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# ZERO DIVISORS FREE $\Gamma-$SEMIRING

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Abstract. In this paper, we introduce the notion of zero divisors free $\Gamma-$semiring. we study the properties of zero divisors free $\Gamma-$semiring and characterize the zero divisors free $\Gamma-$semiring.

## 1. Introduction

Semiring, the algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by American mathematician Vandiver [**12**] in 1934 but non trivial examples of semirings had appeared in the earlier studies on the theory of commutative ideals of rings by German Mathematician Richard Dedekind in 19th century. Semiring is an universal algebra with two binary operations called addition and multiplication where one of them is distributive over the other. Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. A natural example of semiring which is not a ring, is the set of all natural numbers under usual addition and multiplication of numbers. In particular if $I$ is the unit interval on the real line, then $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the multiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In algebraic structure, semirings lie between semigroups and rings. The study of rings shows that multiplicative structure of ring is independent of additive structure whereas in semiring multiplicative structure of semiring is not independent of additive stricture of semiring. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semiring, as the basic algebraic structure, was used in the areas of theoretical computer science as

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well as in the solutions of graph theory and optimization theory and in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches of Mathematics.

The notion of $\Gamma-$ring was introduced by Nobusawa [**10**] as a generalization of ring in 1964. Sen [**11**] introduced the notion of $\Gamma-$semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [**4**] in 1932, Lister [**5**] introduced ternary ring. Dutta & Kar [**3**] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. The notion of $\Gamma-$semiring was introduced by Murali Krishna Rao [**6, 7, 8**] not only generalizes the notion of semiring and $\Gamma-$ring but also the notion of ternary semiring. Murali Krishna rao and Venkateswarlu [**9**] studied regular $\Gamma-$incline and field $\Gamma-$semiring. The notion of $\Gamma-$semifield was introduced by Dutta and Sardar [**2**].

The natural growth of gamma semiring is influenced by two things. One is the generalization of results of gamma rings and another is the generalization of results of semirings and ternary semirings. This notion provides an algebraic back ground to the non positive cones of the totally ordered rings.The set of all negative integers $Z$ is not a semiring with respect to usual addition and multiplication but $Z$ forms a $\Gamma-$semiring where $\Gamma = Z$.The important reason for the development of $\Gamma-$semiring is a generalization of results of rings, $\Gamma-$rings, semirings, semigroups and ternary semirings. In this paper, we introduce the notion of zero divisors free $\Gamma-$semiring. we study the properties of zero divisors free $\Gamma-$semiring and characterize the zero divisors free $\Gamma-$semiring.

## 2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. [1] *A set $S$ together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided*

(i) *addition is a commutative operation.*
(ii) *multiplication distributes over addition both from the left and from the right.*
(iii) *there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.*

Definition 2.2. [6] *Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M$ (images to be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,*

(i) $x\alpha(y + z) = x\alpha y + x\alpha z,$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z,$
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z,$

*then $M$ is called a $\Gamma-$semiring.*

Definition 2.3. [6] *A $\Gamma-$semiring $M$ is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0,$ for all $x \in M$.*

EXAMPLE 2.1. Every semiring $M$ is a $\Gamma-$semiring with $\Gamma = M$ and ternary operation is defined as the usual semiring multiplication

EXAMPLE 2.2. Let $M$ be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and $\Gamma$ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers and ternary operation is defined as usual matrix multiplication. Then $M$ is a $\Gamma-$semiring.

DEFINITION 2.4. [6] *Let $M$ be a $\Gamma-$semiring and $A$ be a non-empty subset of $M$. $A$ is called a $\Gamma-$subsemiring of $M$ if $A$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.*

DEFINITION 2.5. [6] *Let $M$ be a $\Gamma-$semiring. A subset $A$ of $M$ is called a left(right) ideal of $M$ if $A$ is closed under addition and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$. $A$ is called an ideal of $M$ if it is both a left ideal and a right ideal.*

DEFINITION 2.6. [9] *Let $M$ be a $\Gamma-$semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.*

EXAMPLE 2.3. If $M = [0, 1], \Gamma = \{0, 1\}$ then binary operation $+$ is maximum ternary operator $(a, \alpha, b) \to a\alpha b$ is the usual multiplication for $a, b \in M, \alpha \in \Gamma$. Then $M$ is a $\Gamma-$semiring with unity element 1.

EXAMPLE 2.4. Let $S$ be set of all non-negative integers and $\Gamma$ be the set of all non-negative even integers. Then $S$ is a $\Gamma-$semiring with respect to binary operation on usual addition and ternary operation as multiplication of integers.

DEFINITION 2.7. [9] *A $\Gamma-$semiring $M$ is said to be simple $\Gamma-$semiring if it has no proper ideals other than the zero ideal.*

DEFINITION 2.8. [**2**] *A commutative $\Gamma-$semiring $M$ is said to be $\Gamma-$semifield if for any $0 \neq a \in M$ and for any $\alpha \in \Gamma$ there exist $b \in M, \beta \in \Gamma$ such that $a\alpha b\beta d = d$, for all $d \in M$.*

DEFINITION 2.9. *A $\Gamma-$semiring $M$ with unity is said to be division semiring if non-zero elements of $M$ form a multiplicative group.*

## 3. Zero divisors free $\Gamma-$semiring

In this section, we introduce the notion of zero divisors free $\Gamma-$semiring. we study the properties of zero divisors free $\Gamma-$semiring and characterize the zero divisors free $\Gamma-$semiring.

DEFINITION 3.1. *Suppose $M$ is a $\Gamma-$semiring and $r \neq 0$. Then $r$ is said to be left zero divisor if there exist $\alpha \in \Gamma$ and $0 \neq s \in M$ such that $r\alpha s = 0$. Similarly $r$ is said to be right zero divisor if there exist $\alpha \in \Gamma$ and $0 \neq s \in M$ such that $s\alpha r = 0$. A zero divisor of $M$ is an element of $M$ which is either a left zero divisor or a right zero divisor or both.*

DEFINITION 3.2. *A $\Gamma-$semiring $M$ is said to be zero divisors free (ZDF)$\Gamma-$semiring if $a\alpha b = 0$ then either $a = 0$ or $b = 0$, for all $\alpha \in \Gamma$.*

Definition 3.3. *A $\Gamma-$semiring $M$ is said to be semisubtractive if for any elements $b, c \in M$ then there exist $x \in M, y \in M$ such that $b + x = c$ or $c + y = b$.*

Definition 3.4. *A $\Gamma-$semigroup $M$ with unity 1 is called $\Gamma-$group if every non-zero element has inverse.*

Definition 3.5. *A commutative $\Gamma-$semiring $M$ is called a field $\Gamma-$semiring if for each non-zero element of $M$ has multiplication inverse.*

Example 3.1. Let $M$ and $\Gamma$ be the set of all rational numbers and the set of all natural numbers respectively with the usual addition. Define the ternary operation $M \times \Gamma \times M \to M$ by $(a, \alpha, b) \to a\alpha b$, using the usual multiplication. Then $M$ is a field $\Gamma-$semiring.

Definition 3.6. *Let $M$ be a $\Gamma-$semiring is said to satisfy left(right)cancelation law if and only if $r, s, t \in M, r \neq 0, \alpha \in \Gamma$ such that $r\alpha s = r\alpha t$ $(s\alpha r = t\alpha r\ )$ then $s = t$.*

Definition 3.7. *Let $M$ be a $\Gamma-$semiring. An element $t \in M$ is said to be an absorbing if and only if $a\alpha t = t\alpha a = t$ holds for all $a \in M$.*

The proof of the following theorem is a straight forward verification.

Theorem 3.1. *Let $M$ be a $\Gamma-$semiring and 0 an element not containing in $M$. Extend the operations on $M$ to $S = M \cup \{0\}$ by $a+0 = 0+a = 0$ and $a\alpha 0 = 0\alpha a = 0$, for all $a \in M$. Then $S$ is a $\Gamma-$semiring.*

Lemma 3.1. *If $M$ is a $\Gamma-$semiring then the following are equivalent*

(i) *$M$ contains a left zero divisor*
(ii) *$M$ contains a right zero divisor*
(iii) *$M$ contains a zero divisor.*

Proof. Let $M$ be a $\Gamma-$semiring.

(i) By Definition 3.1, (i) $\Rightarrow$ (ii)
(ii) By Definition 3.1, (ii) $\Rightarrow$ (iii)
(iii) Suppose $t$ is a zero divisor and $t$ is a right zero divisor of $M$.
By Definition 3.1, there exist $r \neq 0, \alpha \in \Gamma$ such that $r\alpha t = 0$.
Thus $r$ is a left zero divisor.

□

Lemma 3.2. *Let $M$ be a $\Gamma-$semiring. Then $M$ has the left cancellative property if and only if $M$ contains no left zero divisor.*

Proof. Let $M$ be a $\Gamma-$semiring, $M$ holds the left cancellative property and $M$ contains left zero divisor $r$. By Definition 3.1, there exists $0 \neq s \in M, \alpha \in \Gamma$ such that $r\alpha s = 0 = r\alpha 0 \Rightarrow s = 0$. This contradicts $s \neq 0$.
Hence $M$ contains no left zero divisor. □

Corollary 3.1. *Let $M$ be a $\Gamma-$semiring. $M$ holds the right cancellative property if and only if $M$ contains no right zero divisor.*

THEOREM 3.2. *Let $M$ be a $\Gamma-$semiring. $M$ holds the left cancellative property if and only if $M$ holds right cancellation law.*

PROOF. Let $M$ be a $\Gamma-$semiring. Suppose $M$ holds the left cancellative property.
By Lemma 3.2, $M$ contains no left zero divisor.
By Lemma 3.1, $M$ contains no zero divisor.
By Corollary 3.1, $M$ contains the right cancellative law.
Similarly converse is true. □

DEFINITION 3.8. [**2**] *A commutative $\Gamma-$semiring $M$ is said to be strongly $\Gamma-$semifield if for any $0 \neq a \in M, d \in M$ there exist $b \in M, \alpha$ and $\beta \in \Gamma$ such that $a\alpha b\beta d = d$.*

THEOREM 3.3. *Let $M$ be a $\Gamma-$semiring with unity. $M$ is a strongly $\Gamma-$semifield if and only if $M$ is a field $\Gamma-$semiring.*

PROOF. Let $M$ be a $\Gamma-$semiring. Suppose $M$ is a field $\Gamma-$semiring and $0 \neq a \in M$. Since $M$ is a field $\Gamma-$semiring, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$.
Since 1 is the unity element, there exist $e \in M, \beta \in \Gamma$ such that $1\beta c = c$.
Therefore $a\alpha b\beta c = 1\beta c \Rightarrow a\alpha b\beta c = c$.
Hence $M$ is a strongly $\Gamma-$semifield.

Conversely suppose that $M$ is a strongly $\Gamma-$semifield.
Let $0 \neq a \in M$.Then there exist $b \in M, \alpha$ *and* $\beta \in \Gamma$ such that $a\alpha b\beta 1 = 1$. Therefore $a\alpha(b\beta 1) = 1$.
Hence every nonzero element of $M$ has multiplicative inverse.
Thus $M$ is a field $\Gamma-$semiring. □

The proof of the following theorem follows from theorem in [**9**]

THEOREM 3.4. *Every field $\Gamma-$semiring with $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma$ is a zero divisors free $\Gamma-$semiring.*

THEOREM 3.5. *Let $M$ be a $\Gamma-$semiring with unity and $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma$. Then $M$ is a field $\Gamma-$semiring if and only if $\Gamma-$semiring $M$ is ZDF and $\Gamma-$semigroup $M \setminus \{0\}$ has no proper ideals.*

PROOF. Let $M$ be a $\Gamma-$semiring with unity and $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma$. Suppose $M$ is a field $\Gamma-$semiring.
By Theorem 3.4, $M$ is a ZDF. Let $I$ be an ideal of $\Gamma-$semigroup of $M \setminus \{0\}$ and $a \in I$.
Since $0 \neq a \in M$, there exist $\alpha \in \Gamma, x \in M$ such that $a\alpha x = 1$. Therefore $1 \in I$.
Let $x \in M \setminus \{0\}$. Then $x\alpha 1 \in I$, for all $\alpha \in \Gamma \Rightarrow x \in I$.
Therefore $M \setminus \{0\} = I$.
Thus $\Gamma-$semiring $M \setminus \{0\}$ has no proper ideals.

Conversely suppose that $\Gamma-$semiring $M$ is a ZDF and $\Gamma-$semiring $M \setminus \{0\}$ has no proper ideals. Let $0 \neq a \in M, \alpha \in \Gamma$. Consider $a\alpha M \neq \{0\}$. Then $a\alpha M = M$. Therefore there exists $b \in M$ such that $a\alpha b = 1$. Hence $M$ is a field $\Gamma-$semiring. □

COROLLARY 3.2. *Let $M$ be a $\Gamma-$semiring with unity and $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma$. $M$ is a field $\Gamma-$semiring if and only if $M$ is a ZDF and commutative $\Gamma-$semigroup $M \setminus \{0\}$ is a $\Gamma-$group.*

THEOREM 3.6. *A multiplicative cancellative $\Gamma-$semiring $M$ is a zero divisors free $\Gamma-$semiring.*

PROOF. Let $M$ be a multiplicative cancellative $\Gamma-$semiring and $a\alpha b = 0, a, b \in M, \alpha \in \Gamma$. Suppose $0 \neq b \in M$. Then $a\alpha b = 0\alpha b \Rightarrow a = 0$.
Hence $\Gamma-$semiring $M$ is a zero divisors free $\Gamma-$semiring. □

THEOREM 3.7. *A zero divisors free $\Gamma-$semiring $M$ is a multiplicatives cancellative whenever it is an additively cancellative and a semisubtractive.*

PROOF. Let $M$ be a zero divisors free $\Gamma-$semiring and semisubtractive $\Gamma-$semiring with additively cancellative. Let $a, b, c \in M \setminus \{0\}, \alpha \in \Gamma$ and $a\alpha b = a\alpha c$.
Since $b, c \in M$, there exist $x, y \in M$ such that $b + x = c$ or $y + c = b$.
Suppose $y + c = b$. Then

$$\begin{aligned} &a\alpha(y+c) = a\alpha b \\ \Rightarrow &a\alpha y + a\alpha c = a\alpha b \\ \Rightarrow &a\alpha y + a\alpha c = 0 + a\alpha b \\ \Rightarrow &a\alpha y + a\alpha c = 0 + a\alpha c \\ \Rightarrow &a\alpha y = 0 \\ \Rightarrow &y = 0. \end{aligned}$$

Therefore $c = b$. If $b + x = c$ then similarly we can prove $b = c$.
Hence $M$ is a multiplicatively left cancellative.
Similarly we can prove $M$ is a multiplicative right cancellative.
Thus zero divisors free $\Gamma-$semiring is a multiplicatively cancellative $\Gamma-$semiring. □

DEFINITION 3.9. *Let $M$ be a $\Gamma-$semiring and $\alpha \in \Gamma$. Define a binary operation $*$ on $M$ by $a * b = a\alpha b$, for all $a, b \in M$. Then $(M, +, *)$ is a semiring. It is denoted by $M_\alpha$.*

THEOREM 3.8. *A commutative $\Gamma-$semiring $M$ is a field $\Gamma-$semiring if and only if $M_\alpha$ is a division semiring, for any $\alpha \in \Gamma$.*

PROOF. Let $M$ be a field $\Gamma-$semiring, $a \in M \setminus \{0\}$ and $\alpha \in \Gamma$.
Suppose $b \in M \setminus \{0\}$. Then $a\alpha b \neq 0$, since $M \setminus \{0\}$ is a $\Gamma-$semiring.
By definition of field $\Gamma-$semiring, we have
$(a\alpha b)\beta c = 1,\ \beta \in \Gamma, c \in M$
$\Rightarrow\ a\alpha(b\beta c) = 1.$
Hence $M_\alpha$ is a division semiring.
Converse is obvious. □

THEOREM 3.9. *Let $M$ be a $\Gamma-$semiring. If $M_\alpha$ is a division semiring, for some $\alpha \in \Gamma$ then $M_\beta$ is a division semiring, for all $\beta \in \Gamma$.*

Proof. Suppose $M_\alpha$ is a division semiring for some $\alpha \in \Gamma$. Let $0 \neq a \in M_\beta, \beta \in \Gamma$. Then there exists $b \in M$ such that $a\beta a\alpha b = 1$, since $M_\alpha$is a division semiring . Therefore $a\beta(a\alpha b) = 1$
Hence $M_\beta$ is a division semiring for all $\beta \in \Gamma$. □

Proof of the following theorem follows from Theorems 3.3, 3.4,3.5 , 3.8 ,3.9 and Corollary 3.2 .

Theorem 3.10. *Let $M$ be a commutative $\Gamma-$semiring with unity and $1\alpha 1 \neq 0$ for all $\alpha \in \Gamma$. Then the following are equivalent.*

(i) *$M$ is a zero divisors free $\Gamma-$semiring and for each non-zero element of $M$ has multiplicative inverse.*
(ii) *$M$ is a strongly $\Gamma-$ semifield.*
(iii) *$M$ is field $\Gamma-$semiring.*
(iv) *$M_\alpha$ is a division semiring, for all $\alpha \in \Gamma$.*

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