{if } 3 \leqslant m \leqslant n \end{cases}$$

Proof. Let $U = \{u_1, u_2, \dots, u_m\}$ and $W = \{w_1, w_2, \dots, w_n\}$ be the bipartition of G, where $m \leqslant n$. We prove this theorem by considering four cases.
Case 1. $2 = m = n$. Since G is a cycle of order 4, by Theorem 2.7, we have $f_{tdm}(G) = 3$.
Case 2. $2 = m < n$. For any $j(1 \leqslant j \leqslant n)$, $S_j = U \cup \{w_j\}$ is a minimum total detour monophonic set of G. Since $n \geqslant 3$, then by Theorem 2.2(ii), we have $f_{tdm}(G) = 1$.
Case 3. If $3 = m = n$, then any minimum total detour monophonic set of G is of the following forms: (i) $U \cup \{w_j\}$ for some $j(1 \leqslant j \leqslant n)$, (ii) $W \cup \{u_i\}$ for some $i(1 \leqslant i \leqslant m)$, or (iii) any set got by choosing any two elements from each of U and W. If $3 = m < n$, then any minimum total detour monophonic set of G is either $U \cup \{w_j\}$ for some $j(1 \leqslant j \leqslant n)$, or any set got by choosing any two elements from each of U and W. Hence in both cases, we have $dm_t(G) = 4$. Clearly, no minimum total detour monophonic set of G is the unique minimum total detour monophonic set containing any of its proper subsets. Then by Theorem 2.2(iii), we have $f_{tdm}(G) = dm_t(G) = 4$.
Case 4. $4 = m \leqslant n$. Then any minimum total detour monophonic set is got by choosing any two elements from each of U and W, and G has at least two minimum total detour monophonic sets. Hence $dm_t(G) = 4$. Clearly, no minimum total detour monophonic set of G is the unique minimum total detour monophonic set containing any of its proper subsets. Then by Theorem 2.2(iii), we have $f_{tdm}(G) = dm_t(G) = 4$. □

Theorem 2.10. *For every pair a, b of positive integers with* $0 \leqslant a < b$ *and* $b > 2a+1$, *there exists a connected graph G such that* $f_{tdm}(G) = a$ *and* $dm_t(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 2.8, $f_{tdm}(G) = 0$ and by Theorem 1.2, $dm_t(G) = b$. Thus we assume that $0 < a < b$.

For each i with $1 \leqslant i \leqslant a$, let $C_i : u_{i,1}, u_{i,2}, u_{i,3}, u_{i,4}, u_{i,1}$ be a cycle of order 4. Let $K_{1,b-2a-1}$ be a star with the cutvertex x and $V(K_{1,b-2a-1}) = \{x, v_1, v_2, \cdots, v_{b-2a-1}\}$. The graph G is obtained from $C_i(1 \leqslant i \leqslant a)$ and $K_{1,b-2a-1}$ by joining the vertices x and $u_{i,1}$; and by joining the vertices $u_{i,2}$ and $u_{i,4}$ $(1 \leqslant i \leqslant a)$. The graph G is shown in Figure 2.3. Let $S = \{v_1, v_2, \cdots, v_{b-2a-1}, u_{1,3}, u_{2,3}, \cdots u_{a,3}, x\}$ be the set of all extreme vertices and support vertex of G. By Theorem 1.1, every total detour monophonic set of G contains S. It is easily verified that S is not a total detour monophonic set of G. We observe that every minimum total detour monophonic set of G contains exactly one vertex from $\{u_{i,2}, u_{i,4}\}$ for every $i(1 \leqslant i \leqslant a)$. Hence $dm_t(G) \geqslant b - a + a = b$. On the other hand, $S' = S \cup \{u_{1,2}, u_{2,2}, \cdots, u_{a,2}\}$ is a total detour monophonic set of G, it follows that $dm_t(G) \leqslant b$. Thus $dm_t(G) = b$.

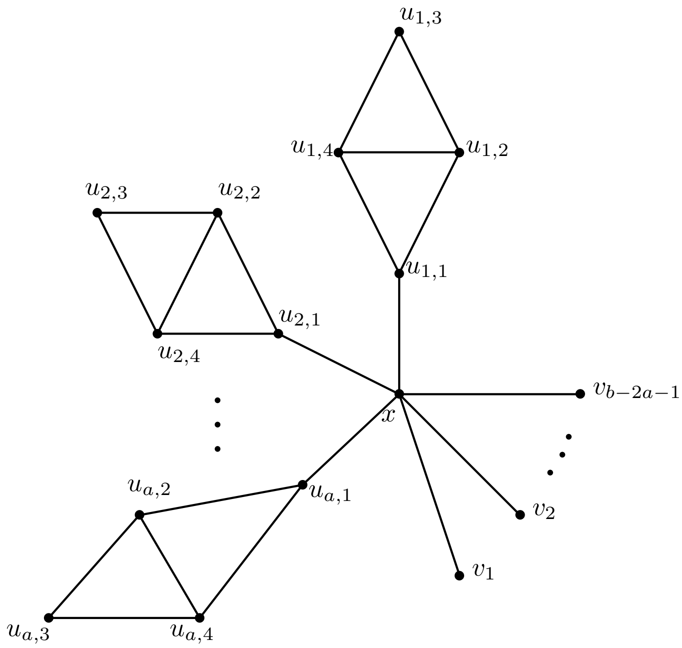

Figure 2.3: G

Next we show that $f_{tdm}(G) = a$. It is observed that S is the set of all total detour monophonic vertices of G. Then by Theorem 2.4, $f_{tdm}(G) \leqslant dm_t(G) - |S| = b - (b - a) = a$. Now, since $dm_t(G) = b$ and every minimum total detour monophonic set of G contains S, it is easily seen that every minimum total detour monophonic set S_1 of G is of the form $S \cup \{x_1, x_2, \cdots, x_a\}$, where $x_i \in \{u_{i,2}, u_{i,4}\}$ for every $i(1 \leqslant i \leqslant a)$. Let T be any proper subset of S_1 with $|T| < a$. Then there is a vertex $x \in S_1 - S$ such that $x \notin T$. If $x = u_{i,2}(1 \leqslant i \leqslant a)$, then $S_2 = (S_1 - \{u_{i,2}\}) \cup \{u_{i,4}\}$ is a minimum total detour monophonic set of G containing T. Similarly, if $x = u_{i,4}(1 \leqslant i \leqslant a)$, then $S_3 = (S_1 - \{u_{i,4}\}) \cup \{u_{i,2}\}$ is a minimum total detour monophonic set of G containing T. Thus S_1 is not the unique minimum total detour monophonic set of G containing T and so T is not a forcing total detour monophonic subset of S_1. Since this is true for all minimum total detour monophonic sets of G, it follows that $f_{tdm}(G) \geqslant a$ and so $f_{tdm}(G) = a$. $\square$

References

[1] F. Buckley and F. Harary. *Distance in Graphs*, Addison-Wesley, Redwood City, CA, 1990.
[2] F. Harary. *Graph Theory*. Addison-Wesley, 1969.
[3] P. Titus and K. Ganesamoorthy. On the Detour Monophonic Number of a Graph. *Ars Combinatoria*, **129**(2016), 33-42.
[4] P. Titus, K. Ganesamoorthy and P. Balakrishnan. The Detour Monophonic Number of a Graph. *J. Combin. Math. Combin. Comput.*, **84**(2013), 179-188.
[5] A.P. Santhakumaran, P. Titus, and K. Ganesamoorthy. The Total Detour Monophonic Number of a Graph, *Proyecciones Journal of Mathematics*, (to appear).

Received by editors 07.10.2016; Available online 08.05.2017.

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