JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
Vol. 7(2017), 37-51
DOI: 10.7251/JIMVI1701037S

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

NORMAL FILTERS IN ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper we introduce normal filters and normlets in an almost distributive lattice with dense elements and reinforce them in both algebraical and topological aspects.

1. Introduction and Preliminaries

The structure of distributive lattice is exponentially enrich and has smooth nature. A vast number of researchers broadly studied the class of distributive lattice in different aspects. In [7, 8, 9, 15, 16, 17, 18], the authors initiated the ideal/ filter/ congruence theory in a distributive lattice and they have showed some special class of distributive lattices like normal lattices, quasi complemented distributive lattices etc. Some of the authors take a broad view of the structure of distributive lattice in different aspects. In this context, U. M. Swamy and G. C. Rao [16] generalized the structure of distributive lattice as a common abstraction of lattice theoretic and ring theoretic aspects called as almost distributive lattice in 1981. Later, the authors [2, 3, 4, 5, 6, 12, 19, 20] analogously extended some concepts to almost distributive lattices which are in distributive lattices.

In this paper we mainly concentrate on filters in an almost distributive lattice with dense elements. In this first section, we collect some preliminary results on almost distributive lattices which are useful in the sequent sections. In second section, we introduce normal filters in an almost distributive lattice and certain examples are given and derive some properties on the class of normal filters. In third section, we study the class of normlets in an almost distributive lattice and obtain several equivalent conditions for a filter to become a normlet. In fourth

2010 *Mathematics Subject Classification.* Primary 06D99; Secondary 06D15.

Key words and phrases. Normal filter, dense element, weak relatively complemented almost distributive lattice and the Hull-kernel topology.

section, we discuss the class of normal prime filters and obtain certain results on them. In last section, we deliberate the space of normal prime filters with the Hull-kernel topology and obtain a good number of equivalent conditions for the space of normal prime filters to become Hausdorff.

Let us first recall the notion of an almost distributive lattice and certain necessary results which are required in consequent sections.

Definition 1.1. ([16]) *By an almost distributive lattice (abbreviated: ADL), we mean an algebra* $(L, \wedge, \vee, 0)$ *of type* $(2, 2, 0)$*, if it satisfies the following conditions;*

(i) $0 \wedge a = 0$
(ii) $a \vee 0 = a$
(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(vi) $(a \vee b) \wedge b = b$*, for all* $a, b, c \in L$.

Throughout this paper by L mean an almost distributive lattice $(L, \wedge, \vee, 0)$ unless otherwise mentioned.

For any $a, b \in L$, we say that a is less than or equal to b and write $a \leqslant b$, if $a \wedge b = a$ or $a \vee b = b$. It can be easy to prove that $\leqslant$ is a partial ordering on L.

Lemma 1.1. ([16]) *For any* $a, b, c \in L$*, we have*

(i) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(ii) $a \wedge a = a \vee a = a$
(iii) $a \vee (b \vee a) = a \vee b$
(iv) $\wedge$ *is associative*
(v) $a \wedge b \wedge c = b \wedge a \wedge c$
(vi) $a \wedge b = 0 \iff b \wedge a = 0$
(vii) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(viii) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(ix) $a \vee b = b \vee a \Longleftrightarrow a \wedge b = b \wedge a$.

A non-empty subset F of L is said to be a filter of L, if for any $a, b \in F$ and $x \in L$, $a \wedge b$, $x \vee a \in F$. For any non-empty subset S of L, $[S) = \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_1, s_2, \ldots\ldots\ldots, s_n \in S,\ x \in L$, n is a positive integer$\}$ is the smallest filter of L containing S. In particular, for any $a \in L$, $[a) = \{x \vee a \mid x \in L\}$ is the principal filter generated by a. The set $\mathcal{F}(L)$ filters of L forms a bounded distributive lattice, where $F \cap G$ is the infimum and $F \vee G = \{f \wedge g \mid f \in F$ and $g \in G\}$ is the supremum of F and G in $\mathcal{F}(L)$. The set $\mathcal{PF}(L)$ principal filters of L forms a sublattice of $\mathcal{F}(L)$, where $[a) \wedge [b) = [a \vee b)$ and $[a) \vee [b) = [a \wedge b)$, for any $a, b \in L$. A non-empty subset I of L is said to be an ideal of L, if for any $a, b \in I$ and $x \in L$, $a \vee b$, $a \wedge x \in I$. In particular, for any $a \in L$, $(a] = \{a \wedge x \mid x \in L\}$ is the principal ideal generated by a. The set $\mathcal{I}(L)$ of ideals of L forms a bounded distributive lattice, where $I \cap J$ is the infimum and $I \vee J = \{i \vee j \mid i \in I$ and $j \in J\}$ is the supremum of I and J in $\mathcal{I}(L)$. The set $\mathcal{PI}(L)$ principal ideals of L forms a sublattice of $\mathcal{I}(L)$, where $(a] \wedge (b] = (a \wedge b]$ and $(a] \vee (b] = (a \vee b]$, for any $a, b \in L$.

For any non-empty subset A of L, the set $A^* = \{x \in L \mid a \wedge x = 0$, for all $a \in A\}$ is an ideal of L. In particular, for any $a \in L$, $\{a\}^* = (a)^*$, where $(a) = (a]$ is the principal ideal generated by a.

LEMMA 1.2. ([4]) *For any $a, b \in L$, we have*

(i) $a \leqslant b \Longrightarrow (b)^* \subseteq (a)^*$
(ii) $(a)^{***} = (a)^*$
(iii) $(a \vee b)^* = (a)^* \cap (b)^*$
(iv) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$
(v) $(a)^* \subseteq (b)^* \Longleftrightarrow (b)^{**} \subseteq (a)^{**}$
(vi) $a \in (a)^{**}$.

An element $d \in L$ is said to be dense, if $(d)^* = \{0\}$. The set D denotes the set of dense elements of L. It is a filter of L, provided D is non-empty. An element $m \in L$ is said to be a maximal element if for any $a \in L$, $m \leqslant a$ implies $m = a$. It is easy to observe that every maximal element is dense. The set M denotes the set of maximal elements of L. It is also a filter of L, provided M is non-empty.

A proper filter (ideal) $F(I)$ of L is said to be a prime, if for any $a, b \in L$, $a \vee b$ $(a \wedge b) \in F(I)$, then $a \in F(I)$ or $b \in F(I)$. For any prime filter F of L is said to be a minimal, if there is no prime filter Q such that $Q \subsetneqq F$.

Here onwards L stands for an ADL with dense elements unless otherwise mentioned.

LEMMA 1.3. ([5]) *Every maximal ideal is prime.*

LEMMA 1.4. ([5]) *P is a prime filter (ideal) of L if and only if $L \backslash P$ is a prime ideal (filter) of L.*

LEMMA 1.5. ([5]) *P is a minimal (maximal) prime filter (ideal) of L if and only if $L \backslash P$ is a maximal (minimal) prime ideal (filter) of L.*

LEMMA 1.6. ([5]) *Let I be an ideal and F be a filter of L such that $I \cap F = \phi$. Then there exists a prime filter P such that $F \subseteq P$ and $P \cap I = \phi$.*

LEMMA 1.7. ([5]) *A prime ideal P of L is minimal prime ideal if and only if for each $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$.*

LEMMA 1.8. ([19]) *Let L be an ADL with maximal element. Then every prime ideal is minimal if and only if every prime ideal is maximal.*

DEFINITION 1.2. ([12]) *L is said to be weak relatively complemented if for any $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $(a \vee x)^* = (a \vee b)^*$.*

2. Normal Filters

In this section we define a normal filter and provide certain examples for it. We observe that the set of normal filters forms a distributive lattice which is not a sub distributive lattice of the set of filters in an almost distributive lattice.

For any filter F of L, let us denote $F^+ = \{x \in L \mid (x)^* \subseteq (a)^*$, for some $a \in F\}$. In particular, for any $a \in L$, $[a)^+ = \{x \in L \mid (x)^* \subseteq (a)^*\}$, where $[a)$ is a principal filter of L.

LEMMA 2.1. *We have*

(i) $L^+ = L = [0)^+$
(ii) $D^+ = D = M^+$
(iii) *For any* $d \in D$, $[d)^+ = D$
(iv) *For any filter* F *of* L, $F \subseteq F^+$ *and* $D \subseteq F^+$.

LEMMA 2.2. *For any filters* F, G *of* L, *we have*

(i) $F \subseteq G$ *implies* $F^+ \subseteq G^+$
(ii) $F^{++} = F^+$
(iii) $(F \cap G)^+ = F^+ \cap G^+$
(iv) $(F \vee G)^+ = (F^+ \vee G^+)^+$.

PROOF. (i) Let $x \in F^+$. Then $(x)^* \subseteq (a)^*$ for some $a \in F \subseteq G$. Therefore $x \in G^+$ and hence $F^+ \subseteq G^+$.

(ii) It is clear that $F^+ \subseteq F^{++}$. Let $x \in F^{++}$. Then $(x)^* \subseteq (a)^*$ for some $a \in F^+$. For this $a \in F^+$, $(a)^* \subseteq (b)^*$ for some $b \in F$. Therefore $(x)^* \subseteq (b)^*$ for some $b \in F$. Hence $x \in F^+$. Thus $F^{++} = F^+$.

(iii) Since $(F \cap G)^+ \subseteq F^+$, G^+ and $(F \cap G)^+ \subseteq F^+ \cap G^+$. Let $x \in F^+ \cap G^+$. Then $(x)^* \subseteq (a)^*$ and $(x)^* \subseteq (b)^*$ for some $a \in F$ and $b \in G$. Therefore $(x)^* \subseteq (a)^* \cap (b)^* = (a \vee b)^*$, where $a \vee b \in F \cap G$. Hence $x \in (F \cap G)^+$. Thus $(F \cap G)^+ = F^+ \cap G^+$.

(iv) Clearly we have $F \vee G \subseteq F^+ \vee G^+$ and $(F \vee G)^+ \subseteq (F^+ \vee G^+)^+$. On the other hand, let $x \in (F^+ \vee G^+)^+$, then there exists $a \wedge b \in F^+ \vee G^+$ such that $(x)^* \subseteq (a \wedge b)^*$, where $a \in F^+$ and $b \in G^+$. For $a \in F^+$ and $b \in G^+$, there exist $f \in F$ and $g \in G$ such that $(a)^* \subseteq (f)^*$ and $(b)^* \subseteq (g)^*$. Therefore $(f)^{**} \subseteq (a)^{**}$, $(g)^{**} \subseteq (b)^{**}$ and $(f)^{**} \cap (g)^{**} \subseteq (a)^{**}, (b)^{**}$. We get $(f \wedge g)^{**} \subseteq (a \wedge b)^{**}$. So that $(x)^* \subseteq (a \wedge b)^* \subseteq (f \wedge g)^*$, where $f \wedge g \in F \vee G$. Hence $x \in (F \vee G)^+$. Thus $(F^+ \vee G^+)^+ = (F \vee G)^+$. □

DEFINITION 2.1. *A filter* F *of* L *is said to be a normal filter, if* $F^+ = F$.

It can be easily observe that F^+ is the smallest normal filter containing F, for any filter F of L. In this regard, we have

LEMMA 2.3. *For any proper filter* F *of* L, F^+ *is always a proper normal filter.*

PROOF. Let F be a proper filter of L. Suppose F^+ is not proper. It means that $F^+ = L$. Since $0 \in L = F^+$, there exists $a \in F$ such that $L = (0)^* \subseteq (a)^*$. Therefore $a = 0$ and $0 \in F$. Hence $F = L$. Which is a contradiction to our assumption. Thus F^+ is proper. □

THEOREM 2.1. *Every maximal filter is a normal filter.*

REMARK 2.1. *The converse of above theorem need not be true. For, see the following example.*

Example 2.1. Let $L = \{0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, d, m\}$ with the operations $\wedge$ and $\vee$ defined as follows:

$\wedge$	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
0	0	0	0	0	0	0	0	0	0	0
b_1	0	b_1	0	b_1	b_1	0	b_1	0	b_1	b_1
b_2	0	0	b_2	b_2	b_2	0	0	b_2	b_2	b_2
b_3	0	b_1	b_2	b_3	b_3	0	b_1	b_2	b_3	b_3
b_4	0	b_1	b_2	b_3	b_4	0	b_1	b_2	b_3	b_4
b_5	0	0	0	0	0	b_5	b_5	b_5	b_5	b_5
b_6	0	b_1	0	b_1	b_1	b_5	b_6	b_5	b_6	b_6
b_7	0	0	b_2	b_2	b_2	b_5	b_5	b_7	b_7	b_7
d	0	b_1	b_2	b_3	b_3	b_5	b_6	b_7	d	d
m	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m

$\vee$	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
0	0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	d	m
b_1	b_1	b_1	b_3	b_3	b_4	b_6	b_6	d	d	m
b_2	b_2	b_3	b_2	b_3	b_4	b_7	d	b_7	d	m
b_3	b_3	b_3	b_3	b_3	b_4	d	d	d	d	m
b_4	b_4	b_4	b_4	b_4	b_4	m	m	m	m	m
b_5	b_5	b_6	b_7	d	m	b_5	b_6	b_7	d	m
b_6	b_6	b_6	d	d	m	b_6	b_6	d	d	m
b_7	b_7	d	b_7	d	m	b_7	d	b_7	d	m
d	d	d	d	d	m	d	d	d	d	m
m	m	m	m	m	m	m	m	m	m	m

Then $(L, \wedge, \vee, 0)$ is an ADL in which $[b_3)$ is a normal filter but not maximal.

From the above example we have the following

Remark 2.2. *Every minimal filter need not be normal. For, see example 2.1., $[m) = \{m\}$ is a minimal filter but not normal (because $[m)^+ = \{d, m\} \neq [m)$).*

Remark 2.3. *Every normal filter need not be prime. For, see example 2.1., $[b_3) = \{b_3, b_4, d, m\}$ is a normal filter but not a prime (because $b_3 = b_1 \vee b_2 \in [b_3)$, but $b_1 \notin [b_3)$ and $b_2 \notin [b_3)$).*

Remark 2.4. *Every prime filter need not be normal. For, see example 2.1., $[b_4) = \{b_4, m\}$ is a prime filter but not normal (because $[b_4)^+ = \{b_3, b_4, d, m\} \neq [b_4)$).*

Remark 2.5. *Every minimal prime filter need not be normal. For, see example 2.1., $[b_4) = \{b_4, m\}$ is a minimal prime filter but not normal (because $[b_4)^+ = \{b_3, b_4, d, m\} \neq [b_4)$).*

Let us denote the set of normal filters of L as $\mathcal{NF}(L)$. It can be observe that $\mathcal{NF}(L)$ is a distributive lattice.

THEOREM 2.2. *$\mathcal{NF}(L)$ can be a distributive lattice with the operations $F^+ \cap G^+ = (F \cap G)^+$ and $F \sqcup G = (F \vee G)^+$, for any $F, G \in \mathcal{NF}(L)$.*

PROOF. Let $F, G \in \mathcal{NF}(L)$. By lemma 2.2., $(F \cap G)^+$ is the infimum of F and G in $\mathcal{NF}(L)$. Also $(F \vee G)^+$ is an upper bound of F and G. Let $H \in \mathcal{NF}(L)$ such that $F^+ \subseteq H$, $G^+ \subseteq H$ and $x \in (F \vee G)^+$. Then $(x)^* \subseteq (a)^*$ for some $a \in F \vee G \subseteq H$. Therefore $x \in H^+ = H$ (since $H \in \mathcal{NF}(L)$). Thus $(F \vee G)^+ = F \sqcup G$ is the supremum of F and G in $\mathcal{NF}(L)$. Let $F, G, H \in \mathcal{NF}(L)$. Then $F \cap (G \sqcup H) = F^+ \cap (G \vee H)^+ = (F \cap (G \vee H))^+ = \{(F \cap G) \vee (F \cap H)\}^+ = (F \cap G) \sqcup (F \cap H)$ (since $\mathcal{F}(L)$ is a distributive lattice). Therefore $(\mathcal{NF}(L), \wedge, \sqcup)$ is a distributive lattice with the greatest element $L^+ = L = [0)^+$. □

THEOREM 2.3. *There is an epimorphism from $\mathcal{F}(L)$ onto $\mathcal{NF}(L)$.*

PROOF. Let $F, G \in \mathcal{F}(L)$. Define a map Φ: $\mathcal{F}(L) \longrightarrow \mathcal{NF}(L)$ by $\Phi(F) = F^+$. Then $\Phi(F \wedge G) = (F \wedge G)^+ = F^+ \cap G^+ = \Phi(F) \cap \Phi(G)$ and $\Phi(F \vee G) = (F \vee G)^+ = (F^+ \vee G^+)^+$ (by lemma 2.2(iv)) $= F^+ \sqcup G^+ = \Phi(F) \sqcup \Phi(G)$. Therefore Φ is a homomorphism. Since $\mathcal{NF}(L) \subseteq \mathcal{F}(L)$, Φ is an onto homomorphism. □

3. Normlets

In this section, we define normlets in an almost distributive lattice. We obtain necessary and sufficient conditions for a filter to become normal in terms of normlets. Finally we obtain necessary and sufficient conditions for an almost distributive lattice to become weak relatively complemented.

DEFINITION 3.1. *A filter F of L is said to be a normlet, if $F = [a)^+$, for some $a \in L$.*

THEOREM 3.1. *Every normlet is a normal filter.*

PROOF. Let $x \in L$ and $t \in [x)^{++}$. Then $(t)^* \subseteq (a)^*$, for some $a \in [x)^+$ and $(a)^* \subseteq (x)^*$. Therefore $(t)^* \subseteq (x)^*$ and hence $t \in [x)^+$. Thus $[x)^+$ is a normal filter. □

LEMMA 3.1. *For any $a, b \in L$, we have*
(i) *$a \leqslant b$ implies $[b)^+ \subseteq [a)^+$*
(ii) *$a \in [b)^+$ implies $[a)^+ \subseteq [b)^+$*
(iii) *$[a)^+ = D$ if and only if $a \in D$*
(iv) *$[a)^+ = L$ if and only if $a = 0$*
(v) *For any maximal element m of L, $[m)^+ = D$*
(vi) *$[a)^+ \cap [b)^+ = [a \vee b)^+$.*

PROOF. (i) Suppose that $a \leqslant b$. Then $[b) \subseteq [a)$. Therefore $[b)^+ \subseteq [a)^+$ (by lemma 2.2(i)).

(ii) Suppose that $a \in [b)^+$. Then $[a) \subseteq [b)^+$. Therefore $[a)^+ \subseteq [b)^{++} = [b)^+$ (since $[b)^+$ is normlet).

(iii) Suppose that $[a)^+ = D$. Then $a \in [a)^+ = D$. On the other hand, let $d \in D$, then $[d)^+ = \{x \in L \mid (x)^* \subseteq (d)^* = \{0\}\} = D$.

(iv) Suppose that $[a)^+ = L$. Then $0 \in L = [a)^+$. Therefore $L = (0)^* \subseteq (a)^*$. Hence $a = 0$. The converse is trivial.

(v) Since every maximal element is dense and from (iii), we have $[m)^+ = D$.

(vi) $[a)^+ \cap [b)^+ = ([a) \cap [b))^+ = [a \vee b)^+$ (by lemma 2.2). □

Lemma 3.2. *For any $a, b \in L$, we have*

(i) $a \wedge b = 0$ *implies* $[a)^+ \vee [b)^+ = L$
(ii) $a \vee b \in D$ *if and only if* $[a)^+ \cap [b)^+ = D$
(iii) *If* $a \neq 0$, *then* $(a)^* \cap [a)^+ = \phi$
(iv) $(a)^* = (b)^*$ *if and only if* $[a)^+ = [b)^+$
(v) $[a)^+ = [b)^+$ *implies* $[a \wedge c)^+ = [b \wedge c)^+$ *for all* $c \in L$
(vi) $[a)^+ = [b)^+$ *implies* $[a \vee c)^+ = [b \vee c)^+$ *for all* $c \in L$.

Proof. (i) Suppose that $a \wedge b = 0$. Then $L = [0) = [a \wedge b) = [a) \vee [b) \subseteq [a)^+ \vee [b)^+ \subseteq L$. Therefore $[a)^+ \vee [b)^+ = L$.

(ii) It can be obtained by Lemma 3.1. (iii) Suppose that $a \neq 0$. Let $x \in (a)^* \cap [a)^+$. Then $(x)^* \subseteq (a)^*$ and $a \wedge x = 0$. Therefore $a \in (x)^* \subseteq (a)^*$. Hence $a \wedge a = 0$. Which is a contradiction. Thus $(a)^* \cap [a)^+ = \phi$.

(iv) Suppose that $(a)^* = (b)^*$. Then $a \in [b)^+$ and $b \in [a)^+$. Therefore $[a)^+ \subseteq [b)^+$ and $[b)^+ \subseteq [a)^+$. Hence $[a)^+ = [b)^+$. On the other hand, suppose that $[a)^+ = [b)^+$. Then $a \in [b)^+$ and $b \in [a)^+$. Therefore $(a)^* \subseteq (b)^*$ and $(b)^* \subseteq (a)^*$ and hence $(a)^* = (b)^*$.

(v) Suppose that $[a)^+ = [b)^+$. For any $t \in L$,

$$\begin{aligned} t \in (a \wedge c)^* &\iff t \wedge a \wedge c = 0 \\ &\iff t \wedge c \in (a)^* = (b)^* \text{ (from (iv))} \\ &\iff t \wedge b \wedge c = 0 \\ &\iff t \in (b \wedge c)^*. \end{aligned}$$

By (iv), we get $[a \wedge c)^+ = [b \wedge c)^+$.

(vi) It can be obtained from (v). □

Theorem 3.2. *For any filter F of L, the following are equivalent;*

(i) F *is normal*
(ii) *For* $x \in L$, $x \in F$ *implies* $[x)^+ \subseteq F$
(iii) *For* $x, y \in L$, $(x)^* = (y)^*$ *and* $x \in F$ *implies* $y \in F$
(iv) *For* $x, y \in L$, $[x)^+ = [y)^+$ *and* $x \in F$ *implies* $y \in F$
(v) $F = \bigcup_{x \in F} [x)^+$.

Proof. (i) $\Longrightarrow$ (ii) Assume(i). Let $x \in F$. Then $[x) \subseteq F$. Therefore $[x)^+ \subseteq F^+ = F$. Thus $[x)^+ \subseteq F$.

(ii) $\Longrightarrow$ (iii) Assume(ii). Let $x, y \in L$ such that $(x)^* = (y)^*$ and $x \in F$. Then $[y)^+ = [x)^+ \subseteq F$ (by our assumption). Therefore $y \in F$.

(iii) $\Longrightarrow$ (iv) It is clear by lemma 3.2.

(iv) $\Longrightarrow$ (v) Assume(iv). Let $x \in F$. Then $x \in [x)^+$. Hence $F \subseteq \bigcup_{x \in F} [x)^+$. On the other hand, let $x \in F$ and $y \in [x)^+$, then $[y)^+ \subseteq [x)^+$. Therefore $[y)^+ =$

$[y)^+ \cap [x)^+ = [y \vee x)^+$ and $y \vee x \in F$. By our assumption, $y \in F$. Therefore $[x)^+ \subseteq F$ and hence $\bigcup_{x \in F} [x)^+ \subseteq F$.

(v) $\Longrightarrow$ (i) Assume(v). Let $x \in F^+$. Then there exists $a \in F$ such that $(x)^* \subseteq (a)^*$. Therefore $x \in [a)^+$ and hence $x \in \bigcup_{y \in F} [y)^+ = F$ (by our assumption). Thus F is normal. □

Let us denote the set of normlets of L as $\mathcal{N}^+\mathcal{F}(L)$. Then we have the following;

Theorem 3.3. *$(\mathcal{N}^+\mathcal{F}(L), \cap, \sqcup)$ is a sublattice of $\mathcal{NF}(L)$ in which $[0)^+$ is the greatest element in $\mathcal{N}^+\mathcal{F}(L)$. Moreover $\mathcal{N}^+\mathcal{F}(L)$ has the smallest element if and only if L has dense element.*

Proof. It can be observed that by theorem 2.2., $(\mathcal{N}^+\mathcal{F}(L), \cap, \sqcup)$ is a sublattice of a distributive lattice $(\mathcal{NF}(L), \cap, \sqcup)$ with the greatest element $[0)^+ = L$. Now, suppose that L has a dense element, say d and let $x \in [d)^+$, then $(x)^* \subseteq (d)^* = \{0\} \subseteq (a)^*$ for all $a \in L$. Therefore $d \in [a)^+$ for all $a \in L$. Hence $[d)^+ \subseteq [a)^+$ for all $a \in L$. Thus $[d)^+$ is the smallest element in $\mathcal{N}^+\mathcal{F}(L)$. Conversely suppose that $\mathcal{N}^+\mathcal{F}(L)$ has the smallest element, say $[a)^+$ for some $a \in L$. Let $x \in (a)^*$. Then $x \wedge a = 0$. Therefore $[x \wedge a)^+ = [x)^+ \sqcup [a)^+ = [x)^+ = L$. Hence $x = 0$. Thus a is dense in L. □

Definition 3.2. [2] : *L is said to be disjunctive, if for any $x, y \in L$, $x \neq y$ implies $(x)^* \neq (y)^*$.*

Theorem 3.4. *If L is a disjunctive ADL, then every filter is normal.*

Proof. Suppose that a filter F of L is not normal. Then there exist $x, y \in L$ such that $[x)^+ = [y)^+$, $x \in F$ and $y \notin F$. Therefore $(x)^* = (y)^*$. Since L is disjunctive, $x = y$. Hence $y \in F$. Which is a contradiction. Thus F is normal. □

Remark 3.1. *The converse of above theorem need not be true. For, see the following example;*

Example 3.1. Let $L = \{0, d_1, d_2, d_3, m_1, m_2\}$ with the operations $\wedge$ and $\vee$ defined as follows

$\wedge$	0	d_1	d_2	d_3	m_1	m_2
0	0	0	0	0	0	0
d_1	0	d_1	d_2	0	d_1	d_2
d_2	0	d_1	d_2	0	d_1	d_2
d_3	0	0	0	d_3	d_3	d_3
m_1	0	d_1	d_2	d_3	m_1	m_2
m_2	0	d_1	d_2	d_3	m_1	m_2

$\vee$	0	d_1	d_2	d_3	m_1	m_2
0	0	d_1	d_2	d_3	m_1	m_2
d_1	d_1	d_1	d_1	m_1	m_1	m_1
d_2	d_2	d_2	d_2	m_2	m_2	m_2
d_3	d_3	m_1	m_2	d_3	m_1	m_2
m_1	m_1	m_1	m_1	m_1	m_1	m_1
m_2	m_2	m_2	m_2	m_2	m_2	m_2

Then $(L, \wedge, \vee, 0)$ is an ADL in which every filter is normal but it is not a disjunctive ADL (because $(m_1)^* = (m_2)^*$ but $m_1 \neq m_2$).

Define a relation ψ on L by $\psi = \{(x, y) \in L \times L \mid [x)^+ = [y)^+\}$. It is easy to observe that ψ is a congruence relation on L(By lemma 3.2).

THEOREM 3.5. *The quotient lattice L/ψ forms a distributive lattice with the operations $x/\psi \wedge y/\psi = (x \wedge y)/\psi$ and $x/\psi \vee y/\psi = (x \vee y)/\psi$. Moreover the least element is $0/\psi = \{0\}$ and the greatest element is $d/\psi = D$.*

S. Ramesh and G. Jogarao [12] introduced the concept of dense complemented ideal in ADL. An ideal I of L is said to be a dense complemented in L, if there exists an ideal J in L such that $I \wedge J = \{0\}$ and $I \vee J$ is an ideal generated by a dense element in L.

THEOREM 3.6. *The following are equivalent;*

(i) *L is a weak relatively complemented*
(ii) *$(\mathcal{N}^+\mathcal{F}(L), \cap, \sqcup, D, L)$ is a Boolean algebra*
(iii) *$(L/\psi, \wedge, \vee, 0/\psi, d/\psi)$ is a Boolean algebra*
(iv) *Every principal ideal of L is dense complemented.*

PROOF. (i) $\Longrightarrow$ (ii) Suppose that L is a weak relatively complemented ADL. Let $x \in L$ and d is a dense element in L. Then by our assumption, there exists $y \in L$ such that $x \wedge y = 0$ and $(x \vee y)^* = (x \vee d)^* = \{0\}$. Therefore $x \vee y$ is dense. Now, $[x)^+ \cap [y)^+ = ([x) \cap [y))^+ = [x \vee y)^+ = D$ and $[x)^+ \sqcup [y)^+ = ([x) \vee [y))^+ = [x \wedge y)^+ = [0)^+ = L$. Therefore $\mathcal{N}^+\mathcal{F}(L)$ is a Boolean algebra.

(ii) $\Longrightarrow$ (iii) Suppose that $(\mathcal{N}^+\mathcal{F}(L), \cap, \sqcup)$ is a Boolean algebra. Let $x \in L$. Then by our assumption, there exists $y \in L$ such that $[x)^+ \cap [y)^+ = D$ and $[x)^+ \sqcup [y)^+ = L$. That is $[x \vee y)^+ = D$ and $L = [x \wedge y)^+$. Therefore $x \vee y$ is dense and $x \wedge y = 0$ and hence $x/\psi \wedge y/\psi = (x \wedge y)/\psi = 0/\psi = \{0\}$ and $x/\psi \vee y/\psi = (x \vee y)/\psi = D$. Thus L/ψ is a Boolean algebra.

(iii) $\Longrightarrow$ (iv) Suppose that $(L/\psi, \wedge, \vee)$ is a Boolean algebra. Let $x \in L$. Then by our assumption, there exists $y \in L$ such that $x/\psi \wedge y/\psi = (x \wedge y)/\psi = 0/\psi$ and $x/\psi \vee y/\psi = (x \vee y)/\psi = d/\psi$. Therefore $x \wedge y = 0$ and $x \vee y$ is dense and hence $(x] \cap (y] = (x \wedge y] = (0]$ and $(x] \vee (y] = (x \vee y]$ is an ideal generated by a dense element $x \vee y$. Thus $(x]$ is a dense complemented ideal.

(iv) $\Longrightarrow$ (i) Let $a, b \in L$. Then there exist $c, d \in L$ such that $(a] \wedge (c] = \{0\} = (b] \wedge (d]$ and $(a] \vee (c]$ and $(b] \vee (d]$ are the principal ideals generated by dense elements. Thus $a \wedge c = 0 = b \wedge d$ and $a \vee c$, $b \vee d$ are dense elements. Take $x = c \wedge b$. Then $a \wedge x = a \wedge c \wedge b = 0$ (since $a \wedge c = 0$) and $(a \vee x) \wedge (a \vee b) = a \vee (x \wedge b) = a \vee (c \wedge b \wedge b) = a \vee x$. So that $(a \vee b)^* \subseteq (a \vee x)^*$. Now, for $t \in L$,

$$\begin{aligned} t \in (a \vee x)^* &\Rightarrow t \wedge (a \vee x) = 0 \\ &\Rightarrow t \wedge a = 0 \text{ and } t \wedge c \wedge b = 0 \\ &\Rightarrow t \wedge b \wedge (a \vee c) = 0 \\ &\Rightarrow t \wedge b = 0 \qquad \text{(since } a \vee c \text{ is dense)} \\ &\Rightarrow t \wedge (a \vee b) = 0 \\ &\Rightarrow t \in (a \vee b)^*. \end{aligned}$$

Therefore $(a \vee x)^* \subseteq (a \vee b)^*$ and hence $(a \vee x)^* = (a \vee b)^*$. Thus L is a weak relatively complemented. □

THEOREM 3.7. *If L is an ADL in which every dense element is maximal, then the following are equivalent;*

(i) *L is quasi complemented*
(ii) *L is a relatively complemented*
(iii) *$(\mathcal{N}^+\mathcal{F}(L), \cap, \sqcup, M, L)$ is a Boolean algebra*
(iv) *$(L/\psi, \wedge, \vee, 0/\psi, m/\psi)$ is a Boolean algebra*
(v) *Every principal ideal of L is complemented.*

THEOREM 3.8. *L is weak relatively complemented if and only if for any $a, b \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $[a \vee x)^+ = [a \vee b)^+$.*

4. Normal Prime filters

In this section, we study the class of normal prime filters in an almost distributive lattice with dense elements.

THEOREM 4.1. *Let F be a filter of L and for any chain of filters $C_1, C_2, C_3, ...$ of L such that $F \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq .... \subseteq F^+$. Then $C_1^+ = C_2^+ = C_3^+ = ... = F^+$.*

PROOF. Suppose that F be a filter of L and for any chain of filters $C_1, C_2, C_3, ..$ of L such that $F \subseteq C_1 \subseteq C_2 \subseteq C_3 \subseteq .... \subseteq F^+$. Then $F^+ \subseteq C_1^+ \subseteq C_2^+ \subseteq C_3^+ \subseteq .... \subseteq F^{++} = F^+$ (since F^+ is normal). Therefore $C_1^+ = C_2^+ = C_3^+ = ... = F^+$. □

THEOREM 4.2. *Let F be a proper filter of L. Then there exists a normal prime filter containing F.*

PROOF. Let F be a proper filter of L. Take $\mathcal{P} = \{G \mid G$ is a proper normal filter of L and $F \subseteq G\}$. By lemma 2.3., F^+ is a proper normal filter containing F. Therefore $F^+ \in \mathcal{P}$ and $\mathcal{P}$ satisfies the hypothesis of Zorn's lemma. Hence $\mathcal{P}$ has a maximal element, say P. Let $a, b \in L$ such that $a \notin P$ and $b \notin P$. Then $P \sqcup [a)^+$ and $P \sqcup [b)^+$ are normal filters, which containing P properly. By the maximality of P, $L = P \sqcup [a)^+ = P \sqcup [b)^+$. Therefore $L = \{P \sqcup [a)^+\} \cap \{P \sqcup [b)^+\} = \{(P \sqcup [a)) \cap (P \sqcup [b))\}^+ = \{P \vee [a \vee b)\}^+$. If $a \vee b \in P$, then $L = P^+ = P$. Which is a contradiction. Hence $a \vee b \notin P$. Thus P is prime. □

THEOREM 4.3. *If P is a minimal in the class of prime filters containing a normal filter F, then P is normal.*

PROOF. Let F be a normal filter of L and P is minimal in the class of prime filters of L containing F. Suppose that P is not a normal. Then there exist $x, y \in L$ such that $[x)^+ = [y)^+$, $x \in P$ and $y \notin P$. Take $I = L - P \vee (x \vee y]$ is an ideal of L. Then $I \cap F = \phi$. If $I \cap F \neq \phi$, then $a \in I \cap F$. Therefore $a = r \vee s$ for some $r \in L - P$ and $s \in (x \vee y]$. So that $r \vee s = r \vee \{(x \vee y) \wedge s\} = r \vee \{(y \vee x) \wedge s\} = \{r \vee (y \vee x)\} \wedge (r \vee s) \in F$(since $r \vee s = a \in F$). So that $r \vee (y \vee x) \in F$. We have $[x)^+ = [y)^+$. Then $[r \vee y \vee x)^+ = [r \vee y \vee y)+ = [r \vee y)^+$. Since F is normal, $r \vee y \in P$. Which is a contradiction. Therefore $I \cap F = \phi$. So that there exists a prime filter Q such that $I \cap Q = \phi$, $F \subseteq Q$ and $Q \subseteq P$. Also $x \vee y \notin Q$ and $x \vee y \in P$. We get $Q \subsetneqq P$. Hence P is not minimal. Which is a contradiction. Thus P is normal prime filter. □

Corollary 4.1. *Every minimal prime filter containing D is normal.*

Proof. Let P be the minimal prime filter of L and $D \subseteq P$. That is P is the minimal in the class of prime filters containing (the normal filter) D. By the above theorem, P is normal. □

Theorem 4.4. *Let F be a normal filter and I is an ideal of L such that $F \cap I = \phi$. Then there exists a normal prime filter P such that $F \subseteq P$ and $P \cap I = \phi$.*

Proof. Let F be a normal filter and I is an ideal of L such that $F \cap I = \phi$. Take $\mathcal{P} = \{G \mid G$ is a normal filter, $F \subseteq G$ and $G \cap I = \phi\}$. Clearly $F \in \mathcal{P}$ and it satisfies the hypothesis of Zorn's Lemma. Therefore $\mathcal{P}$ has a maximal element, say P. Choose $x, y \in L$ such that $x \notin P$ and $y \notin P$. Then $P \subseteq P \sqcup [x)^+ = \{P \vee [x)\}^+$ and $P \subseteq P \sqcup [y)^+ = \{P \vee [y)\}^+$. By the maximality of P, $\{P \vee [x)\}^+ \cap I \neq \phi$ and $\{P \vee [y)\}^+ \cap I \neq \phi$. Let $a \in \{P \vee [x)\}^+ \cap I$ and $b \in \{P \vee [y)\}^+ \cap I$. Then $a \vee b \in I$ and $a \vee b \in \{P \vee [x)\}^+ \cap \{P \vee [y)\}^+ = \{\{P \vee [x)\} \cap \{P \vee [y)\}\}^+ = \{P \vee [x \vee y)\}^+$. If $x \vee y \in P$, then $a \vee b \in P^+ = P$ (since P is normal) and $a \vee b \in I$. Therefore $P \cap I \neq \phi$. Which is a contradiction. So that $x \vee y \notin P$. Hence P is prime. Thus P is a normal prime filter of L such that $F \subseteq P$ and $P \cap I = \phi$. □

Remark 4.1. *If F is not a normal filter, then the above theorem need not be true.*
For, see the example 2.1., let $F = [b_4)$ and $I = (d]$, then $F \cap I = \phi$. But there is no normal prime filter P such that $P \cap I = \phi$ and $F \subseteq P$.

Corollary 4.2. *Let F be a normal filter of L and $x \notin F$. Then there exists a normal prime filter P of L such that $F \subseteq P$ and $x \notin P$.*

Proof. Let F be a normal filter of L and $x \notin F$. Then $(x] \cap F = \phi$. Therefore by theorem 4.4., there exists a normal prime filter P such that $F \subseteq P$ and $(x] \cap P = \phi$. Thus $x \notin P$. □

Theorem 4.5. *For any filter F,*

$$F^+ = \cap\{P \mid P \text{ is a normal prime filter of } L \text{ and } F \subseteq P\}.$$

Proof. Let F be a filter of L and $x \notin F^+$. Put $\mathcal{P} = \{G \mid G$ is a normal filter of L and $x \notin G$ and $F \subseteq G\}$. Clearly $F^+ \in \mathcal{P}$ and it satisfies the hypothesis of Zorn's Lemma. Therefore $\mathcal{P}$ has a maximal element, say P. Let $a, b \in L$ such that $a \notin P$ and $b \notin P$. Then $x \in P \sqcup [a)^+ = \{P \vee [a)^+\}^+ = \{P \vee [a)\}^+$ and $x \in P \sqcup [b)^+ = \{P \vee [b)^+\}^+ = \{P \vee [b)\}^+$. Thus $x \in \{P \vee [a)\}^+ \cap \{P \vee [b)\}^+ = \{P \vee [a \vee b)\}^+$. Suppose that $a \vee b \in P$. Then $x \in P^+ = P$ (since P is normal). So that $x \in P$. Which is a contradiction. Therefore P is prime. Hence P is normal prime filter containing F and $x \notin P$. Thus $F^+ = \cap\{P \mid P$ is a normal prime filter of L and $F \subseteq P\}$. □

Corollary 4.3. *The intersection of normal prime filters of L is equals to D.*

Proof. We have every normal filter of L containing the filter D. Hence from the above theorem it is obvious. □

Corollary 4.4. *For any $a \in L$ and $a \notin D$, there exists a normal prime filter P of L such that $a \notin P$.*

Theorem 4.6. *For any filter F of L, $F^+ \cap F^* = \phi$.*

Proof. Let F be a filter of L. Suppose that $F^+ \cap F^* \neq \phi$. Let $t \in F^+ \cap F^*$. Then there exists $a \in F$ such that $(t)^* \subseteq (a)^*$ and $t \wedge f = 0$ for all $f \in F$. Therefore $a \in (t)^*$(since $t \wedge a = 0$). Hence $a = 0$ and $a \in F$. Which is a contradiction. Thus $F^+ \cap F^* = \phi$. □

Corollary 4.5. *For any filter F of L, there exists a normal prime filter P of L such that $F \subseteq P$ and $P \cap F^* = \phi$.*

Proof. Let F be a filter of L. Then by above theorem, $F^+ \cap F^* = \phi$. By theorem 4.4., there exists a normal prime filter P of L such that $F^+ \subseteq P$ and $P \cap F^* = \phi$. Thus $F \subseteq P$ and $P \cap F^* = \phi$. □

5. The Space of Normal Prime Filters in Almost Distributive Lattices

In this section, we discuss the space of normal prime filters in an almost distributive lattice with the Hull-kernel topology. Finally we obtain necessary and sufficient conditions for the space of normal prime filters to become Hausdorff.

Let us denote Spec$L$ the set of normal prime filters of L. For any $A \subseteq L$, $K(A) = \{P \in \text{Spec}L \mid A \nsubseteq P\}$. In particular, for $a \in L$, $K(a) = \{P \in \text{Spec}L \mid a \notin P\}$.

Lemma 5.1. *For any $a, b \in L$, we have*
- (i) $\bigcup_{a \in L} K(a) = SpecL$
- (ii) $K(a) \cap K(b) = K(a \vee b)$
- (iii) $K(a) \cup K(b) = K(a \wedge b)$
- (iv) $K(a) = \phi$ *if and only if* $a \in D$.
- (v) $K(a) = SpecL$ *if and only if* $a = 0$.

From the above lemma, it can be easy to observe that $\{K(a)\}$, $a \in L$ forms a base for a topology on Spec$L$.

Theorem 5.1. *We have the following*
- (i) *For any $a \in L$, $K(a)$ is compact*
- (ii) *If C is a compact open subset of SpecL, then $C = K(a)$ for some $a \in L$.*

Proof. (i) Let $a \in L$ and $B \subseteq L$ such that $K(a) \subseteq \bigcup_{b \in B} K(b)$ and $F = [B)$ is a normal filter of L generated by B. If $a \notin F$, by the corollary 4.2., there exists a normal prime filter P such that $F \subseteq P$ and $a \notin P$. Therefore $P \in K(a) \subseteq \bigcup_{b \in B} K(b)$. Hence $b \notin P$ for some $b \in B$. Which is a contradiction. So that $a \in F = [B)$ and $a = x \vee (\bigwedge_{i=1}^{n} b_i)$ for some $b_1, b_2, b_3, ..., b_n \in B$ and $x \in L$. By lemma 5.1.,

$K(a) = K(x \vee (\bigwedge_{i=1}^{n} b_i)) = K(x) \cap K(\bigwedge_{i=1}^{n} b_i)) \subseteq K(\bigwedge_{i=1}^{n} b_i) = \bigcup_{i=1}^{n} K(b_i)$ and hence K(a) is compact in Spec$L$.

(ii) Let C is a compact open subset of Spec$L$. Then $C = K(A)$, for some $A \subseteq L$ (since C is open). Therefore $C = \bigcup_{a \in A} K(a)$. Therefore there exist $a_1, a_2, ...a_n \in A$ such that $C = \bigcup_{i=1}^{n} K(a_i) = K(a)$ for some $a \in L$ (since C is compact).

□

Theorem 5.2. *Let L be an ADL in which every prime filter is normal. Then L is a distributive lattice if and only if the map $a \longmapsto K(a)$ is an injection.*

Proof. Suppose that L is a distributive lattice in which every prime filter is normal. Let $a, b \in L$ such that $a \neq b$. Then there exists a prime filter P such that $a \in P$ and $b \notin P$. Hence $K(a) \neq K(b)$. Thus the map is injunction. Conversely suppose that the map is injunction. Let $a, b \in L$. Then $K(a \vee b) = K(b \vee a)$. Therefore $a \vee b = b \vee a$. Hence L is a distributive lattice. □

Lemma 5.2. *Let L be an ADL with maximal elements. Let P be a normal prime filter of L. Then P is minimal if and only if for each $x \in P$ there exists $y \notin P$ such that $x \vee y$ is maximal.*

Proof. Let L be an ADL with maximal elements. Let P be a normal prime filter of L. Suppose that P is a minimal prime filter of L. Let $x \in P$. Then $L \backslash P \vee (x] = L$ (since $L \backslash P$ is maximal ideal). Therefore there exists a maximal element $m \in L$ such that $m = x \vee y$, $x \in P$ and $y \notin P$. On the other hand, clearly we have $L \backslash P$ is a prime ideal of L. Then there exists $x \in L$ such that $x \notin L \backslash P$. Therefore by our assumption there exists $y \notin P$ such that $x \vee y$ is maximal. Hence $L \backslash P$ is maximal. Thus P is minimal prime filter of L. □

For any $A \subseteq L$, denote $H(A) = \{P \in \text{Spec}L \mid A \subseteq P\}$. Then $H(A) = \text{Spec}L \backslash K(A)$. Therefore $H(A)$ is a closed set in Spec$L$ and hence every closed set in Spec$L$ is of the form $H(A)$ for some $A \subseteq L$. Thus we have the following;

Theorem 5.3. *For any $Y \subseteq$ Spec$L$, the closure of Y is given by $\overline{Y} = H(\bigcap_{P \in Y} P)$.*

Proof. Let $Y \subseteq \text{Spec}L$. Let $Q \in Y$. Then $\bigcap_{P \in Y} P \subseteq Q$. Therefore $Q \in H(Q) \subseteq H(\bigcap_{P \in Y} P)$. Hence $H(\bigcap_{P \in Y} P)$ is a closed set containing Y. Let C be a closed set in Spec$L$ containing Y. Then $C = H(A)$, for some $A \subseteq L$. Therefore $A \subseteq \bigcap_{P \in Y} P$. Hence $H(\bigcap_{P \in Y} P) \subseteq H(A) = C$. Thus $\overline{Y} = H(\bigcap_{P \in Y} P)$. □

Theorem 5.4. *Let L be an ADL in which every dense element is maximal. Then the following are equivalent;*

(i) *Every normal prime filter is maximal*
(ii) *Every normal prime filter is minimal*
(iii) *Spec$L$ is a T_1- space*

(iv) *SpecL is a Hausdorff space*
(v) *For any $x, y \in L$, there exists $z \in L$ such that $x \vee z$ is maximal and $K(y) \cap \{SpecL - K(x)\} = K(y \vee z)$.*

PROOF. (i) $\Longrightarrow$ (ii) Let P be a normal prime filter of L. If Q is a normal prime filter of L such that $Q \subseteq P$, then by our assumption $Q = P$. Therefore every normal prime filter is minimal.
(ii) $\Longrightarrow$ (i) Let P be a normal prime filter of L. If Q is a normal prime filter of L such that $P \subseteq Q$, then $P = Q$ (by our assumption). Therefore P is maximal.

(ii) $\Longrightarrow$ (iii) Let P and Q are two distinct normal prime filters of L. Then $P \nsubseteq Q$ and $Q \nsubseteq P$ (since P and Q are minimal). Take $x \in P/Q$ and $y \in Q/P$. Then $Q \in K(x)/K(y)$ and $P \in K(y)/K(x)$. Therefore Spec$L$ is T_1- space.

(iii) $\Longrightarrow$ (iv) Suppose that Spec$L$ is T_1- space. Let $P \in$ Spec$L$. Then $P = \{\overline{P}\} = H(P) = \{Q \in \text{Spec}L \mid P \subseteq Q\}$. Therefore P is maximal. Hence every normal prime filter is maximal. Let $P, Q \in$ Spec$L$ such that $P \neq Q$. Choose $x \in P$ and $x \notin Q$. Then there exists $y \notin P$ such that $x \vee y$ is maximal. Therefore $P \in K(y)$ and $Q \in K(x)$ and $K(x \vee y) = K(x) \cap K(y) = \phi$. Hence Spec$L$ is Hausdorff space.

(iv) $\Longrightarrow$ (v) Let $a \in L$. Then $K(a)$ is a compact subset of the Hausdorff space Spec$L$. Then $K(a)$ is clopen subset of Spec$L$. Let $x, y \in L$ such that $x \neq y$. Then $K(y) \cap \{\text{Spec}L \backslash K(x)\}$ is also a compact open subset of a compact space $K(y)$. Hence $K(y) \cap \{\text{Spec}L \backslash K(x)\}$ is a compact open subset of Spec$L$. By lemma 5.1., there exists $z \in L$ such that $K(z) = K(y) \cap \{\text{Spec}L \backslash K(x)\}$. So that $K(y \vee z) = K(y) \cap K(z) = K(y) \cap \{\text{Spec}L \backslash K(x)\} = K(z)$ and $K(x \vee z) = K(x) \cap K(z) = \phi$. By lemma 5.1., $x \vee z$ is dense. Thus $x \vee z$ is maximal (since every dense is maximal).

(v) $\Longrightarrow$ (ii) Let P is a normal prime filter of L. Let $x, y \in L$ such that $x \in P$ and $y \notin P$. Then by our assumption there exists $z \in L$ such that $x \vee z$ is maximal and $K(y \vee z) = K(y) \cap \{\text{Spec}L \backslash K(x)\}$, $P \notin K(x)$ and $P \in K(y)$. Therefore $P \in K(y) \cap \{\text{Spec}L \backslash K(x)\} = K(y \vee z)$. If $z \in P$, then $y \vee z \in P$, which is a contradiction. Hence for each $x \in L$, there exists a normal prime filter P and $z \notin P$ such that $x \vee z$ is maximal. Thus P is minimal (by lemma 5.2). □

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Received by editors 18.09.2016; Available online 16.01.2017.

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