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ON BANHATTI AND ZAGREB INDICES

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Abstract. Let $G = (V, E)$ be a connected graph. The Zagreb indices were introduced as early as in 1972. They are defined as $M_1(G) = \sum_{uv\in E(G)}[d_G(u)+ d_G(v)]$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$, where $d_G(u)$ denotes the degree of a vertex u. The K Banhatti indices were introduced by Kulli in 2016. They are defined as $B_1(G) = \sum_{ue}[d_G(u) + d_G(e)]$ and $B_2(G) = \sum_{ue} d_G(u)d_G(e)$, where ue means that the vertex u and edge e are incident and $d_G(e)$ denotes the degree of the edge e in G. These two types of indices are closely related. In this paper, we obtain some relations between them. We also provide lower and upper bounds for $B_1(G)$ and $B_2(G)$ of a connected graph in terms of Zagreb indices.

1. Introduction

The graphs considered here are finite, undirected, without loops and multiple edges. Let $G = (V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree $d_G(v)$ of a vertex v is the number of vertices adjacent to v. The edge connecting the vertices u and v will be denoted by uv. Let $d_G(e)$ denote the degree of an edge $e = uv$ in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The vertices and edges of a graph are said to be its elements. For additional definitions and notations, the reader may refer to [**11**].

A molecular graph is a graph in which the vertices correspond to the atoms and the edges to the bonds of a molecule. A single number that can be computed from the molecular graph, and used to characterize some property of the underlying molecule is said to be a topological index or molecular structure descriptor. Numerous such descriptors have been considered in theoretical chemistry, and have found some applications, especially in QSPR/QSAR research, see [**6, 9, 17**].

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In [**12**], Kulli introduced the first and second K Banhatti indices, intending to take into account the contributions of pairs of incident elements. The first K Banhatti index $B_1(G)$ and the second K Banhatti index $B_2(G)$ of a graph G are defined as

$$B_1(G) = \sum_{ue} [d_G(u) + d_G(e)] \qquad \text{and} \qquad B_2(G) = \sum_{ue} d_G(u)\, d_G(e)$$

where ue means that the vertex u and edge e are incident in G.

The first and second K hyper–Banhatti indices of a graph G are defined as

$$HB_1(G) = \sum_{ue} [d_G(u) + d_G(e)]^2 \qquad \text{and} \qquad HB_2(G) = \sum_{ue} [d_G(u)\, d_G(e)]^2 .$$

The K hyper–Banhatti indices were introduced by Kulli in [**13**].

The degree–based graph invariants $M_1(G)$ and $M_2(G)$, called Zagreb indices, were introduced long time ago [**10**] and have been extensively studied. For the their history, applications, and mathematical properties, see [**2, 6, 7, 8, 15**] and the references cited therein.

The first and second Zagreb indices take into account the contributions of pairs of adjacent vertices. The first and second Zagreb indices of a graph G are defined as

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 \qquad \text{or} \qquad M_1(G) = \sum_{uv \in E(G)} \big[d_G(u) + d_G(v)\big]$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)\, d_G(v).$$

In [**14**], Miličević, et al., reformulated the first Zagreb index in terms of edge-degrees instead of vertex-degrees and defined the respective topological index as

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2 .$$

Followed by the first Zagreb index of a graph G, Furtula and one of the present authors [**5**] introduced the so-called forgotten topological index F, defined as

$$F(G) = \sum_{v \in V(G)} d_G(u)^3 = \sum_{uv \in V(G)} \big[d_G(u)^2 + d_G(v)^2\big].$$

In [**16**], Shirdel et al., introduced the first hyper–Zagreb index of G and defined it as

$$HM_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2 .$$

2. Comparison of Banhatti and Zagreb–type indices

THEOREM 2.1. *For any graph G, the first Banhatti index is related to the first Zagreb index as $B_1(G) = 3M_1(G) - 4m$.*

PROOF. Let G be a graph with $n \geqslant 3$ vertices and m edges. Then

$$\begin{aligned}
B_1(G) &= \sum_{ue}[d_G(u) + d_G(e)] \\
&= \sum_{uv\in E(G)} [d_G(u) + d_G(uv)] + \sum_{uv\in E(G)} [d_G(v) + d_G(uv)] \\
&= \sum_{uv\in E(G)} [d_G(u) + d_G(u) + d_G(v) - 2] \\
&+ \sum_{uv\in E(G)} [d_G(v) + d_G(u) + d_G(v) - 2] \\
&= \sum_{uv\in E(G)} [3d_G(u) + 3d_G(v) - 4] = 3M_1(G) - 4m.
\end{aligned}$$

□

THEOREM 2.2. *For any graph G, the second Banhatti index is related to the first Zagreb and hyper–Zagreb indices as $B_2(G) = HM_1(G) - 2M_1(G)$.*

PROOF. Let G be a graph with $n \geqslant 3$ vertices and m edges. Then

$$\begin{aligned}
B_2(G) &= \sum_{ue} d_G(u)\, d_G(e) \\
&= \sum_{uv\in E(G)} d_G(u)\, d_G(uv) + \sum_{uv\in E(G)} d_G(v)\, d_G(uv) \\
&= \sum_{uv\in E(G)} d_G(u)\, [d_G(u) + d_G(v) - 2] \\
&+ \sum_{uv\in E(G)} d_G(v)\, [d_G(u) + d_G(v) - 2] \\
&= \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^2 - 2[d_G(u) + d_G(v)] \\
&= HM_1(G) - 2M_1(G).
\end{aligned}$$

□

THEOREM 2.3. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then $EM_1(G) = HM_1(G) - 4M_1(G) + 4m$.*

PROOF. Let G be a graph with $n \geqslant 3$ vertices and m edges. Then

$$\begin{aligned}
EM_1(G) &= \sum_{e\in E(G)} d_G(e)^2 = \sum_{uv\in E(G)} [d_G(u) + d_G(v) - 2]^2 \\
&= \sum_{uv\in E(G)} \Big([d_G(u) + d_G(v)]^2 - 4[d_G(u) + d_G(v)] + 4\Big) \\
&= HM_1(G) - 4M_1(G) + 4m.
\end{aligned}$$

□

THEOREM 2.4. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then* $B_1(G) = HM_1(G) - EM_1(G) - M_1(G)$.

PROOF.

$$\begin{aligned} EM_1(G) &= HM_1(G) - 4M_1(G) + 4m \\ &= HM_1(G) - M_1(G) - \big[3M_1(G) - 4m\big] \\ &= HM_1(G) - M_1(G) - B_1(G). \end{aligned}$$

□

THEOREM 2.5. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then* $B_2(G) = EM_1(G) + 2M_1(G) - 4m$.

PROOF.

$$\begin{aligned} EM_1(G) &= HM_1(G) - 4M_1(G) + 4m \\ &= HM_1(G) - 2M_1(G) - 2M_1(G) + 4m \\ &= B_2(G) - 2M_1(G) + 4m. \end{aligned}$$

□

COROLLARY 2.1. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then* $B_1(G) + B_2(G) = HM_1(G) + M_1(G) - 4m$.

THEOREM 2.6. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then* $HB_1(G) = 2HM_1(G) - 4M_1(G) + 24m$.

PROOF.

$$\begin{aligned} HB_1(G) &= \sum_{ue} [d_G(u) + d_G(e)]^2 \\ &= \sum_{uv \in E(G)} [d_G(u) + d_G(uv)]^2 + \sum_{uv \in E(G)} [d_G(v) + d_G(uv)]^2 \\ &= \sum_{uv \in E(G)} [d_G(u) + d_G(u) + d_G(v) - 2]^2 \\ &+ \sum_{uv \in E(G)} [d_G(v) + d_G(u) + d_G(v) - 2]^2 \\ &= \sum_{uv \in E(G)} [2(d_G(u) + d_G(v))^2 - 4(d_G(u) + d_G(v)) + 24]. \end{aligned}$$

Theorem 2.6 follows now from the definitions of the hyper–Zagreb and first Zagreb indices, and the fact that $E(G)$ has m elements. □

COROLLARY 2.2. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then* $B_2(G) = \frac{1}{2} HB_1(G) - 12m$.

PROOF.

$$HB_1(G) = 2[HM_1(G) - 2M_1(G)] + 24m = 2B_2(G) + 24m.$$

□

Corollary 2.3. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then*

$$B_1(G) = \frac{1}{2}HB_1(G) - EM_1(G) + M_1(G) - 12m.$$

Proof.

$$\begin{aligned} HB_1(G) &= 2[HM_1(G) - M_1(G)] - 2M_1(G) + 24m \\ &= 2[B_1(G) + EM_1(G)] - 2M_1(G) + 24m \\ &= 2B_1(G) + 2EM_1(G) - 2M_1(G) + 24m. \end{aligned}$$

□

Theorem 2.7. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then $HB_1(G) = 5F(G) + 8M_2(G) - 12M_1(G) + 8m$.*

Proof.

$$\begin{aligned} HB_1(G) &= \sum_{ue}[d_G(u) + d_G(e)]^2 \\ &= \sum_{uv \in E(G)} [d_G(u) + d_G(uv)]^2 + \sum_{uv \in E(G)} [d_G(v) + d_G(uv)]^2 \\ &= \sum_{uv \in E(G)} [d_G(u) + d_G(u) + d_G(v) - 2]^2 \\ &+ \sum_{uv \in E(G)} [d_G(v) + d_G(u) + d_G(v) - 2]^2 \\ &= \sum_{uv \in E(G)} \Big[5\big[d_G(u)^2 + d_G(v)^2\big] + 8\, d_G(u)\, d_G(v) \\ &- 12\,[d_G(u) + d_G(v)] + 8\Big] \\ &= 5F(G) + 8M_2(G) - 12M_1(G) + 8m. \end{aligned}$$

□

In order to prove our next result, we use the earlier established:

Theorem 2.8. [**19**] *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then $EM_1(G) = F(G) + 2M_2(G) - 4M_1(G) + 4m$.*

Corollary 2.4. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then $B_1(G) = F(G) + 2M_2(G) - M_1(G) - EM_1(G)$.*

Proof. From Theorem 2.8, we have

$$\begin{aligned} EM_1(G) &= F(G) + 2M_2(G) - M_1(G) - (3M_1(G) - 4m) \\ &= F(G) + 2M_2(G) - M_1(G) - B_1(G). \end{aligned}$$

□

Corollary 2.5. *Let G be a graph with $n \geqslant 3$ vertices and m edges. Then $B_2(G) = F(G) + 2M_2(G) - 2M_1(G)$.*

Proof. From Theorem 2.5, we have

$$\begin{aligned} B_2(G) &= EM_1(G) + 2M_1(G) - 4m \\ &= F(G) + 2M_2(G) - 4M_1(G) + 4m + 2M_1(G) - 4m \\ &= F(G) + 2M_2(G) - 2M_1(G). \end{aligned}$$

□

3. Bounds on Banhatti and Zagreb–type indices

Theorem 3.1. *For any graph G,*

$$M_1(G) \leqslant B_1(G).$$

Equality is attained if and only if G is totally disconnected or $G \cong mK_2$.

Proof. Let G be a simple graph with n vertices and m edges. Then by Theorem 2.1, we have $B_1(G) = 3M_1(G) - 4m$. Clearly $M_1(G) \leqslant B_1(G)$ follows. Now we prove the second part.

The graph G satisfied the given condition

$$\begin{aligned} &\Leftrightarrow \quad B_1(G) = M_1(G) \\ &\Leftrightarrow \quad 3M_1(G) - 4m = M_1(G) \\ &\Leftrightarrow \quad M_1(G) = 2m. \end{aligned}$$

Since $\sum d_G(u)^2 = 2m = \sum d_G(u)$, and $\sum(d_G(u)^2 - d_G(u)) = 0$, because $d_G(u)^2 - d_G(u) \geqslant 0$.

$$\begin{aligned} &\Leftrightarrow \quad d_G(u)^2 = d_G(u) \\ &\Leftrightarrow \quad d_G(u) = 0 \ \textit{or} \ d_G(u) = 1. \end{aligned}$$

Thus the result follows □

Here, we use the following existing results of the Zagreb and K Banhatti indices of regular graph.

Theorem 3.2. [**15**] *Let G be an r-regular graph. Then*

$$M_1(G) = nr^2 \qquad \textit{and} \qquad M_2(G) = \frac{1}{2}nr^3.$$

Theorem 3.3. [**12**] *Let G be an r-regular graph. Then*

$$B_1(G) = nr(3r-2) \qquad \textit{and} \qquad B_2(G) = 2nr^2(r-1).$$

Theorem 3.4. *For any connected graph G,*

$$B_2(G) \geqslant 4M_2(G) - 2M_1(G).$$

Equality is attained if and only if G is a regular graph.

PROOF.

$$\begin{aligned} B_2(G) &= \sum_{ue} d_G(u)\, d_G(e) \\ &= \sum_{uv\in E(G)} d_G(u)\,\big[d_G(u)+d_G(v)-2\big] \\ &+ \sum_{uv\in E(G)} d_G(v)\,\big[(d_G(u)+d_G(v)-2\big] \\ &= \sum_{uv\in E(G)} \big[d_G(u)^2+d_G(v)^2+2d_G(u)\,d_G(v)\big]-2M_1(G) \\ &\geqslant \sum_{uv\in E(G)} 4d_G(u)\,d_G(v)-2M_1(G). \end{aligned}$$

Since

$$d_G(u)^2+d_G(v)^2 \geqslant 2d_G(u)\,d_G(v)$$

and

$$\sum_{uv\in E(G)} d_G(u)^2+d_G(v)^2 \geqslant \sum_{uv\in E(G)} 2d_G(u)\,d_G(v),$$

the result follows.

The equality case attains directly from Theorems 2.1, 2.2, 3.2, and 3.3. □

Now, we use the following existing results to prove our next result.

THEOREM 3.5. [**19**] *Let G be a simple graph with $n \geqslant 3$ vertices and m edges. Then*

$$M_1(G) \geqslant \frac{4m^2}{n} \qquad \text{and} \qquad M_2(G) \geqslant \frac{4m^3}{n^2}.$$

THEOREM 3.6. *For any connected graph G with $n \geqslant 3$ vertices and m edges,*

$$B_2(G) \geqslant \frac{8m^2(2m-n)}{n^2}.$$

Further, equality is attained if and only if G is a regular graph.

PROOF. From Theorems 3.3–3.5, the desired result follows. □

THEOREM 3.7. *For any connected graph G with $n \geqslant 3$ vertices and m edges,*

$$\frac{4m(3m-n)}{n} \leqslant B_1(G) \leqslant 3m^2-m.$$

The lower bound becomes equality if and only if G is regular. Equality in the upper bound is attained if and only if $G \cong K_{1,n-1}$ or $G \cong K_3$.

PROOF. From Theorems 2.1 and 3.5, bearing in mind that of $M_1(G) \leqslant m(m+1)$, the lower and upper bounds on $B_1(G)$ follow.

The second part is obvious. □

We now obtain lower and upper bounds on $B_1(G)$ in terms of the minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$ of G.

THEOREM 3.8. *For any graph G with $n \geqslant 3$ vertices and m edges,*

$$2m\big[3\delta(G)-2\big] \leqslant B_1(G) \leqslant 2m\big[3\Delta(G)-2\big].$$

Further, equality in both lower and upper bounds is attained if and only if G is regular.

PROOF. Let G be a graph with $n \geqslant 3$ vertices and m edges. Then

$$\begin{aligned} B_1(G) &= \sum_{ue}[d_G(u)+d_G(e)] \\ &= \sum_{uv\in E(G)}[d_G(u)+(d_G(u)+d_G(v)-2)] \\ &+ \sum_{uv\in E(G)}[d_G(v)+(d_G(u)+d_G(v)-2)] \\ &= \sum_{uv\in E(G)} 3(d_G(u)+d_G(v))-4m. \end{aligned}$$

But $2\delta(G) \leqslant d_G(u)+d_G(v) \leqslant 2\Delta(G)$. Bearing this in mind,

$$\begin{aligned} 6\delta(G) &\leqslant 3[d_G(u)+d_G(v)] \leqslant 6\Delta(G) \\ 6\delta(G)-4 &\leqslant 3[d_G(u)+d_G(v)]-4| \leqslant 6\Delta(G)-4 \\ 2m\big[3\delta(G)-2\big] &\leqslant B_1(G) \leqslant 2m\big[3\Delta(G)-2\big]. \end{aligned}$$

Further, equality in both lower and upper bounds holds if and only if $d_G(u)+d_G(v) = 2\delta(G) = 2\Delta(G)$, for each $uv \in E(G)$, which implies that G is a regular graph. □

The following two existing results of hyper–Zagreb index to prove our next two results in terms of $\delta(G)$ and $\Delta(G)$ of G.

THEOREM 3.9. [**4**] *For any simple graph G with $n \geqslant 3$ vertices and m edges,*

$$HM_1(G) \leqslant \frac{[\delta(G)+\Delta(G)]^2}{4m\delta(G)\Delta(G)}\, M_1(G)^2.$$

THEOREM 3.10. [**4**] *For any graph G with $n \geqslant 3$ vertices and m edges,*

$$\delta(G)M_1(G)+2M_2(G) \leqslant HM_1(G) \leqslant \Delta(G)M_1(G)+2M_2(G),$$

with equality if and only if G is a regular graph.

THEOREM 3.11. *For any connected graph G with $n \geqslant 3$ vertices and m edges,*

$$B_2(G) \leqslant \frac{[\delta(G)+\Delta(G)]^2}{4m\delta(G)\Delta(G)}\, M_1(G)^2 - 2M_1(G).$$

PROOF. From Theorem 3.9, we have

$$HM_1(G)-2M_1(G) \quad \leqslant \quad \frac{[\delta(G)+\Delta(G)]^2}{4m\delta(G)\Delta(G)}\, M_1(G)^2 - 2M_1(G)$$

whereas from Theorem 2.2,

$$B_2(G) \leqslant \frac{[\delta(G)+\Delta(G)]^2}{4m\delta(G)\Delta(G)}\, M_1(G)^2 - 2M_1(G).$$

□

THEOREM 3.12. *For any connected graph G with $n \geqslant 3$ vertices,*

$$\big[\delta(G)-2\big]M_1(G)+2M_2(G) \leqslant B_2(G) \leqslant$$
$$\big[\Delta(G)-2\big]M_1(G)+2M_2(G).$$

Further, equality in both lower and upper bounds hold if and only if G is regular.

PROOF. From Theorem 3.10, we have

$$\delta(G)M_1(G)+2M_2(G)-2M_1(G) \leqslant HM_1(G)-2M_1(G) \leqslant$$
$$\Delta(G)M_1(G)+2M_2(G)-2M_1(G).$$

Then from Theorem 2.2, we get the desired result.

Further, equality in both lower and upper bounds will hold if and only if $d_G(u)+d_G(v) = 2\delta(G) = 2\Delta(G)$, for each $uv \in E(G)$, which implies that G is a regular graph. □

Now, we use the following existing results to prove our next result of $B_1(T)$.

THEOREM 3.13. [**7**] *For any tree T with $n \geqslant 3$ vertices and m edges,*

$$4n-6 \leqslant M_1(T) \leqslant n(n-1).$$

THEOREM 3.14. *For any tree T with $n \geqslant 3$ vertices and m edges,*

$$8n-14 \leqslant B_1(T) \leqslant (n-1)(3n-4).$$

Further, equality in the lower bound is attained if and only if $T \cong P_n$ and in the upper bound if and only if $T \cong K_{1,n-1}$.

PROOF. From Theorems 2.1 and 3.13, we have

$$4n-6 \leqslant \frac{1}{3}\big[B_1(T)+4m\big] \leqslant n(n-1)$$

$$12n-18-4m \leqslant B_1(T) \leqslant 3n(n-1)-4m.$$

Since for any tree T, $m = n-1$, the result follows.

Further, the equality in the lower bound is attained if and only if $T \cong P_n$ because $B_1(P_n) = 8n-14$. Equality in the upper bound is attained if and only if $T \cong K_{1,n-1}$ because $B_1(K_{1,n-1}) = (n-1)(3n-4)$. □

In order to prove our next result (upper bound) of $B_1(G)$ via $M_1(G)$, we apply of the Biernacki–Pidek–Ryll–Nardzewski inequality [**1**].

THEOREM 3.15. [**1**] *Let a and b be n-tuples such that $x \leqslant a_i \leqslant X$ and $y \leqslant b_i \leqslant Y$ for $i = 1, 2, \ldots, n$. Then*

$$\left\lfloor \frac{1}{n}\sum_{i=1}^{n} a_i b_i - \frac{1}{n}\sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i \right\rfloor \leqslant \frac{1}{4}(X-x)(Y-y),$$

with $\lfloor \cdot \rfloor$ being the greatest integer function. Equality occurs when n is even.

THEOREM 3.16. *For any connected graph G with $n \geqslant 3$ vertices and m edges,*

$$B_1(G) \leqslant \frac{3n}{4}[\Delta(G)-\delta(G)]^2 + \frac{4m}{n}(3m-n).$$

PROOF. Let $a_i = b_i = d_G(u_i)$ for $i = 1, 2, \ldots, n$ with $x = \delta(G) = y$ and $X = \Delta(G) = Y$. Then

$$\begin{aligned}
\left\lfloor \frac{1}{n}\sum_{i=1}^{n} d_G(u_i)^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} d_G(u_i)\right)^2 \right\rfloor &\leqslant \frac{1}{4}[\Delta(G)-\delta(G)]^2 \\
\left\lfloor \frac{1}{n}M_1(G) - \frac{1}{n^2}(2m)^2 \right\rfloor &\leqslant \frac{1}{4}[\Delta(G)-\delta(G)]^2 \\
\frac{1}{n}M_1(G) - \frac{4m^2}{n^2} &\leqslant \frac{1}{4}[\Delta(G)-\delta(G)]^2.
\end{aligned}$$

Since

$$M_1(G) \geqslant \frac{4m^2}{n} \Rightarrow \frac{1}{n}M_1(G) \geqslant \frac{4m^2}{n^2},$$

we have

$$\begin{aligned}
M_1(G) - \frac{4m^2}{n} &\leqslant \frac{n}{4}[\Delta(G)-\delta(G)]^2 \\
\frac{1}{3}\big[B_1(G)+4m\big] - \frac{4m^2}{n} &\leqslant \frac{n}{4}[\Delta(G)-\delta(G)]^2 \\
B_1(G) + 4m - \frac{12m^2}{n} &\leqslant \frac{3n}{4}[\Delta(G)-\delta(G)]^2.
\end{aligned}$$

Hence the upper bound follows. □

In order to prove our next result (lower bound) of $B_1(G)$ in terms of the minimum degree $\delta(G)$, the maximum degree $\Delta(G)$ and the forgotten topological index $F(G)$, we use of the well known Cassel's inequality [**18**].

THEOREM 3.17. [**18**] *Let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be positive real numbers, satisfying the condition $0 < \ell \leqslant \frac{a_k}{b_k} \leqslant L < \infty$ for each $k \in \{1, 2, \ldots, n\}$, where ℓ and L are some constants. Let $(w_1, w_2, \ldots, w_n)$ be positive weights. Then*

$$\left(\sum_{i=1}^{n} w_k\, a_i^2\right)\left(\sum_{i=1}^{n} w_k\, b_i^2\right) \leqslant \frac{(L+\ell)^2}{4L\ell}\left(\sum_{i=1}^{n} w_k\, a_i\, b_i\right)^2.$$

THEOREM 3.18. *For any connected graph G with $n \geqslant 3$ vertices and m edges,*

$$B_1(G) \geqslant \frac{24m\delta(G)\Delta(G)}{(\delta(G)+\Delta(G))^2}\, F(G) - 4m.$$

PROOF. Let $a_i = d_G(u_i)^{3/2}$ and $b_i = d_G(u_i)^{1/2}$ with $\ell = \delta(G)$, $L = \Delta(G)$ and $w_i = 1$ for all $1 \leqslant i \leqslant n$. By Theorem 3.17 (Cassel's inequality),

$$\begin{aligned} \sum_{i=1}^{n} d_G(u_i)^3 \sum_{i=1}^{n} d_G(u_i) &\leqslant \frac{(\delta(G)+\Delta(G))^2}{4\delta(G)\Delta(G)}\, d_G(u_i)^2 \\ F(G)\, 2m &\leqslant \frac{(\delta(G)+\Delta(G))^2}{4\delta(G)\Delta(G)}\, M_1(G) \\ F(G) &\leqslant \left(\frac{(\delta(G)+\Delta(G))^2}{8m\delta(G)\Delta(G)}\right)\frac{1}{3}\big[B_1(G)+4m\big]. \end{aligned}$$

Thus the result follows. □

Now, we obtain lower and upper bounds on $EM_1(G)$, $B_1(G)$, and $B_2(G)$ in terms of $\delta(G)$, $\Delta(G)$, and $M_1(G)$, using Abel's inequality as follows.

THEOREM 3.19. [**3**] *Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ with*

$$b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$$

be two sequences of real numbers and $S_k = a_1 + a_2 + \cdots + a_k$ for $k = 1, 2, \ldots, n$. If $\omega = min_{1\leqslant k\leqslant n} S_k$ and $\Omega = max_{1\leqslant k\leqslant n} S_k$, then

$$\omega\, b_1 \leqslant a_1b_1 + a_2b_2 + \cdots + a_nb_n \leqslant \Omega\, b_1.$$

In order to prove our next result we make use of the following definition:

The line graph $L(G)$ of the graph G is the graph whose vertices correspond to the edges of G and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent (that is, are incident with a common vertex).

THEOREM 3.20. *For any connected graph G with $n \geqslant 3$ vertices and m edges,*

$$4(\delta(G)-1)^2 \leqslant EM_1(G) \leqslant 2\big[M_1(G) - 2m\big](\Delta(G)-1) \tag{3.1}$$

$$\begin{aligned} &HM_1(G) - M_1(G)(2\Delta(G)-1) + 4m(\Delta(G)-1) \leqslant \\ &B_1(G) \leqslant HM_1(G) - M_1(G) - 4(\delta(G)-1)^2 \end{aligned} \tag{3.2}$$

$$\begin{aligned} &4(\delta(G)-1)^2 + 2M_1(G) - 4m \leqslant B_2(G) \leqslant \\ &\big[2M_1(G) - 4m\big]\Delta(G). \end{aligned} \tag{3.3}$$

PROOF. Inequality (3.1): Let $a_i = d_G(e_i)$ with $e_i = u_iv_j$ for $i \neq j$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$. Clearly, $b_1 = max\ d_G(e_i)$ and $2\delta(G) - 2 \leqslant b_1 \leqslant 2\Delta(G) - 2$, where $S_k = a_1 + a_2 + \cdots + a_k$ for $k = 1, 2, \ldots, n$.

Therefore $\omega = min_{1\leqslant k\leqslant n}\ S_k = min_{1\leqslant i\leqslant n} d_G(e_i) \Rightarrow \omega \geqslant 2(\delta(G)-1)$ and

$$\begin{aligned}
\Omega &= max_{1\leqslant k\leqslant n}\ S_k = max_{1\leqslant i\leqslant n}\ d_G(e_i) = S_n \\
&= \sum_{i=1}^{n} d_G(e_i) = 2|E(L(G))| = 2\left[\frac{1}{2}\sum_{i=1}^{n} d_G(u_i)^2 - m\right] \\
&= 2\left[\frac{1}{2}M_1(G) - m\right] = M_1(G) - 2m.
\end{aligned}$$

By Theorem 3.19 (Abel's inequality), we get

$$\begin{aligned}
\omega\, b_1 &\leqslant a_1b_1 + a_2b_2 + \cdots + a_nb_n \leqslant \Omega\, b_1 \\
(2\delta(G)-2)b_1 &\leqslant a_1b_1 + a_2b_2 + \cdots + a_nb_n \leqslant (2\Delta(G)-2)b_1 \\
4(\delta(G)-1)^2 &\leqslant \sum_{i=1}^{n} d_G(e_i)^2 \leqslant [M_1(G)-2m](2\Delta(G)-2) \\
4(\delta(G)-1)^2 &\leqslant EM_1(G) \leqslant 2\,[M_1(G)-2m]\,(\Delta(G)-1).
\end{aligned}$$

Inequality (3.2): From (3.1) and Theorem 2.4, we get

$$\begin{aligned}
&HM_1(G) - M_1(G)(2\Delta(G)-1) + 4m(\Delta(G)-1) \leqslant B_1(G) \leqslant \\
&HM_1(G) - M_1(G) - 4(\delta(G)-1)^2.
\end{aligned}$$

Inequality (3.3): From (3.1) and Theorem 2.5, we get

$$\begin{aligned}
&4(\delta(G)-1)^2 + 2M_1(G) - 4m \leqslant B_2(G) \leqslant \\
&(2M_1(G)-4m)\Delta(G).
\end{aligned}$$

□

Finally, we obtain the lower and upper bounds on $B_1(G)$ and $B_2(G)$ in terms of the number of pendent vertices and minimal non-pendent vertices of G.

THEOREM 3.21. *For any (n, m)-graph G with η pendent vertices and minimal non-pendent vertex degree $\delta_1(G)$,*

$$\begin{aligned}
&6\delta_1(G)(m-\eta) + 3\eta(1+\delta_1(G)) - 4m \leqslant B_1(G) \leqslant \\
&6\Delta(G)(m-\eta) + 3\eta(1+\Delta(G)) - 4m
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&4\delta_1(G)(\delta_1(G)-1)(m-\eta) + (\delta_1(G)^2-1)\eta \leqslant B_2(G) \leqslant \\
&4\Delta(G)(\Delta(G)-1)(m-\eta) + (\Delta(G)^2-1)\eta.
\end{aligned} \tag{3.5}$$

PROOF. Inequality (3.4):

$$\begin{aligned} B_1(G) &= \sum_{ue}[d_G(u)+d_G(e)] \\ &= \sum_{uv\in E(G)}[d_G(u)+(d_G(u)+d_G(v)-2)] \\ &+ \sum_{uv\in E(G)}[d_G(v)+(d_G(u)+d_G(v)-2)] \\ &= \sum_{uv\in E(G)} 3[d_G(u)+d_G(v)]-4 \\ &= \sum_{uv\in E(G);d_G(u),d_G(v)\neq 1} 3[d_G(u)+d_G(v)] \\ &+ \sum_{uv\in E(G);d_G(u)=1} 3[1+d_G(v)] - \sum_{uv\in E(G)} 4 \\ &\leqslant 6\Delta(G)(m-\eta)+3\eta(1+\Delta(G))-4m. \end{aligned}$$

Thus the upper bound follows.

Similarly,

$$\begin{aligned} B_1(G) &\geqslant \sum_{uv\in E(G);d_G(u),d_G(v)\neq 1} 6\delta_1(G) + \sum_{uv\in E(G);d_G(u)=1} 3\eta(1+\delta_1(G)) - \sum_{uv\in E(G)} 4 \\ &= 6\delta_1(G)(m-\eta)+3\eta(1+\delta_1(G))-4m. \end{aligned}$$

Hence the lower bound follows.

Inequality (3.5):

$$\begin{aligned} B_2(G) &= \sum_{ue} d_G(u)\, d_G(e) \\ &= \sum_{uv\in E(G)} d_G(u)\big[d_G(u)+d_G(v)-2\big] \\ &= \sum_{uv\in E(G);d_G(u),d_G(v)\neq 1} d_G(u)\big[d_G(u)+d_G(v)-2\big] \\ &+ \sum_{uv\in E(G);d_G(u),d_G(v)\neq 1} d_G(v)\big[d_G(u)+d_G(v)-2\big] \\ &+ \sum_{uv\in E(G);d_G(u)=1} 1[d_G(v)-1] + \sum_{uv\in E(G);d_G(u)=1} d_G(v)[d_G(v)-1] \\ &\leqslant \sum_{uv\in E(G);d_G(u),d_G(v)\neq 1} \big[\Delta(G)(2\Delta(G)-2)+\Delta(G)(2\Delta(G)-2)\big] \\ &+ \sum_{uv\in E(G);d_G(u)=1} \big[\Delta(G)-1\big] + \sum_{uv\in E(G);d_G(u)=1} \big[\Delta(G)-1\big]. \end{aligned}$$

Thus the upper bound follows.

Similarly,

$$\begin{aligned} B_2(G) &\geqslant \sum_{uv\in E(G);d_G(u),d_G(v)\neq 1} 2\delta_1(G)\big[2\delta_1(G)-2\big] \\ &+ \sum_{uv\in E(G);d_G(u)=1} \big[\delta_1(G)-1\big] + \sum_{uv\in E(G);d_G(u)=1} \delta_1(G)\big[\delta_1(G)-1\big] \\ &= 6\delta_1(G)(m-\eta)+3\eta(1+\delta_1(G))-4m. \end{aligned}$$

Hence the lower bound follows. □

REMARK 3.1. In the inequalities (3.4) and (3.5), equality is attained if and only if $d_G(u)=d_G(v)=\Delta(G)=\delta_1(G)$ for each $uv\in E(G)$ with $d_G(u),d_G(v)\neq 1$ and $d_G(v)=\Delta(G)=\delta_1(G)$ for each $uv\in E(G)$ with $d_G(u)=1$.

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