BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. 8(2018), 189-202
DOI: 10.7251/BIMVI1801189RA

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ON FIELD Γ-SEMIRING AND COMPLEMENTED Γ-SEMIRING WITH IDENTITY

Marapureddy Murali Krishna Rao

Abstract. In this paper we study the properties of structures of the semigroup $(M, +)$ and the $\Gamma-$semigroup M of field $\Gamma-$semiring M, totally ordered $\Gamma-$semiring M and totally ordered field $\Gamma-$semiring M satisfying the identity $a + a\alpha b = a$ for all $a, b \in M, \alpha \in \Gamma$and we also introduce the notion of complemented $\Gamma-$semiring and totally ordered complemented $\Gamma-$semiring. We prove that, if semigroup $(M, +)$ is positively ordered of totally ordered field $\Gamma-$semiring satisfying the identity $a + a\alpha b = a$ for all $a, b \in M, \alpha \in \Gamma$, then Γ-semigroup M is positively ordered and study their properties.

1. Introduction

In 1995, Murali Krishna Rao [5, 6, 7] introduced the notion of a Γ-semiring as a generalization of Γ-ring, ring, ternary semiring and semiring. The set of all negative integers $\mathbf{Z}^-$ is not a semiring with respect to usual addition and multiplication but $\mathbf{Z}^-$ forms a Γ-semiring where $\Gamma = Z$. Historically semirings first appear implicitly in Dedekind and later in Macaulay, Neither and Lorenzen in connection with the study of a ring. However semirings first appear explicitly in Vandiver, also in connection with the axiomatization of Arithmetic of natural numbers. Semirings have been studied by various researchers in an attempt to broaden techniques coming from semigroup theory, ring theory or in connection with applications. The concept of semirings was first introduced by Vandiver [13] in 1934. However the developments of the theory in semirings have been taking place since 1950. Semirings abound in the Mathematical world around us. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, i.e.,

$$a(b + c) = ab + ac, \ (a + b)c = ac + bc, \text{ for all } a, b, c \in S.$$

2010 *Mathematics Subject Classification.* 20M10, 16Y60.
Key words and phrases. Γ-semigroup, Γ-semiring, field Γ-semiring, totally ordered field Γ-semiring, complemented Γ-semiring, totally ordered complemented Γ-semiring.

A semiring is one of the fundamental structures in Mathematics. Indeed the first Mathematical structure we encounter the set of natural numbers is a semiring. The theory of semirings and ordered semirings have wide applications in linear and combinatorial optimization problems such as path problems, transformation and assignment problems, matching problems and Eigen value problems. The theory of ordered semirings is very popular since it has wide applications in the theory of computer sciences, optimization theory and theoretical physics. Satyanaraya [11] studied additive semigroup structure of semiring and ordered semiring. Complemented elements play an important role in the study of lattices. Such elements play an important part in the semiring representation of the semantics of computer programmes. In structure, semirings lie between semigroups and rings. Vasanthi et al. [14, 15] studied semiring satisfying the identity Hanumanthachari and Venuraju [4] studied the additive semigroup structure of semiring. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semirings are useful in the areas of theoretical computer science as well as in the solutions of graph theory and optimization theory in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches.

As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [10] in 1964. In 1981 Sen [12] introduced the notion of $\Gamma-$semigroup as a generalization of semigroup. The notion of Ternary algebraic system was introduced by Lehmer [2] in 1932, Lister [3] introduced ternary ring. Dutta & Kar [1] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. Also as a generalization of $\Gamma-$ring, ring, ternary semiring and semiring, The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. Murali Krishna Rao and Venkateswarlu [8, 9] introduced the notion $\Gamma-$incline, zero divisors free $\Gamma-$semiring and field $\Gamma-$semiring and studied properties of regular $\Gamma-$incline and field $\Gamma-$semiring.

In this paper we study the properties of additive structure $(M, +)$ and Γ - semigroup structure of field Γ-semiring M satisfying the identity $a + a\alpha b = a$, for all $a, b \in M$, $\alpha \in \Gamma$ and we also introduce the notion of complemented Γ-semiring and totally ordered complemented Γ-semiring and study their properties.

2. Preliminaries

In this section we recall some important definitions introduced by pioneers in this field earlier that will be required to this paper.

Definition 2.1. *A semigroup M is a non-empty set equipped with a binary operation $'\cdot'$, which is associative.*

Definition 2.2. *A semigroup $(M, +)$ is said to be band if $a+a = a$, for all $a \in M$.*

Definition 2.3. *A semigroup $(M, +)$ is said to be rectangular band if $a+b+a = a$, for all $a, b \in M$.*

Definition 2.4. *A semigroup* $(M, \cdot)$ *is said to be partially ordered if there exist a relation* $\leqslant$ *on* M *satisfying the following.*

(i). $a \leqslant a$, *for all* $a \in M$
(ii). $a \leqslant b, b \leqslant a \Rightarrow a = b$, *for all* $a, b \in M$
(iii). $a \leqslant b, b \leqslant c \Rightarrow a \leqslant c$
(iv). $a \leqslant b \Rightarrow ac \leqslant bc$ *and* $ca \leqslant cb$, *for all* $a, b, c \in M$.

Partially ordered semigroup may also be denoted by $(M, \cdot, \leqslant)$.

Definition 2.5. *A partially ordered semigroup in which every two elements are comparable is said to be totally ordered semigroup .*

Definition 2.6. *A totally ordered semigroup* $(M, \cdot)$ *is said to be non-negatively ordered (non-positively ordered) if* $x^2 \geqslant x$ ($x^2 \leqslant x$), *for all* $x \in M$.

Definition 2.7. *A totally ordered semigroup* $(M, +)$ *is said to be non-negatively ordered (non-positively ordered) if* $x + x \geqslant x$ $(x + x \leqslant x)$, *for all* $x \in M$.

Definition 2.8. *A semigroup* $(M, \cdot)$ *is said to be positively ordered (negatively ordered)if* $xy \geqslant x$ *and* $xy \geqslant y$ $(xy \leqslant x$ *and* $xy \leqslant x)$, *for all* $x, y \in M$.

Definition 2.9. *A semiring* $(M, +, \cdot)$ *is an algebra with two binary operation* $' + '$ *and* $' \cdot '$ *such that* $(M, +)$ *and* $(M, \cdot)$ *are semigroups and the following distributive laws hold*

$$x(y + z) = xy + xz, \ (x + y)z = xz + yz,$$

for all $x, y, z \in M$.

Definition 2.10. *A semiring* $(M, +, \cdot)$ *is said to be totally ordered semiring if there exists a partial order* $\leqslant$ *on* M *such that*

(i). $(M, +)$ *is a totally ordered semigroup*
(ii). $(M, \cdot)$ *is a totally ordered semigroup*

It is usually denoted by $(M, +, \cdot, \leqslant)$.

Definition 2.11. *A semiring* $(M, +, \cdot)$ *is said to be mono semiring if* $a + b = ab$, *for all* $a, b \in M$

Definition 2.12. *Let* M *and* Γ *be two non-empty sets. Then* M *is called a* $\Gamma-$*semigroup if it satisfies*

(i) $x\alpha y \in M$
(ii) $x\alpha(y\beta z) = (x\alpha y)\beta z$ *for all* $x, y, z \in M, \alpha, \beta \in \Gamma$.

Definition 2.13. *A* $\Gamma-$*semigroup* M *is said to be commutative if* $a\alpha b = b\alpha a$, *for all* $a, b \in M$ *and* $\alpha \in \Gamma$.

Definition 2.14. *Let* M *be a* $\Gamma-$*semigroup. An element* $a \in M$ *is said to be idempotent of* M *if there exists* $\alpha \in \Gamma$ *such that* $a = a\alpha a$ *and* a *is also said to be* α *idempotent.*

Definition 2.15. *Let* M *be a* $\Gamma-$*semigroup. If every element of* M *is an idempotent of* M *then* $\Gamma-$*semigroup* M *is said to be band*

DEFINITION 2.16. *A $\Gamma-$semigroup M is called a rectangular band if for every pair $a, b \in M$ there exist $\alpha, \beta \in \Gamma$ such that $a\alpha b\beta a = a$.*

DEFINITION 2.17. *A totally ordered $\Gamma-$semigroup M is said to be positively ordered (negatively ordered)if $x\alpha y \geqslant x$ and $x\alpha y \geqslant y$ ($x\alpha y \leqslant x$ and $x\alpha y \leqslant x$), for all $x, y \in M, \alpha \in \Gamma$.*

DEFINITION 2.18. *A $\Gamma-$semigroup M is called $\Gamma-$semiring M if $(M, +), (\Gamma, +)$ are semigroups and satisfies the following conditions.*

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$
(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,*for all* $a, b, c \in M, \alpha, \beta \in \Gamma$.

EXAMPLE 2.1. Let S be a semiring and $M_{p,q}(S)$ denote the additive semigroup of all $p \times q$ matrices whose entries are from S. Then $M_{p,q}(S)$ is a $\Gamma-$semiring with $\Gamma = M_{p,q}(S)$ and the ternary operation defined by the usual matrix multiplication as $x\alpha y = x(\alpha^t)y$, where α^t denotes the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

DEFINITION 2.19. *A $\Gamma-$semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$. and $\alpha \in \Gamma$.*

DEFINITION 2.20. *A $\Gamma-$semiring M is said to be commutative $\Gamma-$semiring if $x\alpha y = y\alpha x, x + y = y + x$ for all $x, y \in M$ and $\alpha \in \Gamma$.*

DEFINITION 2.21. *Let M be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.*

DEFINITION 2.22. *In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1 (a\alpha b = 1)$.*

DEFINITION 2.23. *In a $\Gamma-$semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.*

DEFINITION 2.24. *Let M be a $\Gamma-$semiring is said to satisfy left(right)cancelation law if and only if $r, s, t \in M, r \neq 0, \alpha \in \Gamma$ such that $r\alpha s = r\alpha t$ ($s\alpha r = t\alpha r$) then $s = t$.*

DEFINITION 2.25. *A $\Gamma-$semiring M is called an ordered $\Gamma-$semiring if it admits a compatible relation $\leqslant$, i.e. $\leqslant$ is a partial ordering on M satisfying the following conditions. If $a \leqslant b$ and $c \leqslant d$ then*

(i) $a + c \leqslant b + d$
(ii) $a\alpha c \leqslant b\alpha d$
(iii) $c\alpha a \leqslant d\alpha b$, *for all* $a, b, c, d \in M$ *and* $\alpha \in \Gamma$.

EXAMPLE 2.2. Let $M = [0, 1]$, $\Gamma = N$ and $' + '$ and tbe ternary operation defined by

$$x + y = \max\{x, y\}, \quad x\gamma y = \min\{x, \gamma, y\} \text{ for all } x, y \in M \text{ and } \gamma \in \Gamma.$$

Then M is an ordered $\Gamma-$semiring with respect to usual ordering.

Definition 2.26. *An ordered $\Gamma-$semiring in which every two elements are comparable is said to be totally ordered $\Gamma-$semiring. .*

Definition 2.27. *A $\Gamma-$semiring M is said to be field $\Gamma-$semiring if M is a commutative $\Gamma-$semiring with unity 1 and every nonzero element of M is invertible.*

Example 2.3. Let M be a set of all rational numbers and $\Gamma = M$ be a commutative semigroup with respect to usual addition. If define the mapping $M \times \Gamma \times M \to M$ by $a\alpha b$ as usual multiplication for all $a, b \in M$ and $\alpha \in \Gamma$, then M is a field Γ-semiring.

Definition 2.28. *A $\Gamma-$semiring M is said to be mono $\Gamma-$semiring if M is a commutative $\Gamma-$semiring with unity 1 and for every pair $a, b \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma b = a + b$.*

Definition 2.29. *Let M be a $\Gamma-$semiring. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α - idempotent.*

Definition 2.30. *Let M be a $\Gamma-$semiring.If every element of M is an idempotent of M then M is said to be idempotent $\Gamma-$semiring M.*

Definition 2.31. *A $\Gamma-$semiring M with zero element is said to be zero sum free $\Gamma-$semiring if $x + x = 0$, for all $x \in M$.*

3. Field Γ-semiring and totally ordered Γ-semiring

In this section, we study the properties of additive semigroup structure and $\Gamma-$semigroup structure of field Γ-semiring, totally ordered Γ-semiring and totally ordered field Γ-semiring satisfying the identity.

Definition 3.1. *In a totally ordered Γ-semiring M we define*

(i) *Semigroup $(M, +)$ is said to be positively ordered (negatively ordered) if*
$$a + b \geqslant a, b(a + b \leqslant a, b) \text{ for all } \alpha \in \Gamma, a, b \in M.$$
(ii) *$\Gamma-$Semigroup M is said to be positively ordered (negatively ordered) if*
$$a\alpha b \geqslant a, b(a\alpha b \leqslant a, b) \text{ for all } \alpha \in \Gamma, a, b \in M.$$

Definition 3.2. *In a totally ordered Γ-semiring M we define*

(i) *Semigroup $(M, +)$ is said to be non-negatively ordered(non-positively ordered) if holds*
$$a + a \geqslant a \ \ (a + a \leqslant a) \text{ for all } \alpha \in \Gamma \text{ and } a \in M.$$
(ii) *Γ-semigroup M is said to be non positively ordered (non-negatively ordered) if holds*
$$a\alpha a \geqslant a \ \ (a\alpha a \leqslant a) \text{ for all } \alpha \in \Gamma \text{ and } a \in M.$$

Definition 3.3. *An element x in a totally ordered Γ-semiring M is said to be minimal (maximal) if $x \leqslant a$ $(x \geqslant a)$ holds for all $a \in M$.*

THEOREM 3.1. *Let M be a field Γ-semiring satisfying the identity $a + a\alpha b = a$ for all $0 \neq a$, $b \in M$ and $\alpha \in \Gamma$. Then $1 + a = a$ for all $0 \neq a \in M$.*

PROOF. Let M be a field Γ-semiring satisfying the identity $a + a\alpha b = a$ for all $0 \neq a$, $b \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha a^{-1} = 1$. Thus $a\alpha a^{-1} + a = a$ and $1 + a = a$. Hence the theorem. □

THEOREM 3.2. *Let M be a field Γ-semiring satisfying the identity and $a+a\alpha a = a$ for all $0 \neq a \in M$ and $\alpha \in \Gamma$. Then $1 + a = a$, and for all $0 \neq a \in M$. and for each $b \in M$ there exists $\delta \in \Gamma$ such that $b + a\delta b = b$.*

PROOF. Let M be a Γ-semiring satisfying $a + a\alpha a = a$ for all $a \in M$ and $\alpha \in \Gamma$. Then there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$ and there exists $\delta \in \Gamma$ such that $1\delta b = b$. For all $a \in M$ and $\alpha \in \Gamma$ we have

$$\begin{aligned} a + a\alpha a = a &\Rightarrow a\gamma 1 + a\gamma a = a \\ &\Rightarrow a\gamma(1 + a) = a\gamma 1 \\ &\Rightarrow 1 + a = 1 \\ &\Rightarrow 1\delta b + a\delta b = 1\delta b \\ &\Rightarrow b + a\delta b = b. \end{aligned}$$

Hence the theorem. □

THEOREM 3.3. *Let M be a field $\Gamma-$semiring satisfying the identity $a+a\alpha b = a$ for all $0 \neq a$, $b \in M$ and $\alpha \in \Gamma$. Then $\Gamma-$semigroup M is a band.*

PROOF. Let M be a field $\Gamma-$semiring. By Theorem 3.1 we have $1 + a = a$ for all $a \in M$. Since $a \in M$ there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$.

$$a\alpha a = a\alpha(1 + a) = a\alpha 1 + a\alpha a = a + a\alpha a = a.$$

Hence the theorem. □

THEOREM 3.4. *Let M be a field $\Gamma-$semiring satisfying the identity $a + a\alpha b = a$, for all $0 \neq a, b \in M, \alpha \in \Gamma$. Then additive semigroup $(M, +)$ is a band.*

PROOF. Let M be a field $\Gamma-$semiring. By Theorem 3.1 we have $a + 1 = a$ for all $0 \neq a \in M$. Let $0 \neq a \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$. Further on

$$a = a\alpha 1 = a\alpha(1 + 1) = a\alpha 1 + a\alpha 1 = a + a.$$

Hence an additive semigroup $(M, +)$ of $\Gamma-$semiring M is a band. □

THEOREM 3.5. *Let M be a field $\Gamma-$semiring satisfying the identity $a+a\alpha b = a$ for all a, $b \in M$ and $\alpha \in \Gamma$. Then $a + a\gamma b + b = a$ for all a, $b \in M$ and for some $\gamma \in \Gamma$.*

PROOF. Let M be a field $\Gamma-$semiring. By Theorem 3.1 we have $a + 1 = a$, for all $a \in M$. Suppose $b \in M$. Since $b \in M$ there exists $\gamma \in \Gamma$ such that $1\gamma b = b$. Now we have

$$a + a\gamma b + b = a + a\gamma b + 1\gamma b = a + (a + 1)\gamma b = a + a\gamma b = a.$$

Hence $a + a\gamma b + b = a$, for all $a, b \in M$. □

THEOREM 3.6. *Let M be a field $\Gamma-$semiring satisfying the identity $a\alpha b + a = a$ for all $a \in M$ and $\alpha \in \Gamma$. Then $a = a + a\alpha x\beta a$ for all α, $\beta \in \Gamma$ and $x \in M$.*

PROOF. Let M be a field $\Gamma-$semiring. By Theorem 3.1 we have $a + 1 = a$ for all $0 \neq a \in M$. Let a, $x \in M$ and α, $\beta \in \Gamma$. Since $a \in M$, there exists $\delta \in \Gamma$ such that $a\delta 1 = a$.

$$a + a\alpha x\beta a = a\delta 1 + a\delta 1\alpha x\beta a = a\delta(1 + 1\alpha x\beta a) = a\delta 1 = a.$$

Hence $a = a + a\alpha x\beta a$, for all $\alpha, \beta \in \Gamma, x \in M$. □

THEOREM 3.7. *Let M be a field $\Gamma-$semiring satisfying $a + a\alpha b = a$ for all $0 \neq a$, $b \in M$ and $\alpha \in \Gamma$. If for every pair a, $b \in M$ there exists $\gamma \in \Gamma$ such that $1\gamma a = a$ and $1\gamma b = b$ then M is mono $\Gamma-$semiring.*

PROOF. By Theorem 3.1 we have $1 + a = a$ for all $a \in M$ since M is a field $\Gamma-$semiring. Let a, $b \in M$.. Then there exists $\gamma \in \Gamma$ such that $1\gamma a = a$ and $1\gamma b = b$.

$$\begin{aligned} a\gamma b &= (1 + a)\gamma(1 + b) \\ a\gamma b &= 1\gamma 1 + a\gamma 1 + 1\gamma b + a\gamma b \\ \Rightarrow a\gamma b &= 1\gamma 1 + 1\gamma a + 1\gamma b + a\gamma b \\ &= 1\gamma(1 + a) + b + a\gamma b \\ &= 1\gamma a + b + a\gamma b \\ &= a + b + b\gamma a \\ &= a + b. \end{aligned}$$

Hence M is a mono $\Gamma-$semiring. □

THEOREM 3.8. *If M is a $\Gamma-$semiring with unity satisfying the identity $a\alpha b + a = a$ for all $a \in M$ and $\alpha \in \Gamma$ for all $a \in M$ and suppose $(M, +)$ is right cancellative, then $\mid M \mid = 1$.*

PROOF. Let $a \in M$. Then $a + 1 = 1 \Rightarrow a + 1 = 1 + 1 \Rightarrow a = 1$. Hence $\mid M \mid = 1$. □

THEOREM 3.9. *Let M be a totally ordered $\Gamma-$semiring with unity 1. Then additive semigroup $(M, +)$ is non negatively ordered or non positively ordered .*

PROOF. Let M be a totally ordered $\Gamma-$semiring with unity 1 and $x \in M$. Then $1 + 1 \geqslant 1$ or $1 + 1 \leqslant 1$. Thus

$$\begin{aligned} & x\alpha(1 + 1) \geqslant x\alpha 1 \text{ or } x\alpha(1 + 1) \leqslant x\alpha 1, \text{ for all } \alpha \in \Gamma \\ \Rightarrow\ & x\alpha 1 + x\alpha 1 \geqslant x\alpha 1 \text{ or } x\alpha 1 + x\alpha 1 \leqslant x\alpha 1, \text{ for all } \alpha \in \Gamma \\ \Rightarrow\ & x + x \geqslant x \text{ or } x + x \leqslant x. \end{aligned}$$

Hence additive semigroup $(M, +)$ is non negatively ordered or non positively ordered. □

THEOREM 3.10. *Let M be a totally ordered $\Gamma-$semiring satisfying $a\alpha a + a = a,$ for all $a \in M, \alpha \in \Gamma$. If $(M,+)$ is non-negatively ordered then $\Gamma-$semigroup M is non-positively ordered.*

PROOF. Suppose semigroup $(M,+)$ of $\Gamma-$semiring M is non-negatively ordered. We have $a\alpha a + a = a,$ for all $a \in M, \alpha \in \Gamma$. Now $a = a\alpha a + a \geqslant a\alpha a \Rightarrow a \geqslant a\alpha a$. Hence $\Gamma-$semigroup M is non-positively ordered. $\square$

Proof of the following theorem is similar to above theorem.

THEOREM 3.11. *Let M be a totally ordered $\Gamma-$semiring satisfying $a\alpha a + a = a,$ for all $a, b \in M, \alpha \in \Gamma$. If $(M,+)$ is a non-positively ordered then $\Gamma-$semigroup M is non-negatively ordered.*

THEOREM 3.12. *Let M be a totally ordered $\Gamma-$semiring satisfying the identity $a\alpha a + a = a,$ for all $a \in M, \alpha \in \Gamma$.*
(i). If $(M,+)$ is positively ordered then $\Gamma-$semigroup M is negatively ordered.
(ii). If $(M,+)$ is negatively ordered then $\Gamma-$semigroup M is positively ordered.

PROOF. (i). Let M be a totally ordered $\Gamma-$semiring satisfying the identity $a\alpha b + a = a$ for all $a,$ $b \in M$ and $\alpha \in \Gamma$ and let $(M,+)$ be positively ordered. Then

$$a = a\alpha b + a \geqslant a\alpha b \Rightarrow \; a \geqslant a\alpha b.$$

Suppose $a\alpha b > b$. Then

$$\begin{aligned} &\Rightarrow a\alpha b + a > b + a \\ &\Rightarrow a > b + a \end{aligned}$$

which is a contradiction to semigroup $(M,+)$ is positively ordered. Hence $a\alpha b \leqslant b$. There fore $a\alpha b \leqslant a, b$. Hence $\Gamma-$semigroup M is negatively ordered.
(ii). Proof is similar to (i). $\square$

THEOREM 3.13. *Let M be a totally ordered $\Gamma-$semiring with unity satisfying $a + a\alpha b = a,$ for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M,+)$ is positively ordered then 1 is the maximal element.*

PROOF. Let semigroup $(M,+)$ be positively ordered semigroup of totally ordered $\Gamma-$semiring with $a + a\alpha b = a,$ for all $a, b \in M, \alpha \in \Gamma$.

$$\begin{aligned} &\Rightarrow 1 + 1\alpha a = 1, \text{ for all } a \in M, \alpha \in \Gamma \\ &\Rightarrow 1 + a = 1 \\ &\Rightarrow 1 = 1 + a \geqslant a, \text{ for all } a \in M. \end{aligned}$$

Hence 1 is the maximal element. $\square$

THEOREM 3.14. *Let M be a totally ordered $\Gamma-$semiring with unity satisfying $a + a\alpha b = a,$ for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M,+)$ is negatively ordered semigroup then 1 is the minimal element of a totally ordered $\Gamma-$semiring M.*

PROOF. Let M be a totally ordered $\Gamma-$semiring with $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Suppose $(M, +)$ is negatively ordered. We have $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Since $b \in M$ there exists $\gamma \in \Gamma$ such that $b\gamma 1 = 1\gamma b = b$.

$$\begin{aligned}&\Rightarrow 1 + 1\alpha b = 1, \text{ for all } \alpha \in \Gamma, b \in M\\ &\Rightarrow 1 + 1\gamma b = 1\\ &\Rightarrow 1 + b = 1\end{aligned}$$

Therefore $1 = 1 + b \leqslant b$ for all $b \in M$. Hence 1 is the minimal element of a $\Gamma-$semiring M. □

THEOREM 3.15. *Let M be a totally ordered $\Gamma-$semiring satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. If semigroup $(M, +)$ is positively ordered then $\Gamma-$semigroup M is negatively ordered.*

PROOF. Let M be a totally ordered $\Gamma-$semiring satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. Suppose $(M, +)$ is positively ordered. We have $a = a + a\alpha b \geqslant a\alpha b$, for all $a, b \in M, \alpha \in \Gamma$. Similarly we can prove $b \geqslant a\alpha b$, for all $a, b \in M, \alpha \in \Gamma$. Hence $\Gamma-$semigroup M is negatively ordered. □

THEOREM 3.16. *If M is a totally ordered mono $\Gamma-$semiring with $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$ and semigroup $(M, +)$ is positively ordered then semigroup $(M, +)$ is left singular.*

PROOF. Suppose M is a totally ordered mono $\Gamma-$semiring. We have $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$.

$$a = a + a\alpha b \geqslant a\alpha b \Rightarrow a \geqslant a + b,$$

since M is a mono $\Gamma-$semiring. Since $(M, +)$ is positively ordered, then $a + b \geqslant a$. Therefore $a = a + b$. Hence the Theorem. □

THEOREM 3.17. *Let M be a totally ordered idempotent $\Gamma-$ semiring with unity 1 and zero element 0. satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$. and $\Gamma-$semigroup M is negatively ordered. If $a, b \in M, a \leqslant b$ and $a+b = b$ then there exists $\alpha \in \Gamma$ such that $a = a\alpha b = b\alpha a = a\alpha a$.*

PROOF. Let $a, b \in M$ and $a \leqslant b$. Since M is a totally ordered idempotent $\Gamma-$semiring, there exists $\alpha \in \Gamma$ such that $a\alpha a = a$

$$\begin{aligned}a \leqslant b \Rightarrow\ & a + b = b\\ \Rightarrow\ & a\alpha(a + b) = a\alpha b\\ \Rightarrow\ & a\alpha a + a\alpha b = a\alpha b\\ \Rightarrow\ & a + a\alpha b = a\alpha b\\ \Rightarrow\ & a = a\alpha b\\ \text{and } a \leqslant b \Rightarrow\ & a\alpha a \leqslant b\alpha a\\ \Rightarrow\ & a \leqslant b\alpha a \leqslant a\\ \Rightarrow\ & a = b\alpha a\end{aligned}$$

Hence $a = a\alpha b = b\alpha a = a\alpha a$. □

Theorem 3.18. *If M is a field $\Gamma-$ semiring satisfying the identity $a\alpha b + a = a,$ for all $a, b \in M, \alpha \in \Gamma.$ with unity 1. If semigroup $(M, +)$ is a negatively ordered then 1 is the greatest element of M.*

Proof. Let M be a field $\Gamma-$ semiring satisfying the identity $a\alpha b + a = a,$ for all $a, b \in M, \alpha \in \Gamma.$ and $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x = x\alpha 1 \leqslant 1$. Hence 1 is the greatest element of M. □

Theorem 3.19. *Let M be a totally ordered field $\Gamma-$semiring satisfying the identity $a\alpha b + a = a,$ for all $a, b \in M, \alpha \in \Gamma$.*
(i). If semigroup $(M, +)$ is a negatively ordered then 1 is the maximal element.
(ii). If semigroup $(M, +)$ is a positively ordered then 1 is the minimal element.

Proof. (i). Suppose semigroup $(M, +)$ is a negatively ordered. We have $a\alpha b + a = a,$ for all $a, b \in M, \alpha \in \Gamma$. Since $a \in M$ there exists $a^{-1} \in M$ there exists such that $a\alpha a^{-1} = 1$. Then

$$\begin{aligned} a\alpha a^{-1} = 1 &\Rightarrow a\alpha a^{-1} + a = a \\ &\Rightarrow 1 + a = a \\ &\Rightarrow a = 1 + a \leqslant 1 \\ &\Rightarrow a \leqslant 1. \end{aligned}$$

Hence 1 is the maximal element.
(ii). Proof is similar to (i) □

Theorem 3.20. *Let M be a totally ordered field $\Gamma-$semiring and satisfying the identity $a\alpha b + a = a,$ for all $a, b \in M, \alpha \in \Gamma$.*
(i). If semigroup $(M, +)$ is a non-negatively ordered then $\Gamma-$semigroup M is a non-negatively ordered.
(ii). If semigroup $(M, +)$ is a non-positively ordered then $\Gamma-$semigroup M is a non-positively ordered.

Proof. (i). Let M be a field $\Gamma-$semiring. We have $1 + a = a,$ for all $a \in M$. Let $a \in M$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} 1 + a = a &\Rightarrow a\gamma 1 + a\gamma a = a\gamma a \\ &\Rightarrow a + a\gamma a = a\gamma a. \end{aligned}$$

Suppose $(M, +)$ is a non-negatively totally ordered. Then $a\gamma a = a + a\gamma a \geqslant a \Rightarrow \ a\gamma a \geqslant a$.
Hence $\Gamma-$semigroup M is a non-negatively ordered.
(ii) Proof is similar to (i). □

Theorem 3.21. *Let M be a totally ordered field $\Gamma-$semiring and satisfying the identity $a\alpha b + a = a,$ for all $a, b \in M, \alpha \in \Gamma$.*
(i). *If semigroup $(M, +)$ is positively ordered then $\Gamma-$semigroup M is positively ordered.*
(ii). *If semigroup $(M, +)$ is negatively ordered then $\Gamma-$semigroup M is a negatively ordered.*

PROOF. (i). Let M be a field $\Gamma-$semiring. We have $1+b=b,$ for all $b \in M$. Let $a \in M$. Since $a \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma 1 = a$.

$$\begin{aligned} 1+b=b &\Rightarrow a\gamma 1 + a\gamma b = a\gamma b \\ &\Rightarrow a + a\gamma b = a\gamma b. \end{aligned}$$

Suppose that $(M,+)$ is a positively ordered. Then

$$a\gamma b = a + a\gamma b \geqslant a \ \Rightarrow \ a\gamma b \geqslant a.$$

Similarly we can prove $a\gamma b \geqslant b$. Hence $\Gamma-$semigroup M is a positively ordered.

(ii) Proof is similar to (i). □

4. Complemented $\Gamma-$semiring

In this section, we introduce the notion of complemented $\Gamma-$semiring and totally ordered complemented $\Gamma-$semiring and study their properties.

DEFINITION 4.1. *An element a of $\Gamma-$semiring M is said to be complemented if there exists an element $b \in M$ such that $a+b=1, b\alpha a=0$ and $a\alpha b=0$, for all $\alpha \in \Gamma$.*

DEFINITION 4.2. *A $\Gamma-$semiring M is said to be complemented if for every element of M is complemented.*

THEOREM 4.1. *Let M be a $\Gamma-$semiring. If b is a complement of a and $a+c=1$ then there exists $\alpha \in \Gamma$ such that $b\alpha c = c\alpha b = b\alpha b = b$.*

PROOF. Let b be the complement of a and $c \in M$ such that $a+c=1$.Then there exists a $\alpha \in \Gamma$ such that $b = b\alpha 1 = 1\alpha b$, $b+a=1,$ *and* $b\gamma a = a\gamma b = 0$, for all $\gamma \in \Gamma$.

$$b = b\alpha(1) = b\alpha(a+c) = b\alpha a + b\alpha c = 0 + b\alpha c = b\alpha c.$$

$$b = 1\alpha b = (a+c)\alpha b = a\alpha b + c\alpha b = 0 + c\alpha b = c\alpha b.$$

$$\begin{aligned} b+a&=1 \\ b\gamma(b+a) &= b\gamma 1, \text{ for all } \gamma \in \Gamma \\ \Rightarrow b\gamma b + b\gamma a &= b\gamma 1, \text{ for all } \gamma \in \Gamma \\ \Rightarrow b\alpha b + b\alpha a &= b\alpha 1 \\ \Rightarrow b\alpha b + 0 &= b\alpha 1 \end{aligned}$$

Therefore $b\alpha b = b$. Hence $b\alpha c = c\alpha b = b\alpha b = b$. □

THEOREM 4.2. *Let M be a zero sum free $\Gamma-$semiring. If $a, b \in M$ are complemented elements of M then $a\alpha b\beta c = 0$, for all $\beta \in \Gamma$, for some $\alpha \in \Gamma$, where c is the complement of a.*

Proof. Let M be a zero sum free $\Gamma-$semiring and c, d be complements of a and b respectively and $\beta \in \Gamma$. Since 1 is the unity, there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$.

$$a\alpha b\beta c + a\alpha d\beta c = a\alpha(b + d)\beta c = a\alpha 1\beta c = a\beta c = 0.$$

Hence $a\alpha b\beta c = 0$. □

Theorem 4.3. *Let M be an idempotent $\Gamma-$semiring. If a is a complement of b then there exists $\gamma \in \Gamma$ such that $a + b\gamma b = b\gamma b + a$.*

Proof. Let M be an idempotent $\Gamma-$semiring. Suppose a is a complement of b.Then $a + b = 1$ and $a\alpha b = b\alpha a = 0$, for all $\alpha \in \Gamma$ and there exists $\gamma \in \Gamma$ such that $a\gamma a = a$.

$$\begin{aligned}(a+b)\alpha(b+a) &= a\alpha(b+a) + b\alpha(b+a), \text{ for all } \alpha \in \Gamma\\ &= a\alpha b + a\alpha a + b\alpha b + b\alpha a, \text{ for all } \alpha \in \Gamma\\ &= a + b\gamma b\\ (a+b)\alpha(b+a) &=(a+b)\alpha b + (a+b)\alpha a, \text{ for all } \alpha \in \Gamma\\ &= a\alpha b + b\alpha b + a\alpha a + b\alpha a, \text{ for all } \alpha \in \Gamma\\ &= b\gamma b + a.\end{aligned}$$

Hence $a + b\gamma b = b\gamma b + a$. □

Definition 4.3. *Let M be a $\Gamma-$semiring. An element $a \in M$ is said to be strongly idempotent if $a = a\alpha a$; for all $\alpha \in \Gamma$.*

Definition 4.4. *Let M be a $\Gamma-$semiring.If every element of M is a strongly idempotent of M then M is said to be strongly idempotent $\Gamma-$semiring M.*

Corollary 4.1. *If M is a strongly idempotent complemented $\Gamma-$semiring then semigroup $(M, +)$ is commutative .*

Theorem 4.4. *If $\Gamma-$semigroup of complemented $\Gamma-$semiring M holds cancellation law then every element in M has an unique complement.*

Proof. Let $\Gamma-$semigroup of complemented $\Gamma-$semiring M holds cancellation law. Suppose b and c are complements of a then $a+b = 1, a+c = 1, a\alpha b = 0, a\alpha c = 0$, for all $\alpha \in \Gamma$. Then

$$\begin{aligned}a\alpha b = 0 = a\alpha c &\Rightarrow a\alpha b = a\alpha c\\ &\Rightarrow b = c.\end{aligned}$$

Hence the Theorem. □

Theorem 4.5. *If M is a complemented $\Gamma-$semiring then M is an idempotent $\Gamma-$semiring.*

Proof. Suppose M is a complemented $\Gamma-$semiring and $a \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a, a + b = 1$ and $a\gamma b = 0,$ for all $\gamma \in \Gamma$.

$$\begin{aligned} a + b = 1 &\Rightarrow a\gamma(a + b) = a\gamma 1, \text{ for all } \gamma \in \Gamma \\ &\Rightarrow a\alpha(a + b) = a\alpha 1 \\ &\Rightarrow a\alpha a + a\alpha b = a\alpha 1 \\ &\Rightarrow a\alpha a + 0 = a\alpha 1 \\ &\Rightarrow a\alpha a = a. \end{aligned}$$

Therefore a is an $\alpha-$idempotent. Hence M is an idempotent $\Gamma-$semiring. □

Theorem 4.6. *Let M be a totally ordered complemented $\Gamma-$semiring. If semigroup $(M, +)$ is a positively ordered semigroup then 0 is the minimal element of M and $\Gamma-$semigroup M is negatively ordered.*

Proof. Let M be a totally ordered complemented $\Gamma-$semiring. Let $a, c \in M$. There exists $b \in M$ such that $a+b = 1, b\alpha a = a\alpha b = 0,$ for all $\alpha \in \Gamma$. Now $a+0 = a$ and $a + 0 \geqslant 0$. So, $a \geqslant 0$. Therefore 0 is the minimal element.

Suppose $\Gamma-$semigroup M is a positively ordered. Then

$$a\alpha c \geqslant \ c,\ a \Rightarrow b\alpha(a\alpha c) \geqslant b\alpha c \quad \text{and} \quad (b\alpha a)\alpha c) \geqslant b\alpha c \Rightarrow 0 \geqslant b\alpha c,$$

which is a contradiction. Therefore $\Gamma-$semigroup M is negatively ordered. Hence the Theorem. □

Theorem 4.7. *Let M be a totally ordered complemented Γ - semiring. If Γ - semigroup M is a positively ordered then 1 is the minimal element of M and $(M, +)$ is negatively ordered .*

Proof. Let M be a totally ordered complemented Γ - semiring .Suppose Γ - semigroup M is a positively ordered and $a \in M$. There exists $\beta \in \Gamma$ such that $a\beta 1 = a \Rightarrow a = a\beta 1 \geqslant 1$. Therefore 1 is the minimal element of M. Suppose $(M, +)$ is a positively ordered semigroup and $a \in M$. Then there exists $c \in M$ such that $a + c = 1 \Rightarrow\ 1 \geqslant a$ and $a \geqslant 1$. Therefore $a = 1$, which is a contradiction. Hence $(M, +)$ is negatively ordered. □

Theorem 4.8. *Let M be a totally ordered complemented $\Gamma-$semiring. If $\Gamma-$semigroup M is a positively ordered then semigroup $(M, +)$ is a band.*

Proof. Let M be a totally ordered complemented $\Gamma-$semiring. Suppose $\Gamma-$semigroup M is positively ordered. Then by Theorem 4.7, $(M, +)$ is negatively ordered and 1 is the minimal element of M and $1 + 1 \geqslant 1$ since 1 is the minimal element. Thus $x\alpha(1 + 1) \geqslant x\alpha 1$ for all $x \in M$ and $\alpha \in \Gamma$ since $x \in M$. So, there exists $\gamma \in \Gamma$ such that $x\gamma 1 = x$ and

$$\begin{aligned} x\gamma(1 + 1) \geqslant x\gamma 1 &\Rightarrow x\gamma 1 + x\gamma 1 \geqslant x\gamma 1 \\ &\Rightarrow x + x \geqslant x. \end{aligned}$$

We have $x + x \leqslant x$, since $(M, +)$ is negatively ordered. Hence $x + x = x$. Therefore $(M, +)$ is a band. □

5. Conclusion:

We studied the properties of structures of the semigroup $(M, +)$ and the Γ -semigroup M of field $\Gamma-$semiring M, totally ordered field $\Gamma-$semiring M complemented $\Gamma-$semiring M and totally ordered complemented $\Gamma-$semiring M.We proved,if $(M, +)$ is negatively ordered (positively ordered) of totally ordered $\Gamma-$semiring (field $\Gamma-$semiring) satisfying the identity $a + a\alpha b = a$, for all $a, b \in M, \alpha \in \Gamma$ then $\Gamma-$semigroup M is positively ordered (positively ordered).

References

[1] T. K. Dutta and S. Kar. On regular ternary semirings. In K. P. Shum, Zhexian Wan, J. P. Zhang (Eds.) *Advances in Algebra. Proceedings of the ICM Satellite Conference in Algebra and Related Topics* (pp. 343-355), World Scientific, 2003.

[2] H. Lehmer. A ternary analogue of abelian groups. *American J. Math.*, **54**(2)(1932), 329-338.

[3] W. G. Lister. Ternary rings. *Tran. of American Math. Soc.*, **154**(1971), 37-55.

[4] J. Hanumanthachari and K. Venuraju. The additive semigroup structure of semiring, *Math. Sem. Notes Kobe Univ.*, **11**(2)(1983), 381-386.

[5] M. Murali Krishna Rao. $\Gamma-$semirings-I. *Southeast Asian Bull. Math.*, **19**(1)(1995), 49–54.

[6] M. Murali Krishna Rao. $\Gamma-$semirings-II. *Southeast Asian Bull. Math.*, **21**(5)(1997), 281–287.

[7] M. Murali Krishna Rao. The Jacobson Radical of a $\Gamma-$semiring. *Southeast Asian Bull. Math.*, **23**(1)(1999), 127–134.

[8] M. Murali Krishna Rao and B. Venkateswarlu. Regular $\Gamma-$incline and field $\Gamma-$semiring. *Novi Sad J. Math.*, **45**(2)(2015), 155–171.

[9] M. Murali Krishna Rao and B. Venkateswarlu. Zero divisors free $\Gamma-$semiring. *Bull. Int. Math. Virtual Inst.*, **8**(1)(2018), 37–43 DOI: 10.7251/BIMVI1801037R.

[10] N. Nobusawa. On a generalization of the ring theory. *Osaka. J.Math.*, **1**(1)(1964), 81–89.

[11] M. Satyanarayana. On the additive semigroup of ordered semirings. *Semigroup forum*, **31**(1)(1985), 193-199.

[12] M. K. Sen. On $\Gamma-$semigroup. *Proceedings of International Conference of Algebra and its Applications*, New Delhi, 1981 (pp. 301308), Lecture Notes in Pure and Appl. Math. 91, Dekker, New York, 1984.

[13] H. S. Vandiver. Note on a Simple type of algebra in which the cancellation law of addition does not hold. *Bull. Amer. Math. Soc.*, **40**(12)(1934), 914-920 .

[14] T. Vasanthi and N. Sulochana. Semiring satisfying the identity. *Int, J. Math. Archive*, **3**(9)(2012), 3393-3399 .

[15] T. Vasanthi and C. Venkata Lakshmi. Properties of semirings. *Int. J. Math. Archive*, **4**(6)(2013), 222-227 .

Receibed by editors 25.10.2016; Available online 25.09.2017.

Department of Mathematics, GITAM University,Visakhapatnam, India
E-mail address: mmarapureddy1@gmail.com