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RAINBOW CONNECTION IN BRICK PRODUCT GRAPHS

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Abstract. Let G be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in N$, of the edges of G, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. G is rainbow connected if G contains a rainbow $u - v$ path for every two vertices u and v in it. The minimum k for which there exists such a k-edge coloring is called the rainbow connection number of G, denoted by $rc(G)$. In this paper we determine $rc(G)$ of brick product graphs associated with even cycles. We also discuss the critical property of these graphs with respect to rainbow coloring.

1. Introduction

Connectivity is perhaps the most fundamental graph-theoretic property, both in the combinatorial sense and the algorithmic sense. There are many ways to strengthen the connectivity property, such as requiring hamiltonicity, k-connectivity, imposing bounds on the diameter, requiring the existence of edge-disjoint spanning trees and so on. One among them is rainbow connectivity that strengthens the connectivity requirement, introduced by Chartrand et. al. in 2008 [**2**].

Let G be a nontrivial connected graph with an edge coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in N$, where adjacent edges may be colored the same. A path in G is called a rainbow path if no two edges of it are colored the same. An edge colored graph G is said to be rainbow connected if for any two vertices in G, there is a rainbow path in G connecting them. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected. i.e., a coloring such that each edge has a distinct

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color. The minimum k for which there exist a rainbow k-coloring of G is called the rainbow connection number of G, denoted by $rc(G)$.

For any two vertices u and v in G, $d(u, v)$ is the distance between u and v. Let c be a rainbow coloring of G. For any two vertices u and v of a rainbow $u-v$ geodesic in G is a rainbow $u-v$ path of length G is termed strongly rainbow connected if G contains a rainbow $u-v$ geodesic for every two vertices u and v in G and in this case the coloring c is called a strong rainbow coloring of G. The minimum k for which there exists a coloring $c : E(G) \to \{1, 2, \cdots, k\}$, $k \in N$, of the edges of G such that G is strongly rainbow connected is called the strong rainbow connection number of G, denoted by $src(G)$. Thus $rc(G) \leqslant src(G)$ for every connected graph G.

The rainbow connection number and the strong rainbow connection number are defined for every connected graph G, since every coloring that assigns distinct colors to the edges of G is both a rainbow coloring and a strong rainbow coloring and G is rainbow connected and strongly rainbow connected with respect to some coloring of the edges of G.

In [**2**], Chartrand et.al. determined $rc(G)$ for some classes of graphs like, the cycle graph, the wheel graph etc., and $src(G)$ for complete multipartite graphs. In [**5**] and [**6**] K.Srinivasa Rao and R.Murali, determined $rc(G)$ and $src(G)$ of the stacked book graph, the grid graph, the prism graph etc. Authors also discussed the critical property of these graphs with respect to rainbow coloring. An overview about rainbow connection number can be found in a book of Li and Sun in [**4**] and a survey by Li et.al. in [**3**].

1.1. Definition. A graph G is said to be rainbow critical if the removal of any edge from G increases the rainbow connection number of G, i.e. if $rc(G) = k$ for some positive integer k, then $rc(G - e) > k$ for any edge e in G.

The brick product of even cycles was introduced in a paper by B.Alspach et.al. [**1**] in which the Hamiltonian laceability properties of brick products was explored. In this paper we determine $rc(G)$ of brick product graphs associated with even cycles. We also discuss the critical property of these graphs with respect to rainbow coloring.

1.2. Definition. Let m, n and r be positive integers. Let $C_{2n} = v_0, v_1, v_2, \cdots,$ $v_{(2n-1)}$, v_0 denote a cycle of order $2n$. The (m, r)-brick product of C_{2n} denoted by $C(2n, m, r)$ is defined as follows:

For $m = 1$, we require that r be odd and greater than 1. Then, $C(2n, m, r)$ is obtained from C_{2n} by adding chords $v_{2k}(v_{2k+r})$, $k = 1, 2, \cdots, n$, where the computation is performed under modulo $2n$.

For $m > 1$, we require that $m + r$ be even. Then, $C(2n, m, r)$ is obtained by first taking disjoint union of m copies of C_{2n} namely, $C_{2n}(1), C_{2n}(2), C_{2n}(3), \cdots,$ $C_{2n}(m)$ where for each $i = 1, 2, \cdots, m$, $C_{2n}(i) = v_{i1}, v_{i2}, v_{i3}, , \cdots, v_{i(2n)}$. Next, for each odd $i = 1, 2, \cdots, m-1$ and each even $k = 0, 1, \cdots, 2n-2$, an edge(called a *brick edge*) drawn to join v_{ik} to $v_{(i+1)k}$, whereas, for each even $i = 1, 2, \cdots, m-1$ and each odd $k = 1, 2, \cdots, 2n-1$, an edge (also called a *brick edge*) is drawn to join v_{ik} to $v_{(i+1)k}$.

Finally, for each odd $k = 1, 2, \cdots, 2n-1$, an edge (called a *hooking edge*)is drawn to join v_{1k} to $v_{m(k+r)}$. An edge in $C(2n, m, r)$ which is neither a brick edge nor a hooking edge is called a *flat edge*.

The brick products $C(10, 1, 5), C(10, 2, 4)$ and $C(10, 3, 5)$ are shown in figures 1, 2 and 3.

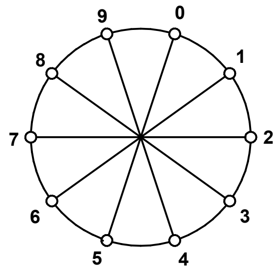

FIGURE 1. The brick product $C(10, 1, 5)$

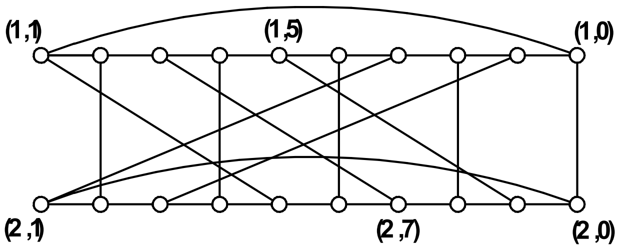

FIGURE 2. The brick product $C(10, 2, 4)$

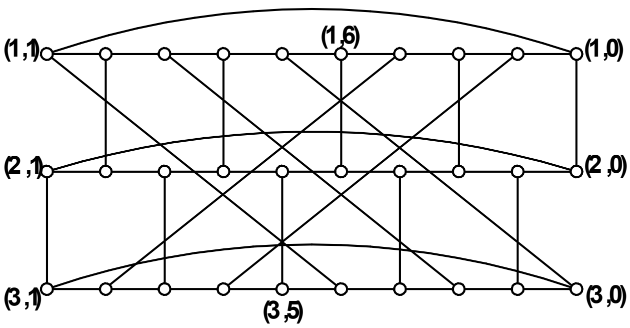

FIGURE 3. The brick product $C(10, 3, 5)$

In the next section, we determine the values of $rc(G)$ for the brick product graph $C(2n, m, r)$ for $m = 1, n \geqslant 3, r = n$ and for $m = 1, n \geqslant 4, r = 3$. In our results, we denote the vertices of the cycle C_{2n} as $v_0, v_1, \cdots, v_{2n-1}, v_{2n} = v_0$.

2. Main Results

THEOREM 2.1. *Let $G = C(2n, m, r)$. Then for $m = 1, n \geqslant 3$ and $r = n$,*

$$rc(G) = \begin{cases} 2 & \text{for } \; n = 3 \\ 3 & \text{for } \; n = 5 \\ \lceil \frac{n}{2} \rceil + 1 & \text{for } \; n \geqslant 7 \text{ and odd} \end{cases}$$

PROOF. We consider the vertex set of G as $V(G) = \{v_0, v_1, \cdots, v_{2n-1}, v_{2n} = v_0\}$ and the edge set of G as $E(G) = \{e_i : 1 \leqslant i \leqslant 2n\} \cup \{e_i^{'} : 1 \leqslant i \leqslant n\}$, where e_i is the edge (v_{i-1}, v_i) and $e_i^{'}$ is the edge (v_{2k}, v_{2k+r}), $k = 0, 1, \cdots, n$. Here $2k + r$ is computed modulo $2n$.

We prove this theorem in different cases as follows.

Case 1: $n = 3$.

Since $diam(G) = 2$, it follows that $rc(G) \geqslant 2$. It remains to show that $rc(G) \leqslant 2$. Define a coloring $c : E(G) \to \{1, 2\}$ and consider the assignment of colors to the edges of G as

$$c(e) = \begin{cases} 1 & \text{if} \quad e = v_0v_1 = v_2v_3 = v_4v_5 = v_0v_3 = v_2v_5 \\ 2 & \text{if} \quad e = v_1v_2 = v_3v_4 = v_5v_0 = v_1v_4 \end{cases}$$

Then, for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in G.

Hence $rc(G) \leqslant 2$.

This proves $rc(G) = 2$.

(An illustration for the assignment of colors in $C(6, 1, 3)$ is provided in figure 4).

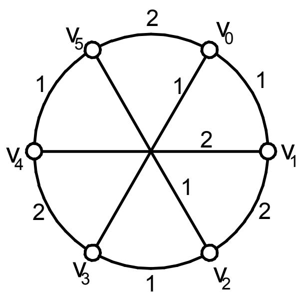

FIGURE 4. Assignment of colors in $C(6, 1, 3)$

Case 2: $n = 5$.

Since $diam(G) = 3$, it follows that $rc(G) \geqslant 3$. It remains to show that $rc(G) \leqslant 3$. Define a coloring $c : E(G) \to \{1, 2, 3\}$ and consider the assignment of colors to the edges of G as

$$c(e) = \begin{cases} 1 & \text{if} \quad e = v_0v_1 = v_2v_3 = v_4v_5 = v_6v_7 = v_8v_9 \\ 2 & \text{if} \quad e = v_1v_2 = v_3v_4 = v_5v_6 = v_7v_8 = v_9v_0 \\ 3 & \text{if} \quad e = v_{2k}v_{2k+r} \quad where \quad k = 0, 1, \cdots, n \end{cases}$$

For any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in G. Hence $rc(G) \leqslant 3$.

This proves $rc(G) = 3$.

(An illustration for the assignment of colors in $C(10, 1, 5)$ is provided in figure 5).

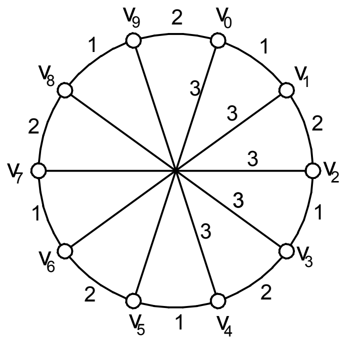

FIGURE 5. Assignment of colors in $C(10, 1, 5)$

Case 3: $n \geqslant 7$ and odd.

In this case $diam(G) = \lceil \frac{n}{2} \rceil$ and hence it follows that $rc(G) \geqslant \lceil \frac{n}{2} \rceil$. But, as in case 2, if we assign the colors to the edges of G, we fail to obtain a rainbow path between the vertices $v_1 - v_{\frac{3(n+1)}{2}}$ $\forall n$. (This is illustrated in figure 6).

Accordingly, we define a coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} i & \text{if} \quad 1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil \\ i - \lceil \frac{n}{2} \rceil & \text{if} \quad \lceil \frac{n}{2} \rceil + 1 \leqslant i \leqslant n + 1 \\ i - (n+1) & \text{if} \quad n + 2 \leqslant i \leqslant \lceil \frac{3n}{2} \rceil + 1 \\ i - (\lceil \frac{3n}{2} \rceil + 1) & \text{if} \quad \lceil \frac{3n}{2} \rceil + 2 \leqslant i \leqslant 2n \end{cases}$$

and

$$c(e_i^{'}) = \lceil \tfrac{n}{2} \rceil + 1$$

From the above assignment, it is easy to verify that for every two distinct vertices $x, y \in V(G)$, there exists an $x - y$ rainbow path with coloring c. Hence $rc(G) \leqslant \lceil \frac{n}{2} \rceil + 1$.

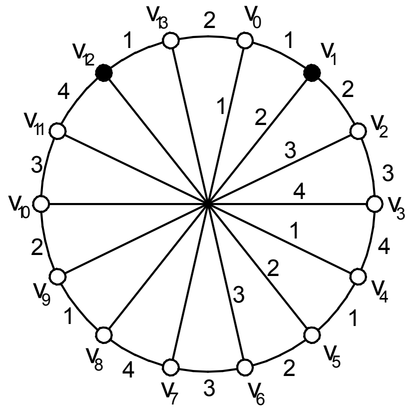

FIGURE 6. Assignment of colors in $C(14, 1, 7)$

This proves $rc(G) = \lceil \frac{n}{2} \rceil + 1$.

Hence the proof.

(An illustration for the assignment of colors in $C(14, 1, 7)$ is provided in figure 7).

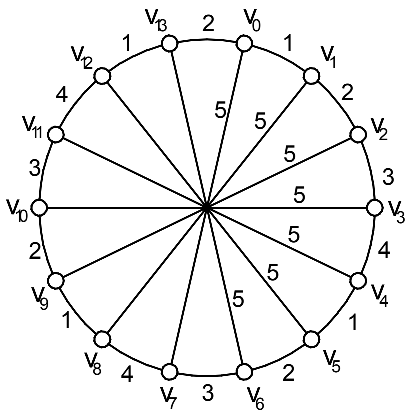

FIGURE 7. Assignment of colors in $C(14, 1, 7)$

□

The critical nature of the brick product graph in theorem 2.1 has been observed. This is illustrated in our next result.

LEMMA 2.1. *Let $G = C(2n, m, r)$, where $m = 1$ and $r = n$. Then G is rainbow critical for $n = 3$.*

$$i.e., \quad rc(G - e) = 3 \quad for \;\; n = 3$$

PROOF. Since $diam(G) = 2$, deletion of any edge in G, increases the diameter by 1. i.e., $diam(G - e) = 3$ and therefore $rc(G - e) \geqslant 3$. It remains to show that $rc(G - e) \leqslant 3$.

Define a coloring $c : E(G) \to \{1, 2, 3\}$ and consider the assignment of colors to the edges of G as

$$c(e) = \begin{cases} 1 & \text{if} \quad e = v_0v_1 = v_4v_5 = v_2v_5 \\ 2 & \text{if} \quad e = v_1v_2 = v_3v_4 = v_5v_0 = v_1v_4 \\ 3 & \text{if} \quad e = v_2v_3 \end{cases}$$

From the above assignment, for any two vertices $x, y \in V(G - e)$, we obtain a rainbow $x - y$ path in $G - e$. This holds $\forall e \in E(G)$.

Hence $rc(G - e) \leqslant 3$.

This proves $rc(G - e) = 3$.

Hence the proof.

□

REMARK 2.1. *Let $G = C(2n, m, r)$ where $m = 1$ and $r = n$. Then, for $n \geqslant 5$ and odd, G is not rainbow critical, since, from theorem 2.1, we have,*

$$rc(G) = \begin{cases} 3 & \textit{for} \quad n = 5 \\ \lceil \frac{n}{2} \rceil + 1 & \textit{for} \quad n \geqslant 7 \textit{ and odd} \end{cases}$$

If we delete of any brick edge $v_{2k}v_{2k+r}$, where $k = 0, 1, \cdots, n$, we immediately obtain the rainbow path between the selected vertices.
(For illustration see figure 8).

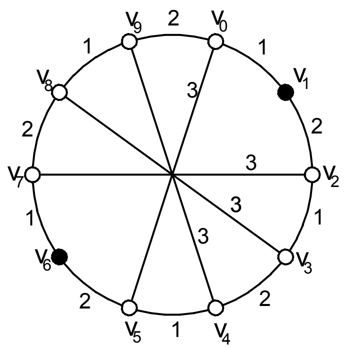

FIGURE 8. Assignment of colors in $C(10, 1, 5)$

When $m = 1$ and $r = 3$, we have the following result.

THEOREM 2.2. *Let $G = C(2n, m, r)$. Then for $m = 1$ and $r = 3$,*

$$rc(G) = \begin{cases} \frac{n}{2} + 1 & \text{for } n \geqslant 4 \text{ and even} \\ 3 & \text{for } n = 5 \\ \lceil \frac{n}{2} \rceil + 1 & \text{for } n \geqslant 7 \text{ and odd} \end{cases}$$

PROOF. We consider the vertex set $V(G)$ and the edge set $E(G)$ defined in Theorem 2.1. We prove this result in different cases as follows.
Case 1: $n = 5$.

Since $diam(G) = 3$, it follows that $rc(G) \geqslant 3$. It remains to show that $rc(G) \leqslant 3$. Define a coloring $c : E(G) \to \{1, 2, 3\}$ and consider the assignment of colors to the edges of G as

$$c(e) = \begin{cases} 1 & \text{if} \quad e = v_0v_1 = v_2v_3 = v_0v_3 = v_5v_6 = v_7v_8 \\ 2 & \text{if} \quad e = v_1v_2 = v_3v_4 = v_6v_7 = v_6v_9 = v_8v_9 \\ 3 & \text{if} \quad e = v_0v_9 = v_1v_8 = v_2v_5 = v_4v_5 = v_4v_7 \end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in G. Hence $rc(G) \leqslant 3$.

This proves $rc(G) = 3$.
(An illustration for the assignment of colors in $C(10, 1, 3)$ is provided in figure 9).

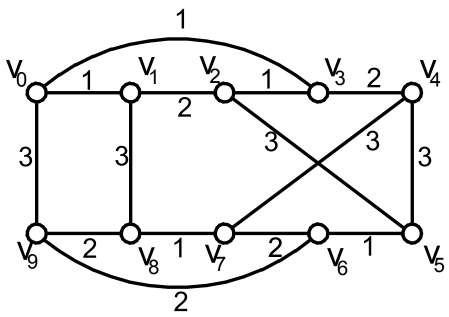

FIGURE 9. Assignment of colors in $C(10, 1, 3)$

Case 2: $n \geqslant 4$ and even.

Since $diam(G) = \frac{n}{2} + 1$, it follows that $rc(G) \geqslant \frac{n}{2} + 1$. In order to show that $rc(G) \leqslant \frac{n}{2} + 1$, we construct an edge coloring $c : E(G) \to \{1, 2, \cdots, \frac{n}{2} + 1\}$ as follows

$$c(e_i) = \begin{cases} \frac{i}{2} & \text{if i is even and} \quad 2 \leqslant i \leqslant n \\ \frac{i-n}{2} & \text{if i is even and} \quad n + 2 \leqslant i \leqslant 2n \\ \frac{n}{2} + 1 & \text{if i is odd and} \quad 1 \leqslant i \leqslant 2n \end{cases}$$

and

$$c(e_i^{'}) = \begin{cases} i & \text{if} \quad 1 \leqslant i \leqslant \frac{n}{2} \\ i - \frac{n}{2} & \text{if} \quad \frac{n}{2} + 1 \leqslant i \leqslant n \end{cases}$$

It is easy to verify that for any two vertices $x, y \in V(G)$, the above assignment gives a rainbow $x - y$ path in G. Hence $rc(G) \leqslant \frac{n}{2} + 1$.

This proves $rc(G) = \frac{n}{2} + 1$.

(An illustration for the assignment of colors in $C(16, 1, 3)$ is provided in figure 10).

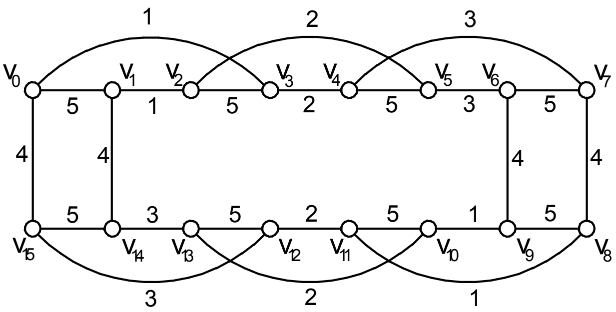

FIGURE 10. Assignment of colors in $C(16, 1, 3)$

Case 3: $n \geqslant 7$ and odd.

In this case, $diam(G) = \lceil \frac{n}{2} \rceil$ and hence it follows that $rc(G) \geqslant \lceil \frac{n}{2} \rceil$. Suppose that c is a $\lceil \frac{n}{2} \rceil$ rainbow coloring, if we color the edges as in case 2, this will not give a rainbow $v_2 - v_{n+2}$ path $\forall n$. (See figure 11).

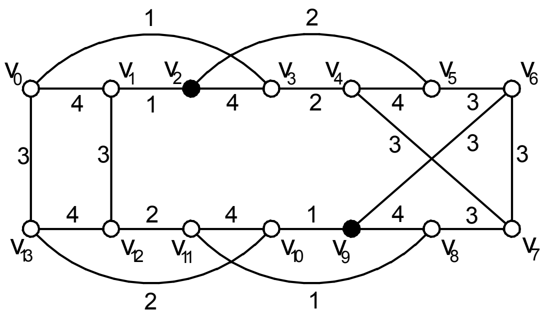

FIGURE 11. Assignment of colors in $C(14, 1, 3)$

Accordingly, we construct an edge coloring $c : E(G) \to \{1, 2, \cdots, \lceil \frac{n}{2} \rceil + 1\}$ and assign the colors to the edges of G as

$$c(e_i) = \begin{cases} \frac{i}{2} & \text{if i is even and} \quad 2 \leqslant i \leqslant n-1 \\ \lceil \frac{i-n}{2} \rceil & \text{if i is even and} \quad n+1 \leqslant i \leqslant 2n-2 \\ \lceil \frac{n}{2} \rceil + 1 & \text{if i is odd} \quad 1 \leqslant i \leqslant n-2 \quad \text{and} \quad n+2 \leqslant i \leqslant 2n-1 \\ \lceil \frac{n}{2} \rceil & \text{elsewhere} \end{cases}$$

and

$$c(e_i^{'}) = \begin{cases} i & \text{if } \ 1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 1 \\ i - n + \lfloor \frac{n}{2} \rfloor + 1 & \text{if } \ \lceil \frac{n}{2} \rceil + 1 \leqslant i \leqslant n-1 \\ \lceil \frac{n}{2} \rceil & \text{elsewhere} \end{cases}$$

From the above assignment, it is easy to verify that for every two distinct vertices $x, y \in V(G)$, there exists an $x - y$ rainbow path with coloring c.

Hence $rc(G) \leqslant \lceil \frac{n}{2} \rceil + 1$.

This proves $rc(G) = \lceil \frac{n}{2} \rceil + 1$.

Hence the proof.

(An illustration for the assignment of colors in $C(14, 1, 3)$ is provided in figure 12).

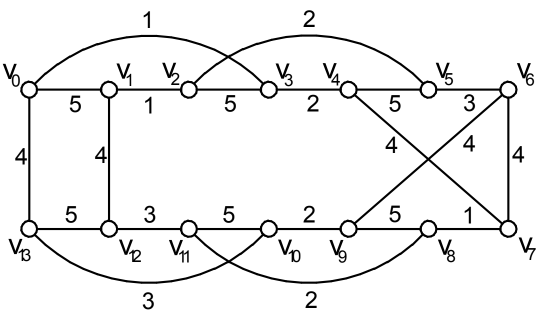

FIGURE 12. Assignment of colors in $C(14, 1, 3)$

□

The critical nature of the brick product graph in theorem 2.2 has been observed for even $n \geqslant 4$. This is illustrated in our next result.

LEMMA 2.2. *Let $G = C(2n, m, r)$, where $m = 1$ and $r = 3$. Then G is rainbow critical for even $n \geqslant 4$.*

$$i.e., \quad rc(G - e) = \tfrac{n}{2} + 2 \ \ for \ \ n \geqslant 4 \ and \ even$$

PROOF. Let $e^{''} = (x, y)$ be any edge in G *(edge in cycle or brick edge)*. Then every e'' is an edge of some sub graph $H = C_4$ in G. If we follow a coloring as in Theorem 2.2, it is clear that the edges of this sub graph can be colored by two colors. Deletion of e'' from any sub graph $H = C_4$ will give $d(x, y) = 3$. Let P be the path from x to y in H. Then, since two edges in P have the same color, a $x - y$ rainbow path in G is not possible. This holds for every e'' in G. Hence, to obtain a rainbow path, one more color is required other than the $\frac{n}{2} + 1$ colors already assigned in G.

This proves $rc(G - e'') = \frac{n}{2} + 2$.

(An illustration for the assignment of colors in $C(16, 1, 3)$ is provided in figure 13).

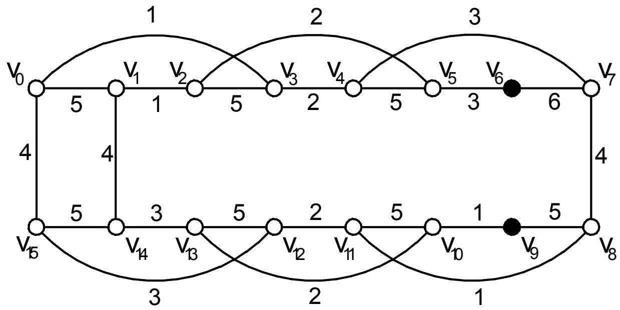

FIGURE 13. Assignment of colors in $C(16, 1, 3)$

□

REMARK 2.2. *Let* $G = C(2n, m, r)$ *where* $m = 1$ *and* $r = 3$. *Then, for* $n \geqslant 5$ *and odd, G is not rainbow critical graph, since, from theorem 2.2, we have,*

$$rc(G) = \begin{cases} 3 & \textit{for} \;\; n = 5 \\ \lceil \frac{n}{2} \rceil + 1 & \textit{for} \;\; n \geqslant 7 \textit{ and odd} \end{cases}$$

If we delete of any brick edge $v_{\lfloor \frac{n}{2} \rfloor} v_n$, we immediately obtain the rainbow path between the selected vertices. This holds for $v_{n-1} - v_{n+2}$ also. (For illustration see figure 14).

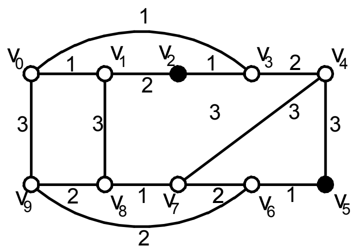

FIGURE 14. Assignment of colors in $C(10, 1, 3)$

3. Conclusion

In this paper, we have determined the rainbow connection number of brick product graphs $C(2n, m, r)$ associated with even cycles. In some cases, the critical property of brick product graphs with respect to rainbow coloring is also investigated.

References

[1] B. Alspach, C.C.Chen and Kevin McAvaney. On a class of Hamiltonian laceable 3-regular graphs. *Discrete Mathematics*, **151**(1-3)(1996), 19-38.
[2] G. Chartrand, G. L. Johns, K. A. McKeon and P. Zhang. Rainbow connection in graphs. *Math. Bohem.*, **133**(1)(2008), 85-98.
[3] X. Li, Y. Shi and Y. Sun. Rainbow connection of graphs: a survey. *Graphs Combin.*, **29**(1)(2013), 1-38.
[4] X. Li and Y. Sun. *Rainbow Connection of Graphs*. New York: Springer-Verlag, 2012.
[5] K. Srinivasa Rao and R. Murali. Rainbow critical graphs. *Int. J. Comp. Appl.*, **4**(4)(2014), 252-259.
[6] K. Srinivasa Rao, R. Murali and S. K. Rajendra. Rainbow and strong rainbow criticalness of some standard graphs. *Int. J. Math. Comp. Research*, **3**(1)(2015), 829-836

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