TL-VAGUE MODULES

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ABSTRACT. In this paper, we investigated concept of TL-vague module and obtained some basic properties of this concept.

1. Introduction

Main notions on fuzzy relations were introduced by Zadeh [**15, 16**]. Simultaneously, Goguen [**7**] introduced L-fuzzy relations on complete lattice. Rosenfeld [**10**] applied this concept to the theory of groupoid and groups. Many authors have worked to present the fuzzy setting of various algebraic concepts based on his approach. In the fuzzy set theory there were many different approaches to the concept of a fuzzy function.

In a number of papers, various kinds of fuzzy functions based on fuzzy equivalence relations have been studied. In particular, such approach has been used in definitions of strong fuzzy functions and perfect fuzzy functions, given by Demirci in [**4, 5, 6**]. Fuzzy functions based on fuzzy equivalence relations have shown oneself to be very useful in many applications in approximate reasoning, fuzzy control,vague algebra and other fields.

In [**3**], Demirci introduced the notion of a vague group based on fuzzy binary relations and concept of fuzzy equality. Later, the concept of vague ring was introduced [**13**]. In this paper, by the use of strong TL-fuzzy function, a new kind of TL-fuzzy modules based on TL-equivalence relation is introduced. The fundamental properties of TL-fuzzy equivalence relation, strong TL-fuzzy function and TL-vague binary operation are presented in Section 2.

Concepts of vague group and ring are given in Section 3. Finally, in Section 4, we focuses on the concept of TL-vague module and investigated some properties.

2. Preliminaries

Let $(L, \leqslant)$ be a complete lattice with with maximal element 1 and minimal element 0. In this section, we give some definitions and preliminaries, most of them are well known, which will be used in the next sections. Some of them were in[**1, 2, 11, 14**].

By an L-subset of X, we mean a mapping from X into L. The set of all L-subsets of X is called the L-power set of X and denoted by L^X. When $L = [0, 1]$ the L-subsets of X are known as fuzzy subsets of X. By an L-relation from X to Y (or an L-relation between X and Y), we mean an L-subset of $X \times Y$. An L-relation from X to X is called a binary L-relation on X, or an L-relation on X for short.

A *t-norm* on L is a commutative, associative mapping $T : L \times L \to L$ which is increasing in both arguments and for which $T(x, 1) = x$ for all $x \in L$. In the future text T will be a t-norm on L.

An L-relation E on a set X is called a TL-equivalence relation on X if, for all $a, b, c \in X$,

(i) $E(a, a) = 1$,

(ii) $E(a, b) = E(b, a)$,

(iii) $E(a, b)TE(b, c) \leqslant E(a, c)$. ($T$-transitivity)

E is called a separable TL-equivalence relation or a TL-equality if, moreover,

(iv) $E(a, b) = 1 \Leftrightarrow a = b$.

Let E and F be two TL-equivalence relations on X and Y, respectively. An ordinary mapping $f : X \to Y$ is called extensional w.r.t. E and F if, for all $x, x' \in X$,

$$E(x, x') \leqslant F(f(x), f(x')).$$

Let F be a TL-equivalence relation on Y and $f : X \to Y$. we define an L-fuzzy relation from X to Y as follows:

$$R_f(x, y) = F(f(x), y) \quad \forall x \in X, \forall y \in Y.$$

PROPOSITION 2.1. *Let E,F be two TL-equivalence relations on X,Y, respectively. Then the L-relation $E \times F$ on $X \times Y$ defined by*

$$E \times F : (X \times Y) \times (X \times Y) \to L,$$

and

$$E \times F((x_1, y_1), (x_2, y_2)) = T(E(x_1, x_2), F(y_1, y_2))$$

is a TL-equivalence relation.

Let E, F be two TL-equivalence relations on X, Y, respectively and $R \in L^{X \times Y}$. Then,

(i) R is called extensional w.r.t. E if, for all $x, x' \in X$ and $y \in Y$,

$$T(R(x, y), E(x, x')) \leqslant R(x', y).$$

(ii) R is called extensional w.r.t. F if, for all $x \in X$ and $y, y' \in Y$,

$$T(R(x, y), F(y, y')) \leqslant R(x, y').$$

Let E, F be two TL-equivalence relations on X, Y, respectively and $R \in L^{X \times Y}$, which is extensional w.r.t. E and F. Then R is called a strong TL-fuzzy function from X to Y

(i) For all $x \in X$ there exists $y \in Y$ such that $R(x, y) = 1$;

(ii) For all $x, x' \in X$ and for all $y, y' \in Y$,

$$R(x,y)TR(x',y')TE(x,x') \leqslant F(y,y').$$

THEOREM 2.1 ([5]). *Let E, F be two TL-equivalence relations on X, Y, respectively. Then:*

(i) *If $f : X \to Y$ is a ordinary extensional function w.r.t. E and F, then the L-relation $R_f \in L^{X \times Y}$ is a strong TL-fuzzy function.*

(ii) *If $R \in L^{X \times Y}$ is a TL-fuzzy function, then there exists an ordinary function $f : X \to Y$ extensional such that $R(x, f(x)) = 1$ and $R(x, y) \leqslant F(f(x), y)$ for all $x \in X, y \in Y$.*

Let F, E be two TL-equivalence relations on $X \times X$, X, respectively.

(i) A strong TL-fuzzy function from $X \times X$ to X is said to be a TL-vague binary operation on X.

(ii) A TL-vague binary operation $\widetilde{o}$ on X is said to be transitive of first order iff
$$\widetilde{o}(a,b,c)TE(c,d) \leqslant \widetilde{o}(a,b,d) \text{ for all } a,b,c,d \in X$$

(iii) A TL-vague binary operation $\widetilde{o}$ on X is said to be transitive of second order iff
$$\widetilde{o}(a,b,c)TE(b,d) \leqslant \widetilde{o}(a,d,c) \text{ for all } a,b,c,d \in X$$

(iv) A TL-vague binary operation $\widetilde{o}$ on X is said to be transitive of third order iff
$$\widetilde{o}(a,b,c)TE(a,d) \leqslant \widetilde{o}(d,b,c) \text{ for all } a,b,c,d \in X.$$

3. TL-vague groups and TL-vague rings

In this section, some new properties are explained about TL-vague groups and TL-vague rings. Although the definitions in this paper are explained on any t-norm and any complete lattice, it seems that they can be restated for the minimum t-norm $\wedge$ and [0,1] instead of any t-norm and any complete lattice, as in [**3, 8, 12, 13**].

DEFINITION 3.1. ([**3**]) Let $\widetilde{o}$ be TL-vague binary operation on G with respect to a TL-fuzzy equivalence relation F on $G \times G$ and a TL-fuzzy equivalence relation E on G. Then

(i) G is a TL- vague semigroup iff for all $a, b, c, d, m, w, q \in G$

$$\widetilde{o}(b,c,d)T\widetilde{o}(a,d,m)T\widetilde{o}(a,b,q)T\widetilde{o}(q,c,w) \leqslant E(m,w).$$

(ii) G is a TL- vague monoid iff G is a TL-vague semigroup and there exists $e \in G$ such that for all $a \in G$ $\widetilde{o}(e,a,a)T\widetilde{o}(a,e,a) = 1$.

(iii) G is a TL- vague group iff G is a TL-vague monoid and for all $a \in G$ there exists $b \in G$ such that $\widetilde{o}(b,a,e)T\widetilde{o}(a,b,e) = 1$.

(iv) G is a TL-vague commutative group iff G is a TL-vague group and for all $a, b, m, w \in G$ $\widetilde{o}(a, b, m) T \widetilde{o}(b, a, w) \leqslant E(m, w)$.

THEOREM 3.1 ([**3**]). *For a given vague group $(G, \widetilde{o})$, there exists a binary operation in the classical sense, denoted by o, on G such that (G, o) is a group in the classical sense. On the other hand, for a given group (G, o), it may be obtained a vague group $(G, \widetilde{o})$.*

LEMMA 3.1. *Let $(G, \widetilde{o})$ be TL-vague group. If $\widetilde{o}(c, c, c) = 1$ for $c \in G$, then $E(e, c) = 1$.*

DEFINITION 3.2. ([**3**]) Let $(G, \widetilde{o})$ be a vague group and A be a nonempty and crisp subset of G. A is called TL-vague closed under $\widetilde{o}$ TL-vague binary operation, if $\widetilde{o}(a, b, c) = 1$ for $a, b \in A$ and $c \in G$, then $c \in A$.

THEOREM 3.2 ([**12**]). *Let $(G, \widetilde{o})$ be TL-vague group. Then, a nonempty and crisp subset A of G is a TL-vague subgroup of G iff*

(i) *A is TL-vague closed under $\widetilde{o}$*
(ii) *For each $a \in A$, $a^{-1} \in A$*

THEOREM 3.3 ([**3**]). *Let $(G, \widetilde{o})$ be a vague group and E be a separable TL-fuzzy equivalence relations on G. Then, the set of TL-vague subgroup of G is a complete lattice according to $\subseteq$ relation.*

THEOREM 3.4 ([**3**]). *Let $(G, \widetilde{o})$ be a vague group and $\widetilde{o}$ TL-vague binary operation be transitive of first order. If $\widetilde{o}(a, b, c) = 1$ for $a, b, c \in G$, $\widetilde{o}(b^{-1}, a^{-1}, c^{-1}) = 1$.*

DEFINITION 3.3. ([**13**]) Let E be TL-fuzzy equivalence relations on R. $(R, \widetilde{+}, \widetilde{*})$ is called TL-vague ring if and only if

(i) $(R, \widetilde{+})$ is commutative TL-vague group,
(ii) $(R, \widetilde{*})$ TL-vague semigroup,
(iii) $(R, \widetilde{+}, \widetilde{*})$ satisfies distributive laws, for $\forall x, y, z, t, a, b, c, d \in R$

$$\widetilde{*}(x, y, a) \, T \widetilde{*}(x, z, b) \, T \widetilde{+}(a, b, c) \, T \widetilde{+}(y, z, d) \, T \widetilde{*}(x, d, t) \leqslant E(t, c)$$

$$\widetilde{*}(x, z, a) \, T \widetilde{*}(y, z, b) \, T \widetilde{+}(a, b, c) \, T \widetilde{+}(x, y, d) \, T \widetilde{*}(d, z, t) \leqslant E(t, c).$$

A TL-vague ring $(R, \widetilde{+}, \widetilde{*})$ is said to be a TL-vague ring with identity if there exists $e \in R$ such that

$$\widetilde{*}(x, e, x) \, T \widetilde{*}(e, x, x) = 1$$

for each $x \in R$.

A TL-vague ring $(R, \widetilde{+}, \widetilde{*})$ is said to be a commutative if for $\forall x, y, a, b \in R$ such that

$$\widetilde{*}(x, y, a) \, T \widetilde{*}(y, x, b) \leqslant E(a, b).$$

In the rest of this paper, the notation $(R, \widetilde{+}, \widetilde{*})$ always stands for the TL-vague ring with respect to a TL-fuzzy equivalence relation F on $R \times R$ and a TL-fuzzy equivalence relation E on R. If $(R, \widetilde{+}, \widetilde{*})$ is a TL-vague ring, then we denote the inverse of a by $-a$ respect to $(R, \widetilde{+})$, additionally if $(R, \widetilde{*})$ is a TL-vague group, then we denote the inverse of a a by a^{-1} respect to $(R, \widetilde{*})$.

THEOREM 3.5. *([**13**]) Let $(R, \widetilde{+}, \widetilde{*})$ be TL-vague ring having $e_{\widetilde{+}}$ identity element and $\widetilde{+}$, $\widetilde{*}$ TL-vague binary operations be transitive of first order. Then, $\widetilde{*}(x, e_{\widetilde{+}}, e_{\widetilde{+}}) = 1$ for all $x \in R$.*

THEOREM 3.6. *For a given TL-vague ring $(R, \widetilde{+}, \widetilde{*})$, there exists two operations in the classical sense, denoted by $+$ and $*$, on R such that $(R, +, *)$ is a ring in the classical sense. On the other hand, for a given group $(R, +, *)$, it may be obtained a TL-vague ring $(R, \widetilde{+}, \widetilde{*})$.*

PROOF. As in Theorem 3.1, it may be seen easily.   □

THEOREM 3.7 ([**13**]). *Let $(R, \widetilde{+}, \widetilde{*})$ be TL-vague ring . Then, a nonempty and crisp subset A of R is a TL-vague subring of R iff*

   (i) *A is TL-vague subgroup of R,*
   (ii) *A is TL-vague closed under $\widetilde{*}$ TL-vague binary operation.*

THEOREM 3.8. *Let $(R, \widetilde{+}, \widetilde{*})$ be TL-vague ring and E be separable TL-fuzzy equivalence relations on R. Then, the set of TL-vague subring of R is a complete lattice according to $\subseteq$ relation.*

PROOF. As in Theorem 3.3, we can easily obtain that the set of *TL*-vague subring of R is a complete lattice according to $\subseteq$ relation.   □

DEFINITION 3.4. ([**8**]) Let $(R, \widetilde{+}, \widetilde{*})$ be *TL*-vague ring and I be a nonempty and crisp subset of R. I, *TL*-vague subring is called left (right) *TL*-vague ideal iff for all $a \in I$ for all $x \in R$ there exists $b \in I$ such that $\widetilde{*}(x, a, b) = 1$ ($\widetilde{*}(a, x, b) = 1$).

PROPOSITION 3.1. *Let E be seperable TL-fuzzy equivalence relation on R, $(R, \widetilde{+}, \widetilde{*})$ be TL-vague ring, I and J be left TL-vague ideals of R. Then, $I \cap J$ is left TL-vague ideal of R.*

PROOF. Since I and J are *TL*-vague subring of R, with theorem 3.8, $I \cap J$ is *TL*-vague subring of R. For $a \in I \cap J$ and $x \in R$, there exists $b \in I$ such that $\widetilde{*}(x, a, b) = 1$ and there exists $c \in J$ such that $\widetilde{*}(x, a, c) = 1$. Since $\widetilde{*}$ is a strong *TL*-fuzzy function,

   $1 = \widetilde{*}(x, a, b) T \widetilde{*}(x, a, c) T E_{R \times R}((x, a), (x, a)) \leqslant E(b, c)$
   $1 = E(b, c)$

Since E is a separable *TL*-fuzzy equivalence relation, b=c. Thus, $b \in I \cap J$. So, $I \cap J$ is left *TL*-vague ideal of R.   □

PROPOSITION 3.2. *Let $(R, \widetilde{+}, \widetilde{*})$ be TL-vague ring having identity element and E be separable TL-fuzzy equivalence relation on R. If I including identity element is a left TL- vague ideal of R, I=R.*

4. *TL*-Vague Module

In a similar fashion to classical algebra, the notion of *TL*-vague module can be given in the following way: In the rest of the study,it will be assumed that F and E are *TL*-fuzzy equivalence relations on R and M , respectively and $F \times E$ is *TL*-fuzzy equivalence relation on $R \times M$.

DEFINITION 4.1. Let $(R, \widetilde{\oplus}, \widetilde{\ast})$ be TL-vague ring. Let $(M, \widetilde{+})$ be TL-vague group. Then $\widetilde{\bullet}$ is called TL-vague scaler product if $\widetilde{\bullet}$ a function of $(R \times M)$ to M is a strong TL-fuzzy function

DEFINITION 4.2. Let $(R, \widetilde{\oplus}, \widetilde{\ast})$ be TL-vague ring. Let $(M, \widetilde{+})$ be TL-vague group. Then M is called a TL-R-vague module if the following three conditions are satisfied $\forall a, b, b_1, b_2, c, d, d_1, k, l, m, m_1, m_2, n, n_1, n_2 \in M$ and $\forall p, r, r_1, r_2, s \in$ R

(i) $\widetilde{\oplus}(m_1, m_2, n)T\widetilde{\bullet}(r, n, k)T\widetilde{\bullet}(r, m_1, n_1)T\widetilde{\bullet}(r, m_2, n_2)T\widetilde{\oplus}(n_1, n_2, l) \leqslant E(k, l)$;

(ii) $\widetilde{+}(r_1, r_2, s)T\widetilde{\bullet}(s, m, a)T\widetilde{\bullet}(r_1, m, b_1)T\widetilde{\bullet}(r_2, m, b_2)T\widetilde{\oplus}(b_1, b_2, b) \leqslant E(a, b)$;

(iii) $\widetilde{\ast}(r_1, r_2, p)T\widetilde{\bullet}(p, m, c)T\widetilde{\bullet}(r_2, m, d_1)T\widetilde{\bullet}(r_1, d_1, d) \leqslant E(c, d)$

Acording to $\widetilde{\oplus}$ TL-vague binary operation adverse of $m \in M$ will show as $-m$.

EXAMPLE 4.1. If I is a left TL-vague ideal of a R TL-vague ring, then I is a left TL-R-vague module with being the ordinary vague binary operation $\widetilde{\ast}$ in R. In particular, $\{e_{\widetilde{+}}\}$ and R are TL-R-vague modules.

PROPOSITION 4.1. *Let M be TL-R-vague module and $\widetilde{\oplus}$ TL-vague binary operation, $\widetilde{\bullet}$ TL-vague skaler product be transitive of the first order. Then*

(i) $\widetilde{\bullet}(e_{\widetilde{+}}, m, e_{\widetilde{\oplus}}) = 1$

(ii) $\widetilde{\bullet}(r, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}) = 1$

PROOF. (i) Since $\widetilde{\bullet}$ is strong a TL-fuzzy function, there exists $l \in M$ for $m \in M$ such that $\widetilde{\bullet}(e_{\widetilde{+}}, m, l) = 1$. Since $\widetilde{\oplus}$ is strong a TL-fuzzy function, there exists $t \in M$ for $l \in M$ such that $\widetilde{\oplus}(l, l, t) = 1$. As $(M, \widetilde{\oplus})$ is TL-vague semigroup

$$1 = \widetilde{+}(e_{\widetilde{+}}, e_{\widetilde{+}}, e_{\widetilde{+}})T\widetilde{\bullet}(e_{\widetilde{+}}, m, l)T\widetilde{\bullet}(e_{\widetilde{+}}, m, l)T\widetilde{\bullet}(e_{\widetilde{+}}, m, l)T\widetilde{\oplus}(l, l, t) \leqslant E(l, t).$$

Since $\widetilde{\oplus}$ is transitive of the first order.

$$1 = \widetilde{\oplus}(l, l, t)TE(l, t) \leqslant \widetilde{\oplus}(l, l, l).$$

By Lemma 3.1 $E(e_{\widetilde{\oplus}}, l) = 1$. Since $\widetilde{\bullet}$ is transitive of the first order.

$$1 = \widetilde{\bullet}(e_{\widetilde{+}}, m, l)TE(e_{\widetilde{\oplus}}, l) \leqslant \widetilde{\bullet}(e_{\widetilde{+}}, m, e_{\widetilde{\oplus}}).$$

Thus $\widetilde{\bullet}(e_{\widetilde{+}}, m, e_{\widetilde{\oplus}}) = 1$

(ii) Since $\widetilde{\bullet}$ is strong a TL-fuzzy function, there exists $k \in M$ for $r \in R$ $\widetilde{\bullet}(r, e_{\widetilde{+}}, k) = 1$. Since $\widetilde{\oplus}$ is strong a TL-fuzzy function, there is $s \in M$ for $k \in M$ $\widetilde{\oplus}(k, k, s) = 1$ In this situation

$$1 = \widetilde{\oplus}(e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}})T\widetilde{\bullet}(r, e_{\widetilde{\oplus}}, k)T\widetilde{\bullet}(r, e_{\widetilde{\oplus}}, k)T\widetilde{\bullet}(r, e_{\widetilde{\oplus}}, k)T\widetilde{\oplus}(k, k, s) \leqslant E(k, s).$$

Since $\widetilde{\oplus}$ is transitive of the first order, then $1 = \widetilde{\oplus}(k, k, s)TE(k, s) \leqslant \widetilde{\oplus}(k, k, k)$. Thus by Lemma 3.1 $E(e_{\widetilde{\oplus}}, k) = 1$

Since $\widetilde{\bullet}$ is transitive of the first order, then

$$1 = \widetilde{\bullet}(r, e_{\widetilde{\oplus}}, k)TE(e_{\widetilde{\oplus}}, k) \leqslant \widetilde{\bullet}(r, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}).$$

Thus $\widetilde{\bullet}(r, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}) = 1$. $\square$

THEOREM 4.1. *Let M be a R-module F and E be a regular TL-fuzzy equivalence relations on R and M, respectively, and $\widetilde{\oplus}, \widetilde{+}, \widetilde{*}$ be TL-vague binary operations and $\widetilde{\bullet}$ be TL-vague scaler product are denoted by following:*

$\widetilde{\oplus}(a, b, c) = E(a \oplus b, c); \widetilde{\bullet}(r, b, c) = E(r \bullet b, c)$ *for all $r \in R$ for all $a, b, c \in M$*

$\widetilde{+}(r, s, p) = F(r + s, p); \widetilde{*}(r, s, p) = F(r * s, p)$ *for all $r, p, s \in R$*

If $F(r, s) \leqslant E(r \bullet m, s \bullet m)$ for all $r, s \in R$ and for all $m \in M$, then M is a TL-R-vague module.

PROOF. Since M is a commutative group, M is a TL-vague commutative group by theorem 3.1. Accordingly, since R is a TL-vague ring, R is a TL-vague ring by theorem 3.6. For all $r, r_1, r_2, s, p \in R$, for all $m, m_1, m_2, k, l, t, y, z \in M$ we have:

$\widetilde{\oplus}(m_1, m_2, k) T \widetilde{\bullet}(r, k, l) T \widetilde{\bullet}(r, m_1, t) T \widetilde{\bullet}(r, m_2, y) T \widetilde{\oplus}(t, y, z)$
$= E(m_1 \oplus m_2, k) T E(r \bullet k, l) T E(r \bullet m_1, t) T E(r \bullet m_2, y) T E(t \oplus y, z)$
$\leqslant E(r \bullet (m_1 \oplus m_2), r \bullet k) T E(r \bullet k, l) T E((r \bullet m_1) \oplus (r \bullet m_2), t \oplus y) T E(t \oplus y, z)$
$\leqslant E(r \bullet (m_1 \oplus m_2), l) T E(r \bullet (m_1 \oplus m_2), z)$
$\leqslant E(l, z).$

$\widetilde{+}(r_1, r_2, s) T \widetilde{\bullet}(s, m, l) T \widetilde{\bullet}(r_1, m, t) T \widetilde{\bullet}(r_2, m, y) T \widetilde{\oplus}(t, y, z)$
$= F(r_1 + r_2, s) T E(s \bullet m, l) T E(r_1 \bullet m, t) T E(r_2 \bullet m, y) T E(t \oplus y, z)$
$\leqslant E((r_1 + r_2) \bullet m, s \bullet m) T E(s \bullet m, l) T E((r_1 \bullet m) \oplus (r_2 \bullet m), t \oplus y) T E(t \oplus y, z)$
$\leqslant E(l, t \oplus y) T E(t \oplus y, z)$
$\leqslant E(l, z).$
Also we obtain
$\widetilde{*}(r_1, r_2, p) T \widetilde{\bullet}(p, m, l) T \widetilde{\bullet}(r_2, m, t) T \widetilde{\bullet}(r_1, t, y)$
$= F(r_1 * r_2, p) T(p \bullet m, l) T E(r_2 \bullet m, t) T E(r_1 \bullet t, y)$
$\leqslant E((r_1 * r_2) \bullet m, p \bullet m) T(p \bullet m, l) T E(r_1 \bullet (r_2 \bullet m), r_1 \bullet t) T E(r_1 \bullet t, y)$
$\leqslant E((r_1 * r_2) \bullet m, l) T E((r_1 \bullet r_2) \bullet m, y) = E((r_1 \bullet r_2) \bullet m, l) T E((r_1 \bullet r_2) \bullet m, y)$
$\leqslant E(l, y).$
Consequently, M is a TL-vague module. $\qquad\square$

DEFINITION 4.3. Let M be a TL-R-vague module and A be a nonempty, crisp subset of M. Then A is a TL-R-vague submodule of M if and only if there exists $a, b \in A$ such that $\widetilde{\oplus}(x, -y, a) = 1$ and $\widetilde{\bullet}(r, x, b) = 1$ for all $r \in R$ and for all $x, y \in A$.

EXAMPLE 4.2. Let $\widetilde{\oplus}$ TL-vague binary operation and $\widetilde{\bullet}$ TL-vague scaler product be first transitive. Then $\{e_{\widetilde{\oplus}}\}$ and M are TL-R-vague submodules of M.

PROPOSITION 4.2. *Let M be a TL-R-vague module and A be a TL-R-vague submodule of M. If E is a separable TL-fuzzy equivalence relation, then $e_{\widetilde{\oplus}} \in A$*

PROOF. Let $x \in A$. Since A is TL-vague submodule of M, for all $r \in R$ there exists $c \in A$ such that $\widetilde{\bullet}(r, x, c) = 1$. Especially for $e_{\widetilde{*}} \in R$ there exists $k \in A$ such that $\widetilde{\bullet}(e_{\widetilde{*}}, x, k) = 1$. Also with Proposition 4.1(i) $\widetilde{\bullet}(e_{\widetilde{*}}, x, e_{\widetilde{\oplus}}) = 1$ and since $\widetilde{\bullet}$ is strong TL-fuzzy function, we obtain $E(e_{\widetilde{\oplus}}, k) = 1$. As E is separable TL-fuzzy equivalence relation, $k = e_{\widetilde{\oplus}} \in A$. $\qquad\square$

DEFINITION 4.4. Let M and N be two TL-R-vague modules. A function $\phi : M \to N$ is called TL-R-vague module homomorphism iff $\forall a, b, c \in M$ and $\forall r \in R$

$$\widetilde{\oplus}(a, b, c) \leqslant \widetilde{\ominus}(\phi(a), \phi(b), \phi(c)) \text{ and } \widetilde{\bullet}(r, b, c) \leqslant \widetilde{\ominus}(r, \phi(b), \phi(c)).$$

In the classical algebra we know that if ϕ and φ are module homomorphisms then $\varphi \circ \phi$ is a module homomorphism. This statement is true for vague algebra as follows.

PROPOSITION 4.3. *Let* $(M, \widetilde{\oplus})$, $(N, \widetilde{\ominus})$ *and* $(K, \widetilde{\odot})$ *be TL-R-vague modules. If* $\phi : M \to N$ *and* $\varphi : N \to K$ *are a TL-R-vague module homomorphism, then* $\varphi \circ \phi : M \to K$ *is a TL-R-vague module homomorphism.*

PROOF. The proof can be easily seen. $\qquad\square$

DEFINITION 4.5. Let M be a TL-R-vague module and A be a crisp subset of M. $\{r \in R | \widetilde{\bullet}(r, a, e_{\widetilde{\oplus}}) = 1 \; for \; all \; a \in A\}$ is called the TL-vague annihilator of A and this set is denoted by $VAnn(A)$.

THEOREM 4.2. *Let* M *be a TL-R-vague module,* A *be a crisp subset of* M *and* $\widetilde{\oplus}$ *TL-vague binary operation,* $\widetilde{\bullet}$ *TL-vague scaler product be transitive of the first order. Then,*

   (i) $VAnn(A)$ *is TL-vague left ideal of R.*
   (ii) *If A is a TL-vague submodule then* $VAnn(A)$ *is TL-vague ideal of R.*

PROOF. (i) Let $a, b \in VAnn(A)$. Since $\widetilde{+}$ is a strong TL-fuzzy function, there exists $c \in R$ such that $\widetilde{+}(a, b, c) = 1$. Then $\widetilde{\bullet}(a, m, e_{\widetilde{\oplus}}) = 1$ and $\widetilde{\bullet}(b, m, e_{\widetilde{\oplus}}) = 1$ for all $m \in A$. Also there exists $l \in M$ such that $\widetilde{\bullet}(c, m, l) = 1$ as $\widetilde{\bullet}$ is a strong TL-fuzzy function. Then

$1 = \widetilde{+}(a, b, c) T \widetilde{\bullet}(c, m, l) T \widetilde{\bullet}(a, m, e_{\widetilde{\oplus}}) T \widetilde{\bullet}(b, m, e_{\widetilde{\oplus}}) T \widetilde{\oplus}(e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}})$
    $\leqslant E(l, e_{\widetilde{\oplus}}).$

Since $\widetilde{\bullet}$ is transitive of the first order, then

$$1 = \widetilde{\bullet}(c, m, l) T E(l, e_{\widetilde{\oplus}}) \leqslant \widetilde{\bullet}(c, m, e_{\widetilde{\oplus}}).$$

Therefore we get $c \in VAnn(A)$. Thus $VAnn(A)$ is TL-vague closed under $\widetilde{+}$ TL-vague binary operation.

Let $a \in VAnn(A)$. Since $\widetilde{\bullet}$ is strong TL-fuzzy function, there exists $t \in M$ such that $\widetilde{\bullet}(-a, m, t) = 1$.Then

$1 = \widetilde{\oplus}(-a, a, e_{\widetilde{+}}) T \widetilde{\bullet}(e_{\widetilde{+}}, m, e_{\widetilde{\oplus}}) T \widetilde{\bullet}(-a, m, t) T \widetilde{\bullet}(a, m, e_{\widetilde{\oplus}}) T \widetilde{\oplus}(t, e_{\widetilde{\oplus}}, t) \leqslant E(t, e_{\widetilde{\oplus}}).$
Since $\widetilde{\bullet}$ is transitive of the first order,

$$1 = \widetilde{\bullet}(-a, m, t) T E(t, e_{\widetilde{\oplus}}) \leqslant \widetilde{\bullet}(-a, m, e_{\widetilde{\oplus}}).$$

We obtain $-a \in VAnn(A)$. Hence $VAnn(A)$ is TL-vague subgroup of R.

Let $a, b \in VAnn(A)$ and $\widetilde{*}(a, b, c) = 1$. Since $\widetilde{\bullet}$ is strong TL-fuzzy function, there exists $l \in M$ such that $\widetilde{\bullet}(c, m, l) = 1$. Then

$$1 = \widetilde{*}(a, b, c) T \widetilde{\bullet}(c, m, l) T \widetilde{\bullet}(b, m, e_{\widetilde{\oplus}}) T \widetilde{\bullet}(a, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}) \leqslant E(l, e_{\widetilde{\oplus}})$$

$$1 = \widetilde{\bullet}(c, m, l) T E(l, e_{\widetilde{\oplus}}) \leqslant \widetilde{\bullet}(c, m, e_{\widetilde{\oplus}}).$$

Hence we obtain $c \in VAnn(A)$. Then $VAnn(A)$ is TL-vague closed under $\widetilde{*}$ TL-vague binary operation. Let $a \in VAnn(A)$, for $r \in R$ $\widetilde{\bullet}(r, a, x) = 1$. Since $\widetilde{\bullet}$ is a strong TL-fuzzy function, there exists $t \in M$ such that $\widetilde{\bullet}(x, m, t) = 1$. Then

$$1 = \widetilde{*}(r, a, x) T \widetilde{\bullet}(x, m, t) T \widetilde{\bullet}(a, m, e_{\widetilde{\oplus}}) T \widetilde{\bullet}(r, e_{\widetilde{\oplus}}, e_{\widetilde{\oplus}}) \leqslant E(t, e_{\widetilde{\oplus}}),$$

$$1 = \widetilde{\bullet}(x, m, t) T E(t, e_{\widetilde{\oplus}}) \leqslant \widetilde{\bullet}(x, m, e_{\widetilde{\oplus}}).$$

Hence $x \in VAnn(A)$ is obtained. That is, $VAnn(A)$ is a TL-vague left ideal of R.

(ii) Let $a \in VAnn(A)$ and $r \in R$. Since A is TL-vague submodule of M, for $m \in A$ there exists $c \in A$ such that $\widetilde{\bullet}(r, m, c) = 1$. We can write $\widetilde{\bullet}(a, c, e_{\widetilde{\oplus}})$ because of $a \in VAnn(A)$. Let $\widetilde{*}(a, r, x) = 1$. Since $\widetilde{\bullet}$ is strong TL-fuzzy function there exists $n \in M$ such that $\widetilde{\bullet}(x, m, n) = 1$. Then

$$1 = \widetilde{*}(a, r, x) T \widetilde{\bullet}(x, m, n) T \widetilde{\bullet}(r, m, c) T \widetilde{\bullet}(a, c, e_{\widetilde{\oplus}}) \leqslant E(n, e_{\widetilde{\oplus}}),$$

$$1 = \widetilde{\bullet}(x, m, n) T E(n, e_{\widetilde{\oplus}}) \leqslant \widetilde{\bullet}(x, m, e_{\widetilde{\oplus}}).$$

Therefore $x \in VAnn(A)$ is obtained. Thus $VAnn(A)$ is TL-vague right ideal of R. Hence $VAnn(A)$ is TL-vague ideal of R. $\square$

EXAMPLE 4.3. Let M be a TL-R-vague module and $\widetilde{\bullet}$ TL-vague scaler product be transitive of the first order. The set B denoted by

$$B = \{x \in M | \exists m \in M \ \widetilde{\bullet}(e_{\widetilde{*}}, m, x) = 1\}$$

is a TL-vague submodule of M.

PROOF. Let $x, y \in B$. There exists $m_1, m_2 \in M$ such that $\widetilde{\bullet}(e_{\widetilde{*}}, m_1, x) = 1$ and $\widetilde{\bullet}(e_{\widetilde{*}}, m_2, y) = 1$. Then $\widetilde{\bullet}(e_{\widetilde{*}}, -m_2, -y) = 1$. Since $\widetilde{\bullet}$ and $\widetilde{\oplus}$ are strong TL-fuzzy functions, there exists $k, q \in M$ such that $\widetilde{\bullet}(e_{\widetilde{*}}, l, q) = 1$ and $\widetilde{\oplus}(x, -y, k) = 1$. Thus

$$1 = \widetilde{\oplus}(m_1, -m_2, l) T \widetilde{\bullet}(e_{\widetilde{*}}, l, q) T \widetilde{\bullet}(e_{\widetilde{*}}, m_1, x) T \widetilde{\bullet}(e_{\widetilde{*}}, -m_2, -y) T \widetilde{\oplus}(x, -y, k)) \leqslant E(q, k)$$

and

$$1 = \widetilde{\bullet}(e_{\widetilde{*}}, l, q) T E(q, k) \leqslant \widetilde{\bullet}(e_{\widetilde{*}}, l, k).$$

We obtained that $k \in B$. For all $x, y \in B$, there exists $k \in B$ such that $\widetilde{\oplus}(x, -y, k) = 1$.

On the other hand, there exists $a \in M$ for all $r \in R$ and $x \in B$ such that $\widetilde{\bullet}(r, x, a) = 1$. Since $x \in B$, there exists $m \in M$ such that $\widetilde{\bullet}(e_{\widetilde{*}}, m, x) = 1$. Since $\widetilde{\bullet}$ is a strong TL-fuzzy function, there exists $s \in M$ such that $\widetilde{\bullet}(e_{\widetilde{*}}, a, s) = 1$.Thus

$$1 = \widetilde{*}(e_{\widetilde{*}}, r, r) T \widetilde{\bullet}(r, x, a) T \widetilde{\bullet}(r, x, a) T \widetilde{\bullet}(e_{\widetilde{*}}, a, s) \leqslant E(a, s)$$

and

$$1 = \widetilde{\bullet}(e_{\widetilde{*}}, a, s) T E(a, s) \leqslant \widetilde{\bullet}(e_{\widetilde{*}}, a, a).$$

Thus, for all $r \in R$ and $x \in B$, there exists $a \in B$ such that $\widetilde{\bullet}(r, x, a) = 1$. As a result B is a TL-vague submodule of M. $\square$

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