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FIXED POINTS OF (φ,ψ) - ALMOST GENERALIZED WEAKLY CONTRACTIVE MAPS WITH RATIONAL EXPRESSIONS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we introduce a notion of (φ,ψ)-almost generalized weakly contractive maps involving rational type expressions in partially ordered metric spaces and prove the existence of fixed points. These results generalize the results of Chandok, Choudhury and Metiya [**16**]. Also we provide examples in support of our results.

1. Inroduction

Banach contraction principle deals with the existence and uniqueness of fixed points of contraction mappings in complete metric spaces. In the direction of generalization of contraction condition, in 1997, Alber and Gurre- Delabrierre [**1**] introduced the concept of weakly contractive maps in the setting of Hilbert spaces and defined weakly contractive maps on a Hilbert spaces and established the corresponding fixed point results. In 2001, Rhoades [**28**] extended this concept to metric spaces. Afterwards, in 2008, Dutta and Choudhury [**18**] introduced (φ,ψ) - weakly contractive maps by applying altering distance functions ψ and φ and proved the existence of fixed points of self maps in complete metric spaces.

In 2004, Berinde [**8**] introduced 'weak contractions' as a generalization of contraction maps, and in 2008, Berinde [**9**] renamed 'weak contractions ' as 'Almost contractions'. For more details on almost contractions and its generalizations, we

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refer to Babu, Sandhya and Kameswari [**10**], Abbas, Babu and Alemayehu [**5**] and the related references cited in these papers.

On the other hand, the notion of fixed points in partially ordered sets was introduced by Brondsted [**7**]. In 2004, Ran and Reurings [**29**] initiated the technique of proving the existence of fixed points of contraction maps in partially ordered complete metric spaces, in which the operator considered is continuous. In 2005, Nieto and Rodriguez-Lopez [**23**] replaced the continuity of the operator by the sequential convergence in X. For more works in this line of research, we refer to Abbas, Nazir, Radenovic [**6**], Agarwal, El-Gebeily and O'Regan [**2**], Altun and Simsek [**3**], Amini-Harandi and Emami [**4**], Choudhury and Kundu [**14**], Ciric, Abbas, Saadati and Hussain [**13**], Ciric, Cakic, Rajovic and Ume [**12**], Harjani and Sadarangani [**20**], Harajani, Lopez and Sadarangani [**21**], Nashine and Altun [**27**], Nashine and Samet [**25**], Nashine, Samet and Kim [**26**], Nieto and Rodriguez [**24**], O'Regan and Petrusel [**30**].

The latest work in this direction is that of Chandok, Choudhury and Metiya [**16**], in which the author established some fixed point results of generalized weakly contractive mappings of rational type in a metric space endowed with a partial order using some auxillary functions.

In Section 2 of this paper, we write preliminaries and introduce (φ,ψ) - almost generalized weakly contractive maps involving rational type expressions, by combining the notation of weakly contractive maps, almost contractions and an altering distance function in the setting of ordered metric spaces and prove the existence of fixed points. In Section 3, we prove our main results. In Section 4, we draw some corollaries to our main results, and provide examples in support of our results.

2. Preliminaries

In 1975, Dass and Gupta [**17**] extended the Banach contraction principle through rational expressions as follows.

THEOREM 2.1. [**17**] *Let (X,d) be a complete metric space and $T : X \to X$ be a self map of X. If there exist $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \leqslant \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y) \tag{2.1}$$

for all x, y in X, then T has a unique fixed point in X.

DEFINITION 2.1. Let $(X,\preceq)$ be a partially ordered set. A mapping $T : X \to X$ is said to be non-decreasing if for any x, y in X, $x \preceq y \implies Tx \preceq Ty$.

DEFINITION 2.2. [**13**] Let X be a nonempty set. Then $(X,\preceq,d)$ is called a partially ordered metric space if:

(i) (X,d) is a metric space, and

(ii) $(X,\preceq)$ is a partially ordered set.

$(X,\preceq,d)$ is called a partially ordered complete metric space if $(X,\preceq,d)$ is a partially ordered metric space in which the metric d is complete.

In [**15**], Cabrera, Harjani and Sadarangani extended Theorem 2.1 to the context of partially ordered metric spaces.

THEOREM 2.2. [**17**] *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that the inequality* (2.1) *is satisfied for all x, y in X with $x \preceq y$. If there exists x_0 in X with $x_0 \preceq Tx_0$ then T has a fixed point.*

THEOREM 2.3. [**17**] *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all n in $\mathbb{N}$. Let $T : X \to X$ be a non-decreasing mapping such that the inequality* (2.1) *is satisfied for all x, y in X with $x \preceq y$. If there exists x_0 in X with $x_0 \preceq Tx_0$ then T has a fixed point.*

THEOREM 2.4. [**17**] *In addition to hypotheses of Theorem 2.2 (or Theorem 2.3), suppose that for any x, y in X, there exists u in$X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.*

DEFINITION 2.3. [**22**] A function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) φ is monotone increasing and continuous

(ii) $\varphi(t) = 0$ if and only if $t = 0$.

We denote the class of all altering distance functions by Φ.

THEOREM 2.5. [**18**] *Let (X, d) be a complete metric space and let $T : X \to X$ be a self map of X. If there exist φ, ψ in Φ such that*

$$\varphi(d(Tx, Ty)) \leqslant \varphi(d(x, y)) - \psi(d(x, y)) \tag{2.2}$$

for all x, y in X. Then T has a unique fixed point in X.

We denote $\varPsi = \{\psi : [0, \infty) \to [0, \infty) \mid \psi$ is lower semi continuous and $\psi(t) = 0$ iff $t = 0\}$. Dorić [**19**] extended Theorem 2.5 to a pair of self maps by replacing the monotonicity and continuity of ψ by lower semi continuity, and Doric's result for the case of single self map is the following.

THEOREM 2.6. [**19**] *Let (X, d) be a complete metric space and let $T : X \to X$ be a self maps of X. If there exist φ in Φ and ψ in $\varPsi$ such that*

$$\varphi(d(Tx, Ty)) \leqslant \varphi(M(x, y)) - \psi(M(x, y)) \tag{2.3}$$

for all x, y in X, where

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y), \tfrac{1}{2}[d(y, Tx) + d(x, Ty)]\}.$$

Then T has a unique fixed point in X.

Recently, Chandok, Choudhury and Metiya [**16**] introduced the following class of functions and used these functions to define weakly contractive maps.

$$\varPsi_1 = \{\psi : [0, \infty) \to [0, \infty) | \text{ for any sequence } \{x_n\} \text{ in } [0, \infty) \\ \text{with } x_n \to t > 0, \liminf_{n\to\infty} \psi(x_n) > 0\}.$$

Example 2.1. We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, by

$$\psi(t) = \begin{cases} 0 & \text{if } t \in [0,1] \\ \frac{t}{2} & \text{if } t \in (1, \infty). \end{cases}$$

♦

Clearly ψ is lower-semi continuous with $\psi(0) = 0$ so that ψ in Ψ. By choosing $t_n = \frac{1}{2} + \frac{1}{n}$, we have $\liminf_{n\to\infty} \psi(t_n) = \dfrac{1}{2} > 0$. But $\psi(t_n) = 0$ for all $n \geqslant 2$ and $\liminf_{n\to\infty} \psi(t_n) = 0$. Hence $\psi \notin \Psi_1$. Therefore $\Psi \not\subseteq \Psi_1$.

Example 2.2. We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{1}{1+t} & \text{if } t \in (0, \frac{1}{3}) \\ \frac{t+1}{4} & \text{if } t \in [\frac{1}{3}, \frac{3}{4}) \\ \frac{t}{2} & \text{if } t \in [\frac{3}{4}, 1] \\ \frac{1}{1+2t} & \text{if } t \in (1, \infty). \end{cases}$$

♦

Then it is easy to see that $\psi \in \Psi_1$. We choose $t_n = 1 + \frac{1}{n}, n = 1, 2, \ldots$. Then $t_n \to 1$ *as* $n \to \infty$. Now

$$\tfrac{1}{2} = \psi(1) \not\leq \tfrac{1}{3} = \liminf_{n\to\infty} \frac{1}{1+2t_n} = \liminf_{n\to\infty} \psi(t_n)$$

so that ψ is not lower semi continuous on $[0.\infty)$. Hence $\psi \notin \Psi$. Hence $\Psi_1 \not\subseteq \Psi$.

From Example 2.1 and Example 2.2, we conclude that the class of functions in Ψ and Ψ_1 are distinct. Here, we note that for any $\psi \in \Psi_1$, it may happen that $\psi(t) > 0$ for $t > 0$, and $\psi(0)$ may not be equal to zero. For more details, we refer to [**16**]. In 2015, Chandok, Choudhury and Metiya [**16**] improved condition 2.3 of Theorem 2.6 by involving rational expressions using $\psi \in \Psi_1$ and proved the following.

Theorem 2.7. [**16**] *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping of X. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi_1$ such that*

$$\varphi(d(Tx, Ty)) \leqslant \varphi(M(x, y)) - \psi(N(x, y)) \tag{2.4}$$

for all x, y in X, where

$$M(x, y) = \max\{\tfrac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \tfrac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x, y)\} \text{ and}$$

$$N(x, y) = \max\{\tfrac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x, y)\}.$$

If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then T has a fixed point.

Remark 2.1. A self map T of a partially ordered metric space X that satisfies the inequality (2.4) is said to be a (φ, ψ) - generalized weakly contractive map involving rational expressions.

THEOREM 2.8. [**16**] *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in \mathbb{N}$. Let $T : X \to X$ be a non-decreasing mapping. Suppose that (2.4) holds, where $M(x,y), N(x,y)$ and the conditions upon φ and ψ are the same as in Theorem 2.7. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then T has a fixed point.*

THEOREM 2.9. [**16**] *In addition to the hypotheses of Theorem 2.7 (or Theorem 2.8), suppose that for each $x, y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.*

In the following, we introduce (φ, ψ) - almost generalized weakly contractive map.

DEFINITION 2.4. Let $(X, \preceq, d)$ be a partially ordered metric space. Let $T : X \to X$ be a self map of X. If there exist $\varphi \in \Phi$ and $\psi \in \Psi_1$ such that

$$\varphi(d(Tx,Ty)) \leqslant \varphi(M_1(x,y)) - \psi(M_2(x,y)) + LN(x,y) \tag{2.5}$$

for all x, y in X with $x \preceq y$, where

$$M_1(x,y) = \max\{\tfrac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, \tfrac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}, d(x,y)\},$$

$$M_2(x,y) = \max\{\tfrac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}, d(x,y)\} \text{ and}$$

$$N(x,y) = \min\{d(x,Tx), d(y,Ty), d(y,Tx), \tfrac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}\},$$

then we say that T is a (φ, ψ) - almost generalized weakly contractive map on X.

EXAMPLE 2.3. Let $X = [0,2]$ with the usual metric. We define partial order $\preceq$ on X as follows:

$$\preceq := \{(x,x) \in X \times X | x \in X\} \cup \{(\tfrac{1}{2}, \tfrac{3}{2}), (1,2), (\tfrac{1}{2}, 2), (\tfrac{3}{2}, 2)\}.$$

We define $T : X \to X$ by

$$T(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{3}{2}] \\ 2 & \text{if } x \in [\frac{3}{2}, 2]. \end{cases}$$

We define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^2$, $t \geqslant 0$. Clearly $\varphi \in \Phi$. We choose $\psi \in \Psi_1$ as in Example 2.2. We now verify that T is a (φ, ψ) - almost generalized weakly contractive map on X.

Case (i): $(x,y) = (\frac{1}{2}, \frac{3}{2})$. In this case,

$$(Tx,Ty) = (1,2),\ M_1(\tfrac{1}{2}, \tfrac{3}{2}) = 1,\ M_2(\tfrac{1}{2}, \tfrac{3}{2}) = 1,\ N(\tfrac{1}{2}, \tfrac{3}{2}) = \tfrac{1}{2},$$

and

$$\begin{aligned} \varphi(d(T(\frac{1}{2}), T(\frac{3}{2}))) = \varphi(1) = 1 &\leqslant 1 - \frac{1}{2} + \frac{1}{2} \\ &= \varphi(1) - \psi(1) + L.\frac{1}{2} \text{ with } L = 1 \\ &= \varphi(M_1(\frac{1}{2}, \frac{3}{2}) - \psi(M_2(\frac{1}{2}, \frac{3}{2}) + L.N(\frac{1}{2}, \frac{3}{2}). \end{aligned}$$

Case (ii): $(x,y) = (1,2)$. In this case,

$$(Tx, Ty) = (\tfrac{3}{2}, 2),\ M_1(1,2) = 1,\ M_2(1,2) = 1,\ N(1,2) = 0,$$

and

$$\begin{aligned}\varphi(d(T(1), T(2))) = \varphi(\frac{1}{2}) = \frac{1}{2} &\leqslant 1 - \frac{1}{2} + 0\\ &= \varphi(1) - \psi(1) + L.0\\ &= \varphi(M_1(1,2) - \psi(M_2(1,2)) + L.N(1,2).\end{aligned}$$

Case (iii): $(x, y) = (\frac{1}{2}, 2)$. In this case,

$$(Tx, Ty) = (1, 2),\ M_1(\tfrac{1}{2}, 2) = \tfrac{3}{2},\ M_2(\tfrac{1}{2}, 2) = \tfrac{3}{2},\ N(\tfrac{1}{2}, 2) = 0,$$

and

$$\begin{aligned}\varphi(d(T(\frac{1}{2}), T(2))) = \varphi(1) &\leqslant \frac{9}{4} - \frac{1}{4} + 0\\ &= \varphi(\frac{3}{2}) - \psi(\frac{3}{2}) + L.0\\ &= \varphi(M_1(\frac{1}{2}, 2) - \psi(M_2(\frac{1}{2}, 2)) + L.N(\frac{1}{2}, 2).\end{aligned}$$

Case (iv): $(x, y) = (\frac{3}{2}, 2)$. In this case,

$$(Tx, Ty) = (2, 2),\ M_1(\tfrac{3}{2}, 2) = \tfrac{1}{2},\ M_2(\tfrac{3}{2}, 2) = \tfrac{1}{2},\ N(\tfrac{3}{2}, 2) = 0,$$

and

$\varphi(d(T(\frac{3}{2}), T(2))) = \varphi(0) = 0$. Hence the inequality (2.5) holds trivially.

♦

Hence T is a (φ, ψ) - almost generalized weakly contractive map on X. Here we observe that T is not a (φ, ψ) - generalized weakly contractive map, for any $\varphi \in \Phi$ and $\psi \in \Psi_1$. For, by choosing $x = \frac{1}{2}$ and $y = \frac{3}{2}$ we have

$$\varphi(d((T(\frac{1}{2}), T(\frac{3}{2}))) = \varphi(1) \nleqslant \varphi(1) - \psi(1) = \varphi(M_1(\frac{1}{2}, \frac{3}{2}) - \psi(M_2(\frac{1}{2}, \frac{3}{2})).$$

Hence the class of (φ, ψ)- almost generalized weakly contractive maps properly contains the class of (φ, ψ)- generalized weakly contractive maps.

We state the following lemma which is useful in proving our main results.

Lemma 2.1. [**11**] *Let (X, d) be a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequence of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) \geqslant \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

$$(i)\ \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) = \epsilon\ ;(ii)\ \lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon;$$

$$(iii)\ \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k}) = \epsilon;\ (iv)\ \lim_{k\to\infty} d(x_{n_k}, x_{m_k+1}) = \epsilon.$$

3. Main results

THEOREM 3.1. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping. Assume that T is a (φ, ψ)- almost generalized weakly contractive map involving rational expressions. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then T has a fixed point.*

PROOF. We choose $x_0 \in X$ such that $x_0 \preceq Tx_0$ (hypothesis). We define $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for each $n = 0, 1, 2, \ldots$. Since $x_0 \preceq Tx_0$ and T is a non-decreasing function, by mathematical induction, it follows that

$x_0 \preceq Tx_0 \preceq Tx_1 \preceq \ldots Tx_{n-1} \preceq Tx_n \ldots$

i.e., $x_0 \preceq x_1 \preceq x_2 \preceq \ldots x_n \preceq x_{n+1} \preceq \ldots$ so that $x_n \preceq x_{n+1}$ for each $n = 0, 1, 2, \ldots$.

If $x_n = x_{n+1}$ for each n then x_n is a fixed point of f. With out loss of generality, we assume that $x_n \neq x_{n+1}$ for each n. Since $x_{n-1} \preceq x_n$ for each $n \geqslant 1$ from (2.5), we have

$$\varphi(d(x_n, x_{n+1})) = \varphi(d(Tx_{n-1}, Tx_n)) \leqslant \varphi(M_1(x_{n-1}, x_n)) - \psi(M_2(x_{n-1}, x_n)) + LN(x_{n-1}, x_n) \tag{3.1}$$

where

$$\begin{aligned} M_1(x_{n-1}, x_n) &= \max\{\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)}, \\ &\qquad \frac{d(x_n, Tx_{n-1})[1 + d(x_{n-1}, Tx_n)]}{1 + d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}, \end{aligned}$$

$$\begin{aligned} M_2(x_{n-1}, x_n) &= \max\{\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)}, d(x_{n-1}, x_n)\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \text{and} \end{aligned}$$

$$\begin{aligned} N(x_{n-1}, x_n) &= \min\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_n, Tx_{n-1}), \\ &\qquad \frac{d(x_n, Tx_{n-1})[1 + d(x_{n-1}, Tx_n)]}{1 + d(x_{n-1}, x_n)}\} \\ &= \min\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), 0, 0\} = 0. \end{aligned}$$

Let $R_n = d(x_{n+1}, x_n)$. Hence from (3.1), we have

$$\varphi(R_n) \leqslant \varphi(\max(R_n, R_{n-1}) - \psi(\max(R_n, R_{n-1}). \tag{3.2}$$

Suppose that $R_n > R_{n-1}$. Then $\varphi(R_n) \leqslant \varphi(R_n) - \psi(R_n)$ which implies that $\psi(R_n) \leqslant 0$, a contradiction. Therefore $R_n \leqslant R_{n-1}$. Therefore $\{R_n\}$ is a decreasing sequence of nonnegative reals, and so there exists $r \geqslant 0$ such that $\lim_{n\to\infty} R_n = r$ $(r \geqslant 0)$. Suppose that $r > 0$. Since $\{R_n\}$ is decreasing, from (3.2) we have

$$\varphi(R_n) \leqslant \varphi(R_{n-1}) - \psi(R_{n-1}). \tag{3.3}$$

On taking limit superior on both sides of (3.3), we get

$$\begin{aligned}\limsup_{n\to\infty}\varphi(R_n)&\leqslant\limsup_{n\to\infty}\varphi(R_{n-1})+\limsup_{n\to\infty}(-\psi(R_{n-1}))\\&\leqslant\limsup_{n\to\infty}\varphi(R_{n-1})-\liminf_{n\to\infty}\psi(R_{n-1})\end{aligned}$$

implies

$$\varphi(r)\leqslant\varphi(r)-\liminf_{n\to\infty}\psi(R_{n-1})$$

implies

$$\liminf_{n\to\infty}\psi(R_{n-1})\leqslant 0,$$

a contradiction. Therefore $r=0$. i.e., $\lim\limits_{n\to\infty}R_n=0$.

Now, we show that $\{x_n\}$ is Cauchy. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 2.1, there exists an $\epsilon>0$ for which we can find sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k>m_k>k$ such that $d(x_{m_k},x_{n_k})>\epsilon$ and $d(x_{m_k},x_{n_{k-1}})\leqslant\epsilon$ and the identities $(i)-(iv)$ of Lemma 2.1 hold. Since $n_k>m_k$, we have $x_{m_{k-1}}\preceq x_{n_{k-1}}$. Now

$$\begin{aligned}\varphi(d(x_{m_k},x_{n_k}))&=\varphi(d(Tx_{m_{k-1}},Tx_{n_{k-1}}))\\&\leqslant\varphi(M_1(x_{m_{k-1}},x_{n_{k-1}}))-\psi(M_2(x_{m_{k-1}},x_{n_{k-1}}))\\&\qquad+LN(x_{m_{k-1}},x_{n_{k-1}})\end{aligned}\tag{3.4}$$

where

$$\begin{aligned}M_1(x_{m_{k-1}},x_{n_{k-1}})&=\max\Big\{\tfrac{d(x_{n_{k-1}},Tx_{n_{k-1}})[1+d(x_{m_{k-1}},Tx_{m_{k-1}})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})},\\&\qquad\tfrac{d(x_{n_{k-1}},Tx_{m_{k-1}})[1+d(x_{m_{k-1}},Tx_{n_{k-1}})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})},d(x_{m_{k-1}},x_{n_{k-1}})\Big\}\\&=\max\Big\{\tfrac{d(x_{n_{k-1}},x_{n_k})[1+d(x_{m_{k-1}},x_{m_k})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})},\\&\qquad\tfrac{d(x_{n_{k-1}},x_{m_k})[1+d(x_{m_{k-1}},x_{n_k})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})},d(x_{m_{k-1}},x_{n_{k-1}})\Big\},\end{aligned}$$

$$M_2(x_{m_{k-1}},x_{n_{k-1}})=\max\Big\{\tfrac{d(x_{n_{k-1}},x_{n_k})[1+d(x_{m_{k-1}},x_{m_k})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})},d(x_{m_{k-1}},x_{n_{k-1}})\Big\}\text{ and}$$

$$\begin{aligned}N(x_{m_{k-1}},x_{n_{k-1}})&=\min\Big\{d(x_{m_{k-1}},Tx_{m_{k-1}}),d(x_{n_{k-1}},Tx_{n_{k-1}}),d(x_{n_{k-1}},Tx_{m_{k-1}}),\\&\qquad\tfrac{d(x_{n_{k-1}},Tx_{m_{k-1}})[1+d(x_{m_{k-1}},Tx_{n_{k-1}})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})}\Big\}\\&=\min\Big\{d(x_{m_{k-1}},x_{m_k}),d(x_{n_{k-1}},x_{n_k}),d(x_{n_{k-1}},x_{m_k}),\\&\qquad\tfrac{d(x_{n_{k-1}},x_{m_k})[1+d(x_{m_{k-1}},x_{n_k})]}{1+d(x_{m_{k-1}},x_{n_{k-1}})}\Big\}.\end{aligned}$$

On letting $k\to\infty$, we get

$$\lim_{k\to\infty}M_1(x_{m_{k-1}},x_{n_{k-1}})=\max\{0,\epsilon,\epsilon\}=\epsilon,\ \lim_{k\to\infty}M_2(x_{m_{k-1}},x_{n_{k-1}})=\max\{0,\epsilon\}=\epsilon$$

and

$$\lim_{k\to\infty}N(x_{m_{k-1}},x_{n_{k-1}})=\min\{0,0,\epsilon,\epsilon\}=0.$$

Now, on taking limit superior on both sides of (3.4), we have

$$\varphi(\epsilon) \leqslant \varphi(\epsilon) - \underline{\lim}\psi(M_2(x_{m_{k-1}}, x_{n_{k-1}})) + L.0,$$

which implies that $\underline{\lim}\psi(M_2(x_{m_{k-1}}, x_{n_{k-1}})) \leqslant 0$, a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now by the continuity of T, we have

$$Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

□

THEOREM 3.2. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a non-decreasing mapping. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \preceq x$ for all $n \in \mathbb{N}$. Suppose that* (2.5) *holds where $M_1(x, y), M_2(x, y), N(x, y)$ and the condition upon (φ, ψ) are same as in Theorem 3.1. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$ then T has a fixed point.*

PROOF. Suppose $x_0 \preceq Tx_0$ and let $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \dots$. Further, we have $x_n \preceq x_{n+1}$ for all n i.e., $\{x_n\}$ is an increasing sequence. Now as in Theorem 3.1, it can be shown that $\{x_n\}$ is a Cauchy sequence in X. Also this sequence converges to u. Then $x_n \preceq u$ for all $n \in \mathbb{N}$. Suppose $u \neq Tu$. i.e, $d(u.Tu) > 0$. Since $x_n \preceq u$ for each n then by condition (2.5), we have

(3.5)
$$\varphi(d(x_{n+1}, u)) = \varphi(d(Tx_n, Tu)) \leqslant \varphi(M_1(x_n, u)) - \psi(M_2(x_n, u))) + LN(x_n, u)$$

where

$$\begin{aligned} M_1(x_n, u) &= \max\{\tfrac{d(u,Tu)[1+d(x_n,Tx_n)]}{1+d(x_n,u)}, \tfrac{d(u,Tx_n)[1+d(x_n,Tu)]}{1+d(x_n,u)}, d(x_n, u)\} \\ &= \max\{\tfrac{d(u,Tu)[1+d(x_n,x_{n+1})]}{1+d(x_n,u)}, \tfrac{d(u,x_{n+1})[1+d(x_n,Tu)]}{1+d(x_n,u)}, d(x_n, u)\}\ , \\ M_2(x_n, u) &= \max\{\tfrac{d(u,Tu)[1+d(x_n,Tx_n)]}{1+d(x_n,u)}, d(x_n, u)\} \\ &= \max\{\tfrac{d(u,Tu)[1+d(x_n,x_{n+1})]}{1+d(x_n,u)}, d(x_n, u)\} \text{ and} \\ N(x_n, u) &= \min\{d(x_n, x_{n+1}), d(u, x_{n+1}), d(u, Tu), \tfrac{d(u,x_{n+1})[1+d(x_n,Tu)]}{1+d(x_n,u)}\} \end{aligned}$$

On letting $n \to \infty$, we get

$$\begin{aligned} \lim_{n\to\infty} M_1(x_n, u) &= \max\{d(u, Tu), 0, 0\} = d(u, Tu), \\ \lim_{n\to\infty} M_2(x_n, u) &= \max\{d(u, Tu), 0, \} = d(u, Tu) \text{ and} \\ \lim_{n\to\infty} N(x_n, u) &= \min\{0, , 0, d(u, Tu), 0\} = 0. \end{aligned}$$

Taking limit superior on both sides of the inequality (3.5) we have

$$\varphi(d(u, Tu)) \leqslant \varphi(d(u, Tu)) - \underline{\lim}\psi(M_2(x_n, u)) + L.0,$$

which implies that $\underline{\lim}\psi(M_2(x_n, u)) \leqslant 0$, a contradiction. Therefore $u = Tu$. □

THEOREM 3.3. *In addition to the hypotheses of either Theorem 3.1 or Theorem 3.2, assume the following condition (H): Suppose that for every $x, y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$ then T has a unique fixed point.*

PROOF. From Theorem 3.1, the set of fixed points of T is nonempty. Suppose x^* and y^* are two fixed points of T i.e., $x^* = Tx^*, y^* = Ty^*$. By the assumption, there exists $u_0 \in X$ such that $u_0 \preceq x^*$ and $u_0 \preceq y^*$. We define the sequence $\{u_n\}$ such that $u_{n+1} = Tu_n$ for $n = 0, 1, 2, \ldots$. Since T is non-decreasing, we have $T^n u = u_n \preceq x^* = T^n x^*$ and $T^n u_0 = u_n \preceq y^* = T^n y^*$. Suppose $x^* = u_n$ for some n then $x^* = Tx^* = Tu_n = u_{n+1}$ for all $n \geqslant m$ then $u_n \to x^*$ as $n \to \infty$. Suppose $x^* \neq u_n$ for all n i.e., $u_n \preceq x^*$ for all n. Let $p_n = d(u_n, x^*)$ for each n. Since $u_n \preceq x^*$, by condition (2.5), we have

(3.6)
$$\varphi(d(u_{n+1}, x^*)) = \varphi(d(Tu_n, Tx^*)) \leqslant \varphi(M_1(u_n, x^*)) - \psi(M_2(u_n, x^*))) + LN(u_n, x^*)$$

where
$$\begin{aligned}
M_1(u_n, x^*) &= \max\{\tfrac{d(x^*,Tx^*)[1+d(u_n,Tu_n)]}{1+d(u_n,x^*)}, \tfrac{d(x^*,Tu_n)[1+d(u_n,Tx^*)]}{1+d(u_n,x^*)}, d(u_n, x^*)\} \\
&= \max\{\tfrac{d(x^*,Tx^*)[1+d(u_n,u_{n+1})]}{1+d(u_n,x^*)}, \tfrac{d(x^*,u_{n+1})[1+d(u_n,Tx^*)]}{1+d(u_n,x^*)}, d(u_n, x^*)\} \\
&= \max\{d(x^*, u_{n+1}), d(u_n, x^*)\}, \\
M_2(u_n, x^*) &= \max\{\tfrac{d(x^*,Tx^*)[1+d(u_n,Tu_n)]}{1+d(u_n,x^*)}, d(u_n, x^*)\} \\
&= \max\{\tfrac{d(x^*,Tx^*)[1+d(u_n,u_{n+1})]}{1+d(u_n,x^*)}, d(u_n, x^*)\} \\
&= \max\{0, d(u_n, x^*)\}, \text{ and} \\
N(u_n, x^*) &= \min\{d(u_n, u_{n+1}), d(x^*, Tx^*), d(x^*, Tu_n), \tfrac{d(x^*,u_{n+1})[1+d(u_n,Tx^*)]}{1+d(u_n,x^*)}\} \\
&= \min\{d(u_n, u_{n+1}), 0, d(x^*, u_{n+1})\} = 0.
\end{aligned}$$
Suppose $p_{n+1} > p_n$. Then from (3.6), we have

$$\varphi(p_{n+1}) \leqslant \varphi(p_{n+1}) - \psi(p_n) + L.0,$$

which implies that $\psi(p_n) \leqslant 0$, a contradiction. Therefore $p_{n+1} \leqslant p_n$. Hence $\{p_n\}$ is a decreasing sequence of nonnegative reals, and so there exists $r > 0$ such that $\lim_{n\to\infty} d(u_n, x^*) = r (\geqslant 0)$.

Again from condition (3.6), we have

$$\varphi(p_{n+1}) \leqslant \varphi(p_n) - \psi(p_n). \tag{3.7}$$

On taking limit superior on both sides of (3.7), we get

$$\varphi(r) \leqslant \varphi(r) - \underline{\lim}\psi(p_n),$$

which implies that $\underline{\lim}\psi(p_n) \leqslant 0$, a contradiction. Therefore $r = 0$. i.e., $u_n \to x^*$ as $n \to \infty$.

Similarly, we can prove that $u_n \to y^*$ as $n \to \infty$. By the uniqueness of the limit we have, $x^* = y^*$, and the conclusion of the theorem follows. $\square$

4. Corollaries and examples

If $M_1(x,y) = M_2(x,y)$ in Theorem 3.1, Theorem 3.2 and Theorem 3.3 then we get the following.

COROLLARY 4.1. *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that for all $x,y \in X$ with $x \preceq y$*

$$\varphi(d(Tx,Ty)) \leqslant \varphi(M_2(x,y)) - \psi(M_2(x,y)) + LN(x,y) \tag{4.1}$$

where $N(x,y), M_2(x,y)$ and the conditions upon (φ,ψ) are same as in Theorem 3.1. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

PROOF. Since the inequality (4.1) implies the inequality (3.1), by Theorem 3.1, the conclusion follows. □

COROLLARY 4.2. *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let $T : X \to X$ be a non-decreasing mapping. Suppose that (4.1) holds, where $N(x,y), M_2(x,y)$ and the conditions upon (φ,ψ) are the same as in theorem 3.1. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.*

PROOF. Since the inequality (4.1) implies the inequality (3.1), by Theorem 3.2, the conclusion follows. □

COROLLARY 4.3. *In addition to the hypotheses of Corollary 4.1 or Corollary 4.2, suppose that for every $x,y \in X$, there exists $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.*

REMARK 4.1. If $L = 0$ in Theorem 3.1, Theorem 3.2 and Theorem 3.3 we get Theorem 2.7, Theorem 2.8 and Theorem 2.9 respectively as corollaries.

We now present an example in support of Theorem 3.1.

EXAMPLE 4.1. Let $X = \{0, \frac{1}{2}, \frac{3}{4}, \frac{9}{8}, \frac{3}{2}\}$ with the usual metric. We define partial order $\preceq$ on X as follows:

$$\preceq := \{(x,x) \in X \times X | x \in X\} \cup \{(\tfrac{1}{2},\tfrac{9}{8}),(\tfrac{3}{4},\tfrac{3}{2}),(\tfrac{3}{4},\tfrac{9}{8}),(\tfrac{9}{8},\tfrac{3}{2}),(\tfrac{1}{2},\tfrac{3}{2})\}.$$

We define $T : X \to X$ by $T(0) = 0, T(\frac{1}{2}) = \frac{3}{4}, T(\frac{3}{4}) = T(\frac{9}{8}) = T(\frac{3}{2}) = \frac{3}{2}$. Then T is a non-decreasing map. We define $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^2$, $t \geqslant 0$. Clearly $\varphi \in \Phi$. We choose $\psi \in \Psi_1$ as in Example 2.2. We now verify the inequality (2.5) for the elements $(\frac{1}{2},\frac{9}{8}),(\frac{1}{2},\frac{3}{2})$ with $L = 3$.

Case (i): $(x,y) = (\frac{1}{2},\frac{9}{8})$. In this case,

$$(Tx,Ty) = (\tfrac{3}{4},\tfrac{3}{2}),\ M_1(\tfrac{1}{2},\tfrac{9}{8}) = \tfrac{5}{8},\ M_2(\tfrac{1}{2},\tfrac{9}{8}) = \tfrac{5}{8},\ N(\tfrac{1}{2},\tfrac{9}{8}) = \tfrac{1}{4},$$

and

$$\varphi(d(T(\frac{1}{2}),T(\frac{9}{8}))) = \varphi(\frac{3}{4}) = \frac{9}{16} \leqslant \frac{25}{64} - \frac{13}{32} + \frac{3}{4}$$
$$= \varphi(\frac{5}{8}) - \psi(\frac{5}{8}) + L.\frac{1}{4} \text{ with } L = 3$$
$$= \varphi(M_1(\frac{1}{2},\frac{9}{8}) - \psi(M_2(\frac{1}{2},\frac{9}{8}) + L.N(\frac{1}{2},\frac{9}{8}).$$

Case (ii): $(x, y) = (\frac{1}{2}, \frac{3}{2})$. In this case,

$$(Tx, Ty) = (\tfrac{3}{4}, \tfrac{3}{2}),\ M_1(\tfrac{1}{2}, \tfrac{3}{2}) = 1,\ M_2(\tfrac{1}{2}, \tfrac{3}{2}) = 1,\ N(\tfrac{1}{2}, \tfrac{3}{2}) = 0,$$

and

$$\varphi(d(T(\frac{1}{2}),T(\frac{3}{2}))) = \varphi(\frac{3}{4}) = \frac{9}{16} \leqslant 1 - \frac{1}{3} + 0$$
$$= \varphi(1) - \psi(1) + L.0$$
$$= \varphi(M_1(\frac{1}{2},\frac{3}{2}) - \psi(M_2(\frac{1}{2},\frac{3}{2})) + L.N(\frac{1}{2},\frac{3}{2}).$$

Hence the inequality (2.5) holds in both the cases. Hence T is a (φ, ψ)-almost generalized weakly contractive map of X. Hence T satisfies all the hypotheses of Theorem 3.1 and T has two fixed points 0 and 1.

♦

Now we present an example in support of Theorem 3.2.

Example 4.2. Let $X = [0, 3]$ with the usual metric. We define partial order $\preceq$ on X as follows:

$$\preceq := \{(x, x) \in X \times X | x \in X\} \cup \{(\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{1}{2}, \tfrac{3}{2}), (1, 3), (2, 3)\}.$$

We define $T : X \to X$ by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0,1] \\ 3 & \text{if } x \in [1,3]. \end{cases}$$

Moreover, we choose $x_0 = 2 \in X$ then $x_0 \preceq Tx_0$. We define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = t^2$, $t \geqslant 0$. Clearly $\varphi \in \Phi$. We choose $\psi \in \Psi_1$ as in Example 2.2. We now verify the inequality (2.5) for the elements $(\frac{1}{4}, \frac{3}{4})$ and $(\frac{1}{2}, \frac{3}{2})$ with $L = 7$ and in the remaining cases the inequality (2.5) holds trivially.

Case (i): $(x, y) = (\frac{1}{4}, \frac{3}{4})$. In this case,

$$(Tx, Ty) = (\tfrac{1}{2}, \tfrac{3}{2}),\ M_1(\tfrac{1}{4}, \tfrac{3}{4}) = \tfrac{5}{8},\ M_2(\tfrac{1}{4}, \tfrac{3}{4}) = \tfrac{5}{8},\ N(\tfrac{1}{4}, \tfrac{3}{4}) = \tfrac{1}{2},$$

and

$$\varphi(d(T(\frac{1}{4}),T(\frac{3}{4})an)) = \varphi(1) = 1 \leqslant \frac{25}{64} - \frac{13}{32} + \frac{7}{4}$$
$$= \varphi(\frac{5}{8}) - \psi(\frac{5}{8}) + L.\frac{1}{4} \text{ with } L = 7$$
$$= \varphi(M_1(\frac{1}{4},\frac{3}{4}) - \psi(M_2(\frac{1}{4},\frac{3}{4}) + L.N(\frac{1}{4},\frac{3}{4}).$$

Case (ii): $(x, y) = (\frac{1}{2}, \frac{3}{2})$. In this case,

$$(Tx,Ty)=(1,3),\ M_1(\tfrac{1}{2},\tfrac{3}{2})=\tfrac{9}{8},\ M_2(\tfrac{1}{2},\tfrac{3}{2})=\tfrac{9}{8},\ N(\tfrac{1}{2},\tfrac{3}{2})=\tfrac{1}{2},$$

and

$$\begin{aligned}\varphi(d(T(\frac{1}{2}),T(\frac{3}{2})))=\varphi(2)=4 &\leqslant \frac{81}{64}-\frac{4}{13}+\frac{7}{2}\\ &=\varphi(\frac{9}{8})-\psi(\frac{9}{8})+L.\frac{1}{2} \text{ with } L=7\\ &=\varphi(M_1(\frac{1}{2},\frac{3}{2})-\psi(M_2(\frac{1}{2},\frac{3}{2})+L.N(\frac{1}{2},\frac{3}{2}).\end{aligned}$$

Hence the inequality (2.5) holds in both the cases with $L=7$. Hence T is a (φ,ψ) - almost generalized weakly contractive map, and T satisfies all the hypotheses of Theorem 3.2 and T has two fixed points 0 and 3. Here we observe that T is not continuous. Further, T fails to satisfy condition (H) of Theorem 3.3. For, we choose $\frac{1}{5},\frac{1}{2}\in X$. Then we have $u\not\preceq\frac{1}{5}$ or $u\not\preceq\frac{1}{2}$ for every $u\in X$.

♦

Now we present an example in support of Theorem 3.3.

Example 4.3. Let $X=\{2,4,6,8\}$ with the usual metric. We define partial order $\preceq$ on X as follows:
$\preceq:=\{(x,x)\in X\times X|x\in X\}\cup\{(2,2),(4,4),(6,6),(8,8),(2,4),(2,6),(4,8),(2,8)\}$.
Hence $x\preceq y \iff x/y$. We define $T:X\to X$ by $T(2)=T(4)=4, T(6)=8, T(8)=4$. Then T is continuous and non-decreasing map. Moreover, we choose $x_0=2\in X$ then $x_0\preceq Tx_0$. We define $\varphi,\psi:[0,\infty)\to[0,\infty)$ by $\varphi(t)=t^2$, $t\geqslant 0$ and

$$\psi(t)=\begin{cases}0 & \text{if } t=0\\ \frac{t+1}{4} & \text{if } t\in(0,\infty).\end{cases}$$

Then $\varphi\in\Phi$ and $\psi\in\Psi_1$. We now verify the inequality (2.5) for the element $(2,6)$ with $L=1$ and in the remaining cases the inequality (2.5) holds trivially. For $(x,y)=(2,6)$, we have $(Tx,Ty)=(4,8)$, $M_1(2,6)=4$, $M_2(2,6)=4$, $N(2,6)=2$, and

$$\begin{aligned}\varphi(d(T(2),T(6)))=\varphi(4)=16 &\leqslant 16-\frac{5}{4}+1.2\\ &=\varphi(4)-\psi(4)+L.2 \text{ with } L=1\\ &=\varphi(M_1(2,6)-\psi(M_2(2,6)+L.N(2,6).\end{aligned}$$

Hence the inequality (2.5) holds with $L=1$. Hence T is a (φ,ψ) - almost generalized weakly contractive map, and T satisfies all the hypotheses of Theorem 3.3 and T has a unique fixed point 4.

♦

References

[1] Ya. I. Alber and S. Gurre-Delabrierre. Principle of weakly contractive maps in Hilbert spaces, in: I. Gohberg, Yu. Lyubich(Eds), *New results in operator theory in: Advances and appl.*, Vol. 98 (1997) of the series Operator Theory: Advances and Applications, (pp. 7-22). Birkhuser Basel 1997.

[2] R. P. Agarwal, M. A. El-Gebeily and D. ORegan. Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87**(1)(2008), 109-116.
[3] I. Altun and H. Simsek. Some fixed point theorems on ordered metric spaces and application, *Fixed Point Theory Appl.*, Volume 2010, Article ID 621469, 17 pages
[4] A. Amini-Harandi and H. Emami. A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, *Nonlinear Anal.: Theory, Meth. Appl.*, **72**(5)(2010), 2238-2242.
[5] M. Abbas, G. V. R. Babu and G. N. Alemayehu. On common fixed points of weakly compatible mappings satisfying generalized condition (B), *Filomat*, **25**(2)(2011), 9-19.
[6] M. Abbas, T. Nazir and S. Radenovic. Common fixed points of four maps in partially ordered metric spaces. *Appl. Math. Letters*, **24**(9)(2011), 1520-1526.
[7] A. Brondsted. Common fixed points and partial orders, *Proc. Amer. Math. Soc.*, **77**(3)(1979), 365-368.
[8] V. Berinde. Approximating fixed points of weak contractions using the Picard iteration, *Nonlinear Anal. Forum*, **9**(1)(2004), 43-53.
[9] V. Berinde. General constructive fixed point theorems for Ciric-type almost contractions in metric spaces, *Carpathian J. Math.,* **24**(2)(2008), 10-19.
[10] G. V. R. Babu, M. L. Sandhya and M. V. R. Kameswari. A note on a fixed point theorem of Berinde on weak contractions. *Carpathian J. Math.*, **24**(1)(2008), 8-12.
[11] G. V .R. Babu and P. D. Sailaja, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, *Thai J. Math.*, **9**(1)(2011), 1-10.
[12] Lj. B. Ciric, N. Cakic, M. Rajovic and J. S. Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, *Fixed point Theory Appl.*, Article ID 131294, (2008), 1-11.
[13] Lj. B. Ciric, M. Abbas, R. Saadati and N. Hussain, Common fixed points of almost generalized contractive mappings in ordered metric spaces, *Appl. Math and Computation*, **217** (12)(2011), 5784-5789.
[14] B. S. Choudhury and Amaresh Kundu, A kannan-like contraction in partially ordred metric spaces, *Demonstratio Mathematics*, 2013, 46123.
[15] I. Cabrera, J. Harjani and K. Sadarangani, A fixed point theorem for contractions of rational type in partially ordered metric spaces, Ann. Univ. Ferrara., **59**(2)(2013), 251-258.
[16] S. Chandok, B. S. Choudhury and N. Metiya. Fixed point results in ordered metric spaces for rational type expressions with auxiliary functions, *J. Egypt. Math. Soc.*, **23**(1)(2015), 95-101.
[17] B. K. Dass and S. Gupta. An extension of Banach contraction principle through rational expressions, *Indian J. pure appl. Math.*, **6**(1975), 1455-1458.
[18] P. N. Dutta and B. S. Choudhury. A generalisation of contraction principle in metric spaces, *Fixed Point Theory and Applications*, Article ID 406368, (2008), 8 pages.
[19] D. Doric. Common fixed point for generalized (ψ, φ)- weak contraction, *Appl. Math. Lett.*, **22**(12)(2009), 1896-1900.
[20] J. Harjani and K. Sadarangani. Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, *Nonlinear Anal.*, **72**(3-4)(2010), 1188-1197.
[21] J. Harjani, B. Lopez, K. Sadarangani. Fixed point theorems for weakly C -contractive mappings in ordered metric spaces. *Comput. Math. Appl.*, **61**(4)(2011), 790-796 .
[22] M. S. Khan, M. Swaleh and S. Sessa. Fixed point theorems by altering distance between points. *Bull. Aust. Math. Soc.*, **30**(1)(1984), 1-9.
[23] J. J. Nieto and R.Rodriguez-Lopez. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order, **22**(3)(2005), 223-239.
[24] J. J. Nieto and R. Rodriguez-Lopez. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, *Acta Math. Sinica*, **23**(12)(2007), 2205-2212 .

[25] H. K. Nashine and B. Samet. Fixed point results for mappings satisfying (ψ, ϕ) -weakly contractive condition in partially ordered metric spaces, *Nonlinear Anal.*, **74**(6)(2011), 2201-2209.

[26] H. K. Nashine, B. Samet and J. K. Kim. Fixed point results for contractions involving generalized altering distances in ordered metric spaces, *Fixed point Theory Appl.*, 2011, **2011**: 5.

[27] H. K. Nashine and I. Altun. A common fixed point theorem on ordered metric spaces, *Bull. Iranian Math. Soc.*, **38**(4)(2012), 925-934.

[28] B. E. Rhoades, Some theorems on weakly contractive maps, *Nonlinear Anal.*, **47**(4)(2001), 2683-2693.

[29] A. C. M. Ran and M. C. B. Reurings. A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132**(5)(2004), 1435-1443.

[30] D. O'Regan and A. Petrutel. Fixed point theorems for generalized contractions in ordered metric spaces, *J. Math. Anal. Appl.*, **341**(2)(2008), 1241-1252.

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