JOURNAL OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4866, ISSN (o) 2303-4947
www.imvibl.org /JOURNALS / JOURNAL
Vol. 7(2017), 101-118
DOI: 10.7251/JIMVI1701101M

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

DECOMPOSITION OF $g\omega$-CONTINUITY VIA IDEALIZATION

K. Meena, R. Premkuma, and O. Ravi

Abstract. In this paper we introduce the notions of αg-$\mathcal{I}_\omega$-open sets, gp-$\mathcal{I}_\omega$-open sets, gs-$\mathcal{I}_\omega$-open sets, $g\omega_t$-$\mathcal{I}$-sets, $g\omega_{\alpha^*}$-$\mathcal{I}$-sets and $g\omega_S$-$\mathcal{I}$-sets in ideal topological spaces and investigate some of their properties. Using these notions we obtain decompositions of $g\omega$-continuity.

1. Introduction

In 1961, Levine [**7**] obtained a decomposition of continuity which was later improved by Rose [**16**]. Tong [**18**] decomposed continuity into A-continuity and showed that his decomposition is independent of Levine's. The concept of $g\omega$-continuity was introduced and studied by Khalid Y. Al-Zoubi [**4**]. In 2000, Sundaram and Rajamani [**17**] obtained two different decompositions of g-continuity by introducing the notions of C(S)-sets and C*-sets in topological spaces. Noiri et al [**11**] introduced αg-$\mathcal{I}$-open sets, gp-$\mathcal{I}$-open sets, gs-$\mathcal{I}$-open sets, C(S)-$\mathcal{I}$-sets, C*-$\mathcal{I}$-sets and S*-$\mathcal{I}$-sets to obtain decompositions of g-continuity. In this paper we introduce αg-$\mathcal{I}_\omega$-open sets, gp-$\mathcal{I}_\omega$-open sets, gs-$\mathcal{I}_\omega$-open sets, $g\omega_t$-$\mathcal{I}$-sets, $g\omega_{\alpha^*}$-$\mathcal{I}$-sets and $g\omega_S$-$\mathcal{I}$-sets to obtain decompositions of $g\omega$-continuity.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}, \mathbb{Q}, \mathbb{R}-\mathbb{Q}, (\mathbb{R}-\mathbb{Q})_+$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers, the set of all positive irrational numbers).

2010 *Mathematics Subject Classification.* 26D05, 26D07.

Key words and phrases. αg-$\mathcal{I}_\omega$-open set, gp-$\mathcal{I}_\omega$-open set, gs-$\mathcal{I}_\omega$-open set, $g\omega_t$-$\mathcal{I}$-set, $g\omega_{\alpha^*}$-$\mathcal{I}$-set, $g\omega_S$-$\mathcal{I}$-set, αg-$\mathcal{I}_\omega$-continuity, gp-$\mathcal{I}_\omega$-continuity, $g\omega_t$-$\mathcal{I}$-continuity, $g\omega_{\alpha^*}$-$\mathcal{I}$-continuity, $g\omega_S$-$\mathcal{I}$-continuity.

By a space (X,τ), we always mean a topological space (X,τ) with no separation axioms assumed. For a subset A of a space (X,τ), $cl(A)$ and $int(A)$ denote the closure of A and the interior of A respectively.

Definition 2.1. *A subset A of a space (X,τ) is called semi-open* [**6**] *(resp. preopen* [**8**], *α-open* [**9**]*) if $A \subseteq cl(int(A))$ (resp. $A \subseteq int(cl(A))$, $A \subseteq int(cl(int(A)))$). The complement of a semi-open (resp. a preopen, an α-open) set is called semi-closed (resp. preclosed, α-closed).*

Definition 2.2. [**11**] *The largest semi-open (resp. preopen, α-open) set contained in A is called the semi-interior (resp. pre-interior, α-interior) of A and is denoted by s-$int(A)$ (resp. p-$int(A)$, α-$int(A)$). The smallest semi-closed (resp. preclosed, α-closed) set containing A is called the semi-closure (resp. preclosure, α-closure) of A and is denoted by s-$cl(A)$ (resp. p-$cl(A)$, α-$cl(A)$).*

Definition 2.3. [**21**] *Let A be a subset of a space (X,τ), a point p in X is called a condensation point of A if for each open set U containing p, $U \cap A$ is uncountable.*

Definition 2.4. [**3**] *A subset A of a space (X,τ) is called ω-closed if it contains all its condensation points.*

The complement of an ω-closed set is called ω-open.

It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X,τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 2.1. [**3**] *Let A be a subset of a space (X,τ). Then*
1. *A is ω-closed in X if and only if $A = cl_\omega(A)$.*
2. *$cl_\omega(X\backslash A) = X\backslash int_\omega(A)$.*
3. *$cl_\omega(A)$ is ω-closed in X.*
4. *$x \in cl_\omega(A)$ if and only if $A \cap G \neq \phi$ for each ω-open set G containing x.*
5. *$cl_\omega(A) \subseteq cl(A)$.*
6. *$int(A) \subseteq int_\omega(A)$.*

Remark 2.1. [**2, 3**] *In a space (X,τ), every closed set is ω-closed but not conversely.*

Definition 2.5. [**10**] *A subset A of a space (X,τ) is said to be*
1. *α-ω-open if $A \subseteq int_\omega(cl(int_\omega(A)))$.*
2. *pre-ω-open if $A \subseteq int_\omega(cl(A))$.*

Definition 2.6. [**12**] *A subset A of a space (X,τ) is said to be semi-ω-open if $A \subseteq cl(int_\omega(A))$.*

In the light of these definitions, we have

$$\alpha\text{-}int_\omega(A) = A \cap int_\omega(cl(int_\omega(A))),$$
$$p\text{-}int_\omega(A) = A \cap int_\omega(cl(A)) \text{ and}$$
$$s\text{-}int_\omega(A) = A \cap cl(int_\omega(A)),$$

where α-$int_\omega(A)$ denotes α-ω-interior of A in (X,τ) which is the union of all α-ω-open sets of (X,τ) contained in A. p-$int_\omega(A)$ and s-$int_\omega(A)$ have similar meanings.

REMARK 2.2. [**19**] *Let A be a subset of a space (X,τ). Then the following results are true:*

(1) *A is α-ω-open $\Rightarrow$ A is pre-ω-open.*
(2) *A is α-ω-open $\Rightarrow$ A is semi-ω-open.*
where (1) and (2) follow from Definitions 2.5 and 2.6.
(3) *α-$int_\omega(A) \subseteq p$-$int_\omega(A)$ by (1).*
(4) *α-$int_\omega(A) \subseteq s$-$int_\omega(A)$ by (2).*

DEFINITION 2.7. *A subset A of a space (X,τ) is said to be*

(1) *$g\omega$-open if $F \subseteq int_\omega(A)$ whenever $F \subseteq A$ and F is closed in (X,τ).* [**4**]
(2) *αg-ω-open if $F \subseteq \alpha$-$int_\omega(A)$ whenever $F \subseteq A$ and F is closed in (X,τ).* [**19**]
(3) *gp-ω-open if $F \subseteq p$-$int_\omega(A)$ whenever $F \subseteq A$ and F is closed in (X,τ).* [**19**]
(4) *gs-ω-open if $F \subseteq s$-$int_\omega(A)$ whenever $F \subseteq A$ and F is closed in (X,τ).* [**19**]
(5) *a ω^*-t-set if $int_\omega(A) = int_\omega(cl(A))$.* [**12**]
(6) *an α^*-ω^*-set if $int_\omega(A) = int_\omega(cl(int_\omega(A)))$.* [**19**]
(7) *a $g\omega_t$-set if $A = U \cap V$, where U is $g\omega$-open and V is a ω^*-t-set in (X,τ).* [**19**]
(8) *an $g\omega_{\alpha^*}$-set if $A = U \cap V$, where U is $g\omega$-open and V is an α^*-ω^*-set in (X,τ).* [**19**]

The collection of all $g\omega_t$-sets (resp. $g\omega_{\alpha^*}$-sets) of X is denoted by $g\omega_t(X,\tau)$ (resp. $g\omega_{\alpha^*}(X,\tau)$).

PROPOSITION 2.1. [**19**] *In a space (X,τ), the following statements hold:*

(1) *Every αg-ω-open set is gp-ω-open but not conversely.*
(2) *Every αg-ω-open set is gs-ω-open but not conversely.*

PROOF. It follows from Remark 2.2 and Definition 2.7. □

EXAMPLE 2.1. [**19**]

(1) *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R}-\mathbb{Q}\}$, the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is gp-ω-open but not αg-ω-open.*
(2) *In $(\mathbb{R}, \tau = \tau_u)$, the subset $[0,1]$ is gs-ω-open but not αg-ω-open.*

SOLUTION.
(1) For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, $\mathbb{Q}$ is the only closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq A$. Also $int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R}$. Thus

$$p - int_\omega(A) = A \cap int_\omega(cl(A)) = A \cap \mathbb{R} = A$$

and

$$\mathbb{Q} \subseteq p - int_{\omega}(A).$$

Hence A is gp-ω-open. On the other hand, $int_{\omega}(cl(int_{\omega}(A))) = int_{\omega}(cl(\phi)) = \phi$. Thus

$$\alpha - int_{\omega}(A) = A \cap int_{\omega}(cl(int_{\omega}(A))) = \phi$$

and so $\mathbb{Q} \nsubseteq \alpha$-$int_{\omega}(A)$. Hence A is not αg-ω-open.

(2) For the subset $A = [0, 1]$, we have

$$cl(int_{\omega}(A)) = cl((0, 1)) = [0, 1] = A$$

and

$$s - int_{\omega}(A) = A \cap cl(int_{\omega}(A)) = A \cap A = A.$$

If B is any closed subset such that $B \subseteq A$, then $B \subseteq A = s$-$int_{\omega}(A)$ and hence A is gs-ω-open. On the other hand,

$$int_{\omega}(cl(int_{\omega}(A))) = int_{\omega}(cl((0, 1))) = int_{\omega}([0, 1]) = (0, 1).$$

Thus

$$\alpha - int_{\omega}(A) = A \cap int_{\omega}(cl(int_{\omega}(A))) = A \cap (0, 1) = (0, 1)$$

and so $A \nsubseteq \alpha$-$int_{\omega}(A)$. Hence A is not αg-ω-open.

Proposition 2.2. [**19**] *Let S be a subset of (X, τ). If S is an $g\omega$-open set in X, then $S \in g\omega_t(X, \tau)$ and $S \in g\omega_{\alpha^*}(X, \tau)$.*

Remark 2.3. [**19**] *The converse of Proposition 2.2 need not be true.*

Example 2.2. [**19**] *In $(\mathbb{R}, \tau_u)$, the subset $(0, 1]$ is a $g\omega_t$-set as well as a $g\omega_{\alpha^*}$-set, but not $g\omega$-open.*

Solution. For the subset $A = (0, 1]$,

$$int_{\omega}(cl(A)) = int_{\omega}([0, 1]) = (0, 1) = int_{\omega}(A)$$

and hence A is a ω^*-t-set and so a $g\omega_t$-set. Also

$$int_{\omega}(cl(int_{\omega}(A))) = int_{\omega}(cl((0, 1))) = int_{\omega}([0, 1]) = (0, 1) = int_{\omega}(A)$$

and hence A is an α^*-ω^*-set and so a $g\omega_{\alpha^*}$-set. On the other hand, $[\frac{1}{2}, 1]$ is a closed subset of $(\mathbb{R}, \tau_u)$ such that $[\frac{1}{2}, 1] \subseteq A$ whereas $[\frac{1}{2}, 1] \nsubseteq (0, 1) = int_{\omega}(A)$. Hence A is not $g\omega$-open.

Definition 2.8. *An ideal $\mathcal{I}$ on a space (X, τ) is a non-empty collection of subsets of X which satisfies the following two conditions.*

(1) *$A \in \mathcal{I}$ and $B \subseteq A$ imply $B \in \mathcal{I}$ and*
(2) *$A \in \mathcal{I}$ and $B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$.*

Given a space (X, τ) with an ideal $\mathcal{I}$ on X if $\mathbb{P}(X)$ is the set of all subsets of X, a set operator $(.)^* : \mathbb{P}(X) \to \mathbb{P}(X)$, called a local function [**5**] *of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$.*

A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [**20**].

We will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal topological space or an ideal space.

Proposition 2.3. [**1**] *Let $(X,\tau,\mathcal{I})$ be an ideal topological space and A a subset of X. If $\mathcal{I}=\{\phi\}$ (resp. $\mathbb{P}(X)$), then $A^\star = cl(A)$ (resp. ϕ) and $cl^\star(A) = cl(A)$ (resp. A).*

Definition 2.9. *A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be*

(1) *pre-$\mathcal{I}_\omega$-open* [**15**] *if $A \subseteq int_\omega(cl^*(A))$.*

(2) *semi-$\mathcal{I}_\omega$-open* [**14**] *if $A \subseteq cl^*(int_\omega(A))$.*

(3) *α-$\mathcal{I}_\omega$-open* [**15**] *if $A \subseteq int_\omega(cl^*(int_\omega(A)))$.*

(4) *a t-$\mathcal{I}_{\omega^*}$-set* [**14**] *if $int_\omega(cl^*(A)) = int_\omega(A)$.*

(5) *an α^*-$\mathcal{I}_\omega$-set* [**13**] *if $int_\omega(cl^*(int_\omega(A))) = int_\omega(A)$.*

(6) *an S-$\mathcal{I}_\omega$-set* [**13**] *if $cl^*(int_\omega(A)) = int_\omega(A)$.*

In the light of these definitions, we have

$$\alpha - \mathcal{I} - int_\omega(A) = A \cap int_\omega(cl^*(int_\omega(A))),$$

$$p - \mathcal{I} - int_\omega(A) = A \cap int_\omega(cl^*(A))$$

and

$$s - \mathcal{I} - int_\omega(A) = A \cap cl^*(int_\omega(A)),$$

where α-$\mathcal{I}$-$int_\omega(A)$ denotes α-$\mathcal{I}$-ω-interior of A in $(X,\tau,\mathcal{I})$ which is the union of all α-$\mathcal{I}_\omega$-open sets of $(X,\tau,\mathcal{I})$ contained in A. p-$\mathcal{I}$-$int_\omega(A)$ and s-$\mathcal{I}$-$int_\omega(A)$ have similar meanings.

3. Properties and relationships of sets

Proposition 3.1. *In an ideal space $(X,\tau,\mathcal{I})$, the following results are true for a subset A of X.*

(1) *A is α-$\mathcal{I}_\omega$-open $\Rightarrow$ A is α-ω-open.*

(2) *A is pre-$\mathcal{I}_\omega$-open $\Rightarrow$ A is pre-ω-open.*

(3) *A is semi-$\mathcal{I}_\omega$-open $\Rightarrow$ A is semi-ω-open.*

(4) *α-$\mathcal{I}$-$int_\omega(A) \subseteq \alpha$-$int_\omega(A)$.*

(5) *p-$\mathcal{I}$-$int_\omega(A) \subseteq p$-$int_\omega(A)$.*

(6) *s-$\mathcal{I}$-$int_\omega(A) \subseteq s$-$int_\omega(A)$.*

Proof. (1), (2) and (3) follow from the fact that, in any ideal space $(X,\tau,\mathcal{I})$, $cl^*(A) \subseteq cl(A)$ and definitions 2.6 and 2.9. Also (4), (5) and (6) follow respectively from (1), (2) and (3). □

Proposition 3.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A a subset of X. Then the following statements hold:*

(1) *If A is a ω^*-t-set, then A is a t-$\mathcal{I}_{\omega^*}$-set.*

(2) *If A is a t-$\mathcal{I}_{\omega^*}$-set, then A is an α^*-$\mathcal{I}_\omega$-set.*

(3) *If A is an α^*-ω^*-set, then A is an α^*-$\mathcal{I}_\omega$-set.*

(4) *If A is a S-$\mathcal{I}_\omega$-set, then A is an α^*-$\mathcal{I}_\omega$-set.*

Remark 3.1. *The separate converses of Proposition 3.2 are not necessarily true as seen from the following example.*

Example 3.1. **(1)** *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R}-\mathbb{Q}\}$ and ideal $\mathcal{I} = \mathbb{P}(\mathbb{R})$, the subset $(\mathbb{R}-\mathbb{Q})_+$ is a t-$\mathcal{I}_{\omega^*}$-set but not a ω^*-t-set.*

(2) *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R}-\mathbb{Q}\}$ and ideal $\mathcal{I} = \{\phi\}$, the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is an α^*-$\mathcal{I}_\omega$-set but not a t-$\mathcal{I}_{\omega^*}$-set.*

(3) *In $(\mathbb{R}, \tau = \tau_u, \mathcal{I} = \mathbb{P}(\mathbb{R}))$, the subset $\mathbb{R}-\mathbb{Q}$ is an α^*-$\mathcal{I}_\omega$-set but not an α^*-ω^*-set.*

(4) *In $(\mathbb{R}, \tau = \tau_u, \mathcal{I} = \{\phi\})$, the subset $[0,1]$ is α^*-$\mathcal{I}_\omega$-set but not a S-$\mathcal{I}_\omega$-set.*

Solution. **(1)** For the subset $A = (\mathbb{R}-\mathbb{Q})_+$, $int_\omega(cl^*(A)) = int_\omega(A)$. Hence A is a t-$\mathcal{I}_{\omega^*}$-set. On the other hand,

$$int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq \phi = int_\omega(A).$$

Hence A is not a ω^*-t-set.
(2) For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*(\phi)) = \phi = int_\omega(A).$$

Hence A is an α^*-$\mathcal{I}_\omega$-set. On the other hand,

$$int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq int_\omega(A)$$

and hence A is not a t-$\mathcal{I}_{\omega^*}$-set.
(3) For the subset $A = (\mathbb{R}-\mathbb{Q})$,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*(A)) = int_\omega(A).$$

Hence A is an α^*-$\mathcal{I}_\omega$-set. On the other hand,

$$int_\omega(cl(int_\omega(A))) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq (\mathbb{R}-\mathbb{Q}) = int_\omega(A)$$

and hence A is not an α^*-ω^*-set.
(4) For the subset $A = [0,1]$,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*((0,1))) = int_\omega(cl((0,1))) = int_\omega([0,1]) = (0,1) = int_\omega(A).$$

Hence A is an α^*-$\mathcal{I}_\omega$-set. On the other hand,

$$cl^*(int_\omega(A)) = cl^*((0,1)) = cl((0,1)) = [0,1] \neq (0,1) = int_\omega(A).$$

Hence A is not a S-$\mathcal{I}_\omega$-set.

Remark 3.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, the following results are true for a subset A of X.*

(1) *A is ω-open $\Rightarrow$ A is α-$\mathcal{I}_\omega$-open.*
(2) *A is α-$\mathcal{I}_\omega$-open $\Rightarrow$ A is pre-$\mathcal{I}_\omega$-open.*
(3) *A is α-$\mathcal{I}_\omega$-open $\Rightarrow$ A is semi-$\mathcal{I}_\omega$-open.*

(4) $int_\omega(A) \subseteq \alpha$-$\mathcal{I}$-$int_\omega(A)$.
(5) α-$\mathcal{I}$-$int_\omega(A) \subseteq p$-$\mathcal{I}$-$int_\omega(A)$.
(6) α-$\mathcal{I}$-$int_\omega(A) \subseteq s$-$\mathcal{I}$-$int_\omega(A)$.

PROOF. (1), (2) and (3) follow directly from definition 2.9. Also (4), (5) and (6) respectively follow from (1), (2) and (3). □

4. αg-$\mathcal{I}_\omega$-open sets, gp-$\mathcal{I}_\omega$-open sets and gs-$\mathcal{I}_\omega$-open sets

DEFINITION 4.1. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be*

(1) *αg-$\mathcal{I}_\omega$-open if $F \subseteq \alpha$-$\mathcal{I}$-$int_\omega(A)$ whenever $F \subseteq A$ and F is closed in X.*
(2) *gp-$\mathcal{I}_\omega$-open if $F \subseteq p$-$\mathcal{I}$-$int_\omega(A)$ whenever $F \subseteq A$ and F is closed in X.*
(3) *gs-$\mathcal{I}_\omega$-open if $F \subseteq s$-$\mathcal{I}$-$int_\omega(A)$ whenever $F \subseteq A$ and F is closed in X.*

PROPOSITION 4.1. *In an ideal space $(X, \tau, \mathcal{I})$, the following statements hold:*

(1) *Every αg-$\mathcal{I}_\omega$-open set is αg-ω-open.*
(2) *Every gp-$\mathcal{I}_\omega$-open set is gp-ω-open.*
(3) *Every gs-$\mathcal{I}_\omega$-open set is gs-ω-open.*

PROOF. (1), (2) and (3) follow respectively from (4), (5) and (6) of Proposition 3.1. □

PROPOSITION 4.2. *In an ideal topological space, the following statements hold:*

(1) *Every $g\omega$-open set is αg-$\mathcal{I}_\omega$-open.*
(2) *Every αg-$\mathcal{I}_\omega$-open set is gp-$\mathcal{I}_\omega$-open.*
(3) *Every αg-$\mathcal{I}_\omega$-open set is gs-$\mathcal{I}_\omega$-open.*

PROOF. (1), (2) and (3) follow respectively from (4), (5) and (6) of Remark 3.2. □

REMARK 4.1. *The converses of Propositions 4.1 and 4.2 need not be true as seen from the next six Examples.*

EXAMPLE 4.1. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{N}, \mathbb{Q} - \mathbb{N}\}$ and ideal $\mathcal{I} = \mathbb{P}(\mathbb{R})$, the subset $\mathbb{Q}$ is αg-ω-open but not αg-$\mathcal{I}_\omega$-open.*

SOLUTION. For the subset $A = \mathbb{Q}$, $\mathbb{N}$ is the only closed subset of $\mathbb{R}$ such that $\mathbb{N} \subseteq A$. Also

$$int_\omega(cl(int_\omega(A))) = int_\omega(cl(A - \mathbb{N})) = int_\omega(\mathbb{R}) = \mathbb{R}.$$

Thus

$$\alpha - int_\omega(A) = A \cap int_\omega(cl(int_\omega(A))) = A \cap \mathbb{R} = A$$

and $\mathbb{N} \subseteq \alpha$-$int_\omega(A)$. Hence A is αg-ω-open. On the other hand

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(int_\omega(A)) = int_\omega(A) = A - \mathbb{N}.$$

Thus

$$\alpha - \mathcal{I} - int_\omega(A) = A \cap int_\omega(cl^*(int_\omega(A))) = A \cap (A - \mathbb{N}) = A - \mathbb{N}$$

and so $\mathbb{N} \nsubseteq A - \mathbb{N} = \alpha$-$\mathcal{I}$-$int_\omega(A)$. Hence A is not αg-$\mathcal{I}_\omega$-open.

EXAMPLE 4.2. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(1). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is gp-ω-open, but not gp-$\mathcal{I}_\omega$-open.*

SOLUTION. The subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$ is gp-ω-open by (1) of Example 2.1. On the other hand, $int_\omega(cl^*(A)) = int_\omega(A) = \phi$. Thus p-$\mathcal{I}$-$int_\omega(A) = A \cap int_\omega(cl^*(A)) = \phi$. The closed subset $\mathbb{Q}$ is such that $\mathbb{Q} \subseteq A$ whereas $\mathbb{Q} \nsubseteq \phi = p$-$\mathcal{I}$-$int_\omega(A)$ and hence A is not gp-$\mathcal{I}_\omega$-open.

EXAMPLE 4.3. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(3). Then the subset $[0, 1]$ is gs-ω-open, but not gs-$\mathcal{I}_\omega$-open.*

SOLUTION. The subset $A = [0, 1]$ is gs-ω-open by (2) of Example 2.1. On the other hand, $cl^*(int_\omega(A)) = cl^*((0, 1)) = (0, 1)$. Thus s-$\mathcal{I}$-$int_\omega(A) = A \cap cl^*(int_\omega(A)) = (0, 1)$. The closed subset A is such that $A \subseteq A$ whereas $A \nsubseteq s$-$\mathcal{I}$-$int_\omega(A)$ and hence A is not gs-$\mathcal{I}_\omega$-open.

EXAMPLE 4.4. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{N}, \mathbb{Q} - \mathbb{N}\}$ and ideal $\mathcal{I} = \{\phi\}$, the subset $\mathbb{Q}$ is αg-$\mathcal{I}_\omega$-open but not $g\omega$-open.*

SOLUTION. For the subset $A = \mathbb{Q}$, $\mathbb{N}$ is the only closed subset of $\mathbb{R}$ such that $\mathbb{N} \subseteq A$. Also

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl(\mathbb{Q} - \mathbb{N})) = int_\omega(\mathbb{R}) = \mathbb{R}.$$

Thus

$$\alpha - \mathcal{I} - int_\omega(A) = A \cap int_\omega(cl^*(int_\omega(A))) = A \cap \mathbb{R} = A$$

and $\mathbb{N} \subseteq \alpha$-$\mathcal{I}$-$int_\omega(A)$. Hence A is αg-$\mathcal{I}_\omega$-open. On the other hand, $\mathbb{N} \subseteq A$ whereas $\mathbb{N} \nsubseteq \mathbb{Q} - \mathbb{N} = int_\omega(A)$. Hence A is not $g\omega$-open.

EXAMPLE 4.5. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(2). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is gp-$\mathcal{I}_\omega$-open, but not αg-$\mathcal{I}_\omega$-open.*

SOLUTION. For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, $\mathbb{Q}$ is the only closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq A$. Also

$$int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R}.$$

Thus

$$p - \mathcal{I} - int_\omega(A) = A \cap int_\omega(cl^*(A)) = A.$$

Thus $\mathbb{Q} \subseteq A = p$-$\mathcal{I}$-$int_\omega(A)$ and hence A is gp-$\mathcal{I}_\omega$-open. On the other hand,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*(\phi)) = \phi.$$

Thus α-$\mathcal{I}$-$int_\omega(A) = A \cap \phi = \phi$. Since the closed set $\mathbb{Q}$ is such that $\mathbb{Q} \subseteq A$ whereas $\mathbb{Q} \nsubseteq \phi = \alpha$-$\mathcal{I}$-$int_\omega(A)$, A is not αg-$\mathcal{I}_\omega$-open.

EXAMPLE 4.6. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(4). Then the subset $[0, 1]$ is gs-$\mathcal{I}_\omega$-open, but neither αg-$\mathcal{I}_\omega$-open nor $g\omega$-open.*

SOLUTION. For the subset $A = [0, 1]$,

$$cl^*(int_\omega(A)) = cl^*((0, 1)) = cl((0, 1)) = [0, 1].$$

Thus $s\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap cl^*(int_\omega(A)) = A$. If B is any closed subset such that $B \subseteq A$, then $B \subseteq A = s\text{-}\mathcal{I}\text{-}int_\omega(A)$. Hence A is $gs\text{-}\mathcal{I}_\omega$-open. On the other hand,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega([0, 1]) = (0, 1).$$

Thus $\alpha\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap (0, 1) = (0, 1)$. Since A is closed and $A \subseteq A$, but $A \nsubseteq (0, 1) = \alpha\text{-}\mathcal{I}\text{-}int_\omega(A)$, A is not $\alpha g\text{-}\mathcal{I}_\omega$-open. Also A is closed and $A \subseteq A$ but $A \nsubseteq (0, 1) = int_\omega(A)$. Hence A is not $g\omega$-open.

REMARK 4.2. *The following two Examples show that the concepts of $gs\text{-}\mathcal{I}_\omega$-openness and $gp\text{-}\mathcal{I}_\omega$-openness are independent.*

EXAMPLE 4.7. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $[0, 1]$ is $gs\text{-}\mathcal{I}_\omega$-open, but not $gp\text{-}\mathcal{I}_\omega$-open.*

SOLUTION. The subset $A = [0, 1]$ is $gs\text{-}\mathcal{I}_\omega$-open by Example 4.6. On the other hand,

$$int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega([0, 1]) = (0, 1).$$

Thus

$$p - \mathcal{I} - int_\omega(A) = A \cap int_\omega(cl^*(A)) = A \cap (0, 1) = (0, 1).$$

Since A is closed and $A \subseteq A$ whereas $A \nsubseteq (0, 1) = p\text{-}\mathcal{I}\text{-}int_\omega(A)$, A is not $gp\text{-}\mathcal{I}_\omega$-open.

EXAMPLE 4.8. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(2). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is $gp\text{-}\mathcal{I}_\omega$-open, but not $gs\text{-}\mathcal{I}_\omega$-open.*

SOLUTION. The subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$ is $gp\text{-}\mathcal{I}_\omega$-open by Example 4.5. On the other hand, $cl^*(int_\omega(A)) = cl^*(\phi) = \phi$. Thus $s\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap \phi = \phi$. Since $\mathbb{Q}$ is closed and $\mathbb{Q} \subseteq A$ whereas $\mathbb{Q} \nsubseteq \phi = s\text{-}\mathcal{I}\text{-}int_\omega(A)$, A is not $gs\text{-}\mathcal{I}_\omega$-open.

REMARK 4.3. *We have the following Diagram-I from Propositions 2.1, 4.1 and 4.2.*

Diagram-I

$$\begin{tikzcd}
g\omega\text{-}open \arrow[r] \arrow[d] & \alpha g\text{-}\mathcal{I}_\omega\text{-}open \arrow[r] \arrow[dl] \arrow[d] & gp\text{-}\mathcal{I}_\omega\text{-}open \arrow[d] \\
gs\text{-}\mathcal{I}_\omega\text{-}open \arrow[dr] & \alpha g\text{-}\omega\text{-}open \arrow[r] \arrow[d] & gp\text{-}\omega\text{-}open \\
 & gs\text{-}\omega\text{-}open &
\end{tikzcd}$$

In Diagram-I, none of the implications is reversible as seen from examples 4.1 to 4.6 in Remark 4.1 and (1), (2) in Example 2.1.

5. $g\omega_t$-$\mathcal{I}$-sets, $g\omega_{\alpha^*}$-$\mathcal{I}$-sets and $g\omega_S$-$\mathcal{I}$-sets

DEFINITION 5.1. *A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be*

(1) *a $g\omega_t$-$\mathcal{I}$-set if $A = U \cap V$, where U is $g\omega$-open and V is a t-$\mathcal{I}_{\omega^*}$-set.*
(2) *a $g\omega_{\alpha^*}$-$\mathcal{I}$-set if $A = U \cap V$, where U is $g\omega$-open and V is an α^*-$\mathcal{I}_\omega$-set.*
(3) *a $g\omega_S$-$\mathcal{I}$-set if $A = U \cap V$, where U is $g\omega$-open and V is a S-$\mathcal{I}_\omega$-set.*

PROPOSITION 5.1. *In an ideal topological space, the following statements hold:*

(1) *Every t-$\mathcal{I}_{\omega^*}$-set is a $g\omega_t$-$\mathcal{I}$-set.*
(2) *Every α^*-$\mathcal{I}_\omega$-set is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*
(3) *Every S-$\mathcal{I}_\omega$-set is a $g\omega_S$-$\mathcal{I}$-set.*
(4) *Every $g\omega$-open set is a $g\omega_t$-$\mathcal{I}$-set.*
(5) *Every $g\omega$-open set is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*
(6) *Every $g\omega$-open set is a $g\omega_S$-$\mathcal{I}$-set.*

PROOF. The proof is obvious. $\square$

REMARK 5.1. *The converses of Proposition 5.1 need not be true as seen from the following examples.*

EXAMPLE 5.1. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $\mathbb{R} - \mathbb{Q}$ is a $g\omega_t$-$\mathcal{I}$-set, but not a t-$\mathcal{I}_{\omega^*}$-set.*

SOLUTION. The subset $A = \mathbb{R} - \mathbb{Q}$ is ω-open and hence $g\omega$-open. It shows that A is a $g\omega_t$-$\mathcal{I}$-set. On the other hand,

$$int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq A = int_\omega(A).$$

Hence A is not a t-$\mathcal{I}_{\omega^*}$-set.

EXAMPLE 5.2. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $\mathbb{R} - \mathbb{Q}$ is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set, but not an α^*-$\mathcal{I}_\omega$-set.*

SOLUTION. The subset $A = \mathbb{R} - \mathbb{Q}$ is $g\omega$-open and hence a $g\omega_{\alpha^*}$-$\mathcal{I}$-set. On the other hand,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R} \neq A = int_\omega(A).$$

Hence A is not an α^*-$\mathcal{I}_\omega$-set.

EXAMPLE 5.3. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $\mathbb{R} - \mathbb{Q}$ is a $g\omega_S$-$\mathcal{I}$-set, but not a S-$\mathcal{I}_\omega$-set.*

SOLUTION. The subset $A = \mathbb{R} - \mathbb{Q}$ is $g\omega$-open and hence a $g\omega_S$-$\mathcal{I}$-set. On the other hand,

$$cl^*(int_\omega(A)) = cl^*(A) = cl(A) = \mathbb{R} \neq (\mathbb{R} - \mathbb{Q}) = int_\omega(A).$$

Hence A is not a S-$\mathcal{I}_\omega$-set.

EXAMPLE 5.4. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $[0, 1]$ is a $g\omega_t$-$\mathcal{I}$-set as well as a $g\omega_{\alpha^*}$-$\mathcal{I}$-set, but not $g\omega$-open.*

SOLUTION. For the subset $A = [0, 1]$,

$$int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega([0, 1]) = (0, 1) = int_\omega(A).$$

Hence A is a t-$\mathcal{I}_{\omega^*}$-set and hence a $g\omega_t$-$\mathcal{I}$-set. Also,

$$int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*((0, 1))) = int_\omega(cl((0, 1))) = int_\omega([0, 1]) = (0, 1) = int_\omega(A).$$

Hence A is an α^*-$\mathcal{I}_\omega$-set and hence a $g\omega_{\alpha^*}$-$\mathcal{I}$-set. On the other hand, A is not $g\omega$-open, since A is closed and $A \subseteq A$ whereas $A \nsubseteq (0, 1) = int_\omega(A)$.

EXAMPLE 5.5. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $\mathbb{Q}$ is a $g\omega_S$-$\mathcal{I}$-set, but not $g\omega$-open.*

SOLUTION. For the subset $A = \mathbb{Q}$, $cl^*(int_\omega(A)) = cl^*(\phi) = \phi = int_\omega(A)$. Hence A is a S-$\mathcal{I}_\omega$-set and hence a $g\omega_S$-$\mathcal{I}$-set. On the other hand, $\{1\}$ is closed and $\{1\} \subseteq \mathbb{Q}$ whereas $\{1\} \nsubseteq \phi = int_\omega(A)$. Hence A is not $g\omega$-open.

PROPOSITION 5.2. *In an ideal topological space, the following statements hold:*

(1) *Every $g\omega_t$-set is a $g\omega_t$-$\mathcal{I}$-set.*
(2) *Every $g\omega_t$-set is a $g\omega_{\alpha^*}$-set.*
(3) *Every $g\omega_{\alpha^*}$-set is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*
(4) *Every $g\omega_t$-$\mathcal{I}$-set is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*
(5) *Every $g\omega_S$-$\mathcal{I}$-set is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*

PROOF. (1), (2), (3) and (4) follow from Proposition 3.2.
(5) If K is a $g\omega_S$-$\mathcal{I}$-set then $K = U \cap A$ where U is $g\omega$-open and A is a S-$\mathcal{I}_\omega$-set and so A is an α^*-$\mathcal{I}_\omega$-set by (4) of Proposition 3.2. Hence K is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set. □

REMARK 5.2. *The converses of Proposition 5.2 need not be true as seen from the next four Examples.*

EXAMPLE 5.6. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(1). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is a $g\omega_t$-$\mathcal{I}$-set, but not a $g\omega_t$-set.*

SOLUTION. For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, $int_\omega(cl^*(A)) = int_\omega(A)$. Thus A is a t-$\mathcal{I}_{\omega^*}$-set and hence a $g\omega_t$-$\mathcal{I}$-set. On the other hand if $A = U \cap V$, where U is $g\omega$-open and V is a ω^*-t-set, then $A \subseteq V$. So $int_\omega(cl(A)) \subseteq int_\omega(cl(V)) = int_\omega(V) \subseteq V$. But $int_\omega(cl(A)) = \mathbb{R}$ which implies $\mathbb{R} \subseteq V$. Hence $\mathbb{R} = V$. Thus $A = U \cap \mathbb{R} = U$ which implies that A is $g\omega$-open which is a contradiction since A is not $g\omega$-open, for the closed subset $\mathbb{Q} \subseteq A$ whereas $\mathbb{Q} \nsubseteq \phi = int_\omega(A)$.

EXAMPLE 5.7. *Let $\mathbb{R}$ and τ be as in Example 2.1(1). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is a $g\omega_{\alpha^*}$-set, but not a $g\omega_t$-set.*

SOLUTION. For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, $int_\omega(cl(int_\omega(A))) = int_\omega(cl(\phi)) = \phi = int_\omega(A)$. Thus A is an α^*-ω^*-set and hence a $g\omega_{\alpha^*}$-set. But A is not a $g\omega_t$-set by Example 5.6.

EXAMPLE 5.8.
(1) *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{N}, \mathbb{Q} - \mathbb{N}\}$ and ideal $\mathcal{I} = \mathbb{P}(\mathbb{R})$, the subset $\mathbb{Q}$ is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set but not a $g\omega_{\alpha^*}$-set.*
(2) *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(2). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set, but not a $g\omega_t$-$\mathcal{I}$-set.*

SOLUTION.
For the subset $A = \mathbb{Q}$, $int_\omega(cl^*(int_\omega(A))) = int_\omega(int_\omega(A)) = int_\omega(A)$. Thus A is an α^*-$\mathcal{I}_\omega$-set and hence a $g\omega_{\alpha^*}$-$\mathcal{I}$-set. On the other hand if $A = U \cap V$ where U is $g\omega$-open and V is an α^*-ω^*-set, then $A \subseteq V$. This implies $int_\omega(cl(int_\omega(A))) \subseteq int_\omega(cl(int_\omega(V))) = int_\omega(V)$ by assumption. Thus $\mathbb{R} \subseteq int_\omega(V) \subseteq V$ and so $V = \mathbb{R}$. Then $A = U$ which implies that A is $g\omega$-open which is a contradiction since A is not $g\omega$-open for $\mathbb{N}$ is closed and $\mathbb{N} \subseteq A$ whereas $\mathbb{N} \nsubseteq A - \mathbb{N} = int_\omega(A)$. Thus A is not a $g\omega_{\alpha^*}$-set.

For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, A is an α^*-$\mathcal{I}_\omega$-set by (2) of Example 3.1 and hence a $g\omega_{\alpha^*}$-$\mathcal{I}$-set. On the other hand, if $A = U \cap V$, where U is $g\omega$-open and V is a t-$\mathcal{I}_{\omega^*}$-set, then $A \subseteq V$. This implies $int_\omega(cl^*(A)) \subseteq int_\omega(cl^*(V)) = int_\omega(V) \subseteq V$. But $int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega(\mathbb{R}) = \mathbb{R}$ which implies that $\mathbb{R} \subseteq V$ and $V = \mathbb{R}$. Hence $A = U$ which implies that A is $g\omega$-open by assumption on U. This is a contradiction since A is not $g\omega$-open for the closed subset $\mathbb{Q} \subseteq A$ whereas $\mathbb{Q} \nsubseteq \phi = int_\omega(A)$. This proves that A is not a $g\omega_t$-$\mathcal{I}$-set.

EXAMPLE 5.9. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{R} - \mathbb{Q}, \mathbb{N} \cup (\mathbb{R} - \mathbb{Q})\}$ and ideal $\mathcal{I} = \{\phi\}$, the subset $\mathbb{Q}$ is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set but not a $g\omega_S$-$\mathcal{I}$-set.*

SOLUTION. For the subset $A = \mathbb{Q}$, $int_\omega(cl^*(int_\omega(A))) = int_\omega(cl(\mathbb{N})) = int_\omega(\mathbb{Q}) = \mathbb{N} = int_\omega(A)$. Thus A is an α^*-$\mathcal{I}_\omega$-set and hence a $g\omega_{\alpha^*}$-$\mathcal{I}$-set. On the other hand if $A = U \cap V$ where U is $g\omega$-open and V is a S-$\mathcal{I}_\omega$-set, then $A \subseteq U$ and $A \subseteq V$. Since $\mathbb{Q}$ is closed and $\mathbb{Q} = A \subseteq U$ and U is $g\omega$-open, $\mathbb{Q} \subseteq int_\omega(U)$. Also $A \subseteq V$ implies $cl^*(int_\omega(A)) \subseteq cl^*(int_\omega(V)) = int_\omega(V)$ by assumption. Thus $\mathbb{Q} \subseteq int_\omega(V)$. Then $\mathbb{Q} \subseteq int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(A) = \mathbb{N}$ which is a contradiction. This proves that A is not a $g\omega_S$-$\mathcal{I}$-set.

By Propositions 2.2, 3.2, 5.1 and 5.2, we have the following Diagram-II.

$$\begin{tikzcd}
t\text{-}\mathcal{I}_{\omega^*}\text{-set} \arrow[r] \arrow[d] & g\omega_t\text{-}\mathcal{I}\text{-set} \arrow[d] & g\omega_t\text{-set} \arrow[l] \arrow[d] & \\
\alpha^*\text{-}\mathcal{I}_\omega\text{-set} \arrow[r] & g\omega_{\alpha^*}\text{-}\mathcal{I}\text{-set} & g\omega_{\alpha^*}\text{-set} \arrow[l] & g\omega\text{-open} \arrow[l] \arrow[ul] \arrow[dll] \\
S\text{-}\mathcal{I}_\omega\text{-set} \arrow[r] \arrow[u] & g\omega_S\text{-}\mathcal{I}\text{-set} \arrow[u] & &
\end{tikzcd}$$

None of the implications in Diagram-II is reversible as seen from the above Examples 2.2, (2) and (4) of 3.1, 5.1 to 5.5, 5.6 to 5.9.

REMARK 5.3. (1) *The notions of $g\omega_t$-$\mathcal{I}$-sets and $g\omega_{\alpha^*}$-sets are independent as seen from Examples 5.10 and 5.11.*

(2) *The notions of $g\omega_t$-$\mathcal{I}$-sets and $g\omega_S$-$\mathcal{I}$-sets are independent as seen from Examples 5.12 and 5.13.*

(3) *The notions of gp-$\mathcal{I}_\omega$-open sets and $g\omega_t$-$\mathcal{I}$-sets are independent as seen from Examples 5.14 and 5.15.*

(4) *The notions of αg-$\mathcal{I}_\omega$-open sets and $g\omega_{\alpha^*}$-$\mathcal{I}$-sets are independent as seen from Examples 5.16 and 5.17.*

(5) *The notions of gs-$\mathcal{I}_\omega$-open sets and $g\omega_S$-$\mathcal{I}$-sets are independent as seen from Examples 5.18 and 5.19.*

EXAMPLE 5.10. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(2). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is a $g\omega_{\alpha^*}$-set, but not a $g\omega_t$-$\mathcal{I}$-set.*

SOLUTION. For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, $int_\omega(cl(int_\omega(A))) = int_\omega(A) = \phi$. Thus A is an α^*-ω^*-set and hence a $g\omega_{\alpha^*}$-set. But A is not a $g\omega_t$-$\mathcal{I}$-set by (2) of Example 5.8.

EXAMPLE 5.11. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.1. Then the subset $\mathbb{Q}$ is a $g\omega_t$-$\mathcal{I}$-set but not a $g\omega_{\alpha^*}$-set.*

SOLUTION. For the subset $A = \mathbb{Q}$, $int_\omega(cl^*(A)) = int_\omega(A)$. So A is a t-$\mathcal{I}_{\omega^*}$-set and hence a $g\omega_t$-$\mathcal{I}$-set. On the other hand if $A = U \cap V$ where U is $g\omega$-open and V is an α^*-ω^*-set, then $A \subseteq V$. Since $A \subseteq V$, $int_\omega(cl(int_\omega(A))) \subseteq int_\omega(cl(int_\omega(V))) = int_\omega(V)$ by assumption. Thus $\mathbb{R} \subseteq int_\omega(V) = V$. So $V = \mathbb{R}$ and $A = U \cap \mathbb{R} = U$ which means A is $g\omega$-open. This is a contradiction since A is not $g\omega$-open for $\mathbb{N}$ is closed and $\mathbb{N} \subseteq \mathbb{Q} = A$ whereas $\mathbb{N} \nsubseteq \mathbb{Q} - \mathbb{N} = int_\omega(A)$. Thus A is not a $g\omega_{\alpha^*}$-set.

EXAMPLE 5.12. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(2). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is a $g\omega_S$-$\mathcal{I}$-set, but not a $g\omega_t$-$\mathcal{I}$-set.*

SOLUTION. For the subset $A = \mathbb{Q} \cup \{\sqrt{2}\}$, $cl^*(int_\omega(A)) = cl(int_\omega(A)) = cl(\phi) = \phi = int_\omega(A)$. Hence A is a S-$\mathcal{I}_\omega$-set and hence a $g\omega_S$-$\mathcal{I}$-set. But A is not a $g\omega_t$-$\mathcal{I}$-set by (2) of Example 5.8.

EXAMPLE 5.13. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.6. Then the subset $[0,1]$ is a $g\omega_t$-$\mathcal{I}$-set but not a $g\omega_S$-$\mathcal{I}$-set.*

SOLUTION. For the subset $A = [0,1]$,

$$int_\omega(cl^*(A)) = int_\omega(cl(A)) = int_\omega([0,1]) = (0,1) = int_\omega(A).$$

So A is a t-$\mathcal{I}_{\omega^*}$-set and hence a $g\omega_t$-$\mathcal{I}$-set. On the other hand if $A = U \cap V$ where U is $g\omega$-open and V is a S-$\mathcal{I}_\omega$-set, then $A \subseteq U$ and $A \subseteq V$. Since A is closed and $A \subseteq U$, U being $g\omega$-open by assumption, $A \subseteq int_\omega(U)$. Also $A \subseteq V$ implies $cl^*(int_\omega(A)) \subseteq cl^*(int_\omega(V)) = int_\omega(V)$ by assumption. Thus $cl(int_\omega(A)) \subseteq int_\omega(V)$. Since $cl(int_\omega(A)) = cl((0,1)) = [0,1] = A$, $A \subseteq int_\omega(V)$. Thus $A \subseteq int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(A) = (0,1)$ which is a contradiction. Hence A is not a $g\omega_S$-$\mathcal{I}$-set.

EXAMPLE 5.14. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(2). Then the subset $\mathbb{Q} \cup \{\sqrt{2}\}$ is gp-$\mathcal{I}_\omega$-open by Example 4.8, but not a $g\omega_t$-$\mathcal{I}$-set by (2) of Example 5.8.*

EXAMPLE 5.15. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(3). Then the subset $[0,1]$ is a $g\omega_t$-$\mathcal{I}$-set, but not gp-$\mathcal{I}_\omega$-open.*

SOLUTION. For the subset $A = [0,1]$, $int_\omega(cl^*(A)) = int_\omega(A)$. Thus A is a t-$\mathcal{I}_{\omega^*}$-set and hence a $g\omega_t$-$\mathcal{I}$-set. On the other hand, $int_\omega(cl^*(A)) = int_\omega(A) = (0,1)$ and p-$\mathcal{I}$-$int_\omega(A) = A \cap (0,1) = (0,1)$. Since A is closed and $A \subseteq A$ whereas $A \nsubseteq (0,1) = p$-$\mathcal{I}$-$int_\omega(A)$, A is not gp-$\mathcal{I}_\omega$-open.

EXAMPLE 5.16. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 4.4. Then the subset $\mathbb{Q}$ is αg-$\mathcal{I}_\omega$-open but not a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*

SOLUTION. The subset $A = \mathbb{Q}$ is αg-$\mathcal{I}_\omega$-open by Example 4.4. On the other hand if $A = U \cap V$ where U is $g\omega$-open and V is an α^*-$\mathcal{I}_\omega$-set, then $A \subseteq V$. This implies $int_\omega(cl^*(int_\omega(A))) \subseteq int_\omega(cl^*(int_\omega(V))) = int_\omega(V)$ by assumption. Also $int_\omega(cl^*(int_\omega(A))) = int_\omega(cl(int_\omega(A))) = int_\omega(cl(A - \mathbb{N})) = int_\omega(\mathbb{R}) = \mathbb{R}$ and thus $\mathbb{R} \subseteq V$. So $V = \mathbb{R}$ and $A = U \cap \mathbb{R} = U$. This means A is $g\omega$-open which is a contradiction since A is not $g\omega$-open by Example 4.4. Hence A is not a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.

EXAMPLE 5.17. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(3). Then the subset $[0,1]$ is an $g\omega_{\alpha^*}$-$\mathcal{I}$-set, but not αg-$\mathcal{I}_\omega$-open.*

SOLUTION. For the subset $A = [0,1]$, $int_\omega(cl^*(int_\omega(A))) = int_\omega(cl^*((0,1))) = int_\omega((0,1)) = (0,1) = int_\omega(A)$. Hence A is an α^*-$\mathcal{I}_\omega$-set and hence an $g\omega_{\alpha^*}$-$\mathcal{I}$-set. On the other hand, α-$\mathcal{I}$-$int_\omega(A) = A \cap int_\omega(cl^*(int_\omega(A))) = A \cap (0,1) = (0,1)$. Then A is not αg-$\mathcal{I}_\omega$-open for A is closed and $A \subseteq A$ whereas $A \nsubseteq (0,1) = \alpha$-$\mathcal{I}$-$int_\omega(A)$.

EXAMPLE 5.18. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 5.9. Then the subset $\mathbb{Q}$ is gs-$\mathcal{I}_\omega$-open but not a $g\omega_S$-$\mathcal{I}$-set*

Solution. For the subset $A = \mathbb{Q}$, $\mathbb{Q} - \mathbb{N}$ and $\mathbb{Q}$ are the only closed subsets of $\mathbb{R}$ such that $\mathbb{Q} \subseteq A$ and $\mathbb{Q} - \mathbb{N} \subseteq A$. Also $s\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap cl^*(int_\omega(A)) = A \cap cl(int_\omega(A)) = A \cap cl(\mathbb{N}) = A \cap \mathbb{Q} = \mathbb{Q}$. Thus $\mathbb{Q} - \mathbb{N} \subseteq s\text{-}\mathcal{I}\text{-}int_\omega(A)$ and $\mathbb{Q} \subseteq s\text{-}\mathcal{I}\text{-}int_\omega(A)$. Hence A is gs-$\mathcal{I}_\omega$-open. But $A = \mathbb{Q}$ is not a $g\omega_S$-$\mathcal{I}$-set by Example 5.9.

Example 5.19. *Let $\mathbb{R}, \tau$ and $\mathcal{I}$ be as in Example 3.1(3). Then the subset $[0, 1]$ is an $g\omega_S$-$\mathcal{I}$-set, but not gs-$\mathcal{I}_\omega$-open set.*

Solution. For the subset $A = [0, 1]$, $cl^*(int(A)) = int(A)$. Thus A is a S-$\mathcal{I}_\omega$-set and hence a $g\omega_S$-$\mathcal{I}$-set. But A is not gs-$\mathcal{I}_\omega$-open by Example 4.3.

Proposition 5.3. *A subset A of $(X, \tau, \mathcal{I})$ is $g\omega$-open if and only if it is gp-$\mathcal{I}_\omega$-open and a $g\omega_t$-$\mathcal{I}$-set.*

Proof. Necessity is trivial from Diagram-I and Diagram-II.

To prove sufficiency, let A be gp-$\mathcal{I}_\omega$-open and a $g\omega_t$-$\mathcal{I}$-set in X. Let $F \subseteq A$ where F is closed in X. Since A is a $g\omega_t$-$\mathcal{I}$-set in X, $A = U \cap V$, where U is $g\omega$-open and V is a t-$\mathcal{I}_{\omega^*}$-set. Then $A \subseteq U$ and $F \subseteq U$. Since F is closed and U is $g\omega$-open, $F \subseteq int_\omega(U)$. Also A is gp-$\mathcal{I}_\omega$-open implies $F \subseteq p\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap int_\omega(cl^*(A))$. Hence $F \subseteq int_\omega(cl^*(A)) \subseteq int_\omega(cl^*(V)) = int_\omega(V)$ for $A \subseteq V$ and V is a t-$\mathcal{I}_{\omega^*}$-set. Thus $F \subseteq int_\omega(V)$ and so $F \subseteq int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(A)$. Hence A is $g\omega$-open in X. □

Proposition 5.4. *A subset A of $(X, \tau, \mathcal{I})$ is $g\omega$-open if and only if it is αg-$\mathcal{I}_\omega$-open and a $g\omega_{\alpha^*}$-$\mathcal{I}$-set.*

Proof. Necessity is trivial from Diagram-I and Diagram-II.

To prove sufficiency, let A be αg-$\mathcal{I}_\omega$-open and a $g\omega_{\alpha^*}$-$\mathcal{I}$-set in X. Let $F \subseteq A$ where F is closed in X. Since A is a $g\omega_{\alpha^*}$-$\mathcal{I}$-set in X, $A = U \cap V$, where U is $g\omega$-open and V is an α^*-$\mathcal{I}_\omega$-set. Then $A \subseteq U$ and $F \subseteq U$. Since F is closed and U is $g\omega$-open, $F \subseteq int_\omega(U)$. Also A is αg-$\mathcal{I}_\omega$-open implies $F \subseteq \alpha\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap int_\omega(cl^*(int_\omega(A)))$. Hence $F \subseteq int_\omega(cl^*(int_\omega(A))) \subseteq int_\omega(cl^*(int_\omega(V))) = int_\omega(V)$ for $A \subseteq V$ and V is an α^*-$\mathcal{I}_\omega$-set. Thus $F \subseteq int_\omega(V)$ and so $F \subseteq int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(A)$. Hence A is $g\omega$-open in X. □

Proposition 5.5. *A subset A of $(X, \tau, \mathcal{I})$ is $g\omega$-open if and only if it is gs-$\mathcal{I}_\omega$-open and a $g\omega_S$-$\mathcal{I}$-set.*

Proof. Necessity is trivial from Diagram-I and Diagram-II.

To prove sufficiency, let A be gs-$\mathcal{I}_\omega$-open and a $g\omega_S$-$\mathcal{I}$-set in X. Let $F \subseteq A$ where F is closed in X. Since A is a $g\omega_S$-$\mathcal{I}$-set in X, $A = U \cap V$, where U is $g\omega$-open and V is a S-$\mathcal{I}_\omega$-set. Then $A \subseteq U$ and $F \subseteq U$. Since F is closed and U is $g\omega$-open, $F \subseteq int_\omega(U)$. Also A is gs-$\mathcal{I}_\omega$-open implies $F \subseteq s\text{-}\mathcal{I}\text{-}int_\omega(A) = A \cap cl^*(int_\omega(A))$. Hence $F \subseteq cl^*(int_\omega(A)) \subseteq cl^*(int_\omega(V)) = int_\omega(V)$ for $A \subseteq V$ and V is a S-$\mathcal{I}_\omega$-set. Thus $F \subseteq int_\omega(V)$ and so $F \subseteq int_\omega(U) \cap int_\omega(V) = int_\omega(U \cap V) = int_\omega(A)$. Hence A is $g\omega$-open in X. □

Remark 5.4. *Proposition 5.3 together with Remark 5.3(3) proves that neither* gp-$\mathcal{I}_\omega$*-openness nor being a* $g\omega_t$-$\mathcal{I}$*-set implies* $g\omega$*-openness and so gives a decomposition of a* $g\omega$*-open set into a* gp-$\mathcal{I}_\omega$*-open set and a* $g\omega_t$-$\mathcal{I}$*-set.*

Remark 5.5. *Proposition 5.4 together with Remark 5.3(4) proves that neither* αg-$\mathcal{I}_\omega$*-openness nor being a* $g\omega_{\alpha^*}$-$\mathcal{I}$*-set implies* $g\omega$*-openness and so gives a decomposition of a* $g\omega$*-open set into an* αg-$\mathcal{I}_\omega$*-open set and a* $g\omega_{\alpha^*}$-$\mathcal{I}$*-set.*

Remark 5.6. *Proposition 5.5 together with Remark 5.3(5) proves that neither* gs-$\mathcal{I}_\omega$*-openness nor being a* $g\omega_S$-$\mathcal{I}$*-set implies* $g\omega$*-openness and so gives a decomposition of a* $g\omega$*-open set into a* gs-$\mathcal{I}_\omega$*-open set and a* $g\omega_S$-$\mathcal{I}$*-set.*

6. Decompositions of $g\omega$-continuity via idealization

Definition 6.1. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called* $g\omega$*-continuous* [**4**] *if for every* $V \in \sigma$, $f^{-1}(V)$ *is* $g\omega$*-open in* (X, τ).

Definition 6.2. [**19**] *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called* αg-ω*-continuous (resp.* gp-ω*-continuous) if for every* $V \in \sigma$, $f^{-1}(V)$ *is* αg-ω*-open (resp.* gp-ω*-open) in* (X, τ).

Definition 6.3. [**19**] *A function* $f : (X, \tau) \to (Y, \sigma)$ *is called*

(1) $g\omega_t$*-continuous if for every* $V \in \sigma$, $f^{-1}(V) \in g\omega_t(X, \tau)$.
(2) $g\omega_{\alpha^*}$*-continuous if for every* $V \in \sigma$, $f^{-1}(V) \in g\omega_{\alpha^*}(X, \tau)$.

Definition 6.4. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is called* αg-$\mathcal{I}_\omega$*-continuous (resp.* gp-$\mathcal{I}_\omega$*-continuous,* gs-$\mathcal{I}_\omega$*-continuous,* $g\omega_t$-$\mathcal{I}$*-continuous,* $g\omega_{\alpha^*}$-$\mathcal{I}$*-continuous,* $g\omega_S$-$\mathcal{I}$*-continuous) if for every* $V \in \sigma$, $f^{-1}(V)$ *is* αg-$\mathcal{I}_\omega$*-open (resp.* gp-$\mathcal{I}_\omega$*-open,* gs-$\mathcal{I}_\omega$*-open, a* $g\omega_t$-$\mathcal{I}$*-set, a* $g\omega_{\alpha^*}$-$\mathcal{I}$*-set, a* $g\omega_S$-$\mathcal{I}$*-set) in* $(X, \tau, \mathcal{I})$.

From Propositions 5.3, 5.4 and 5.5, we have the following

Theorem 6.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. For a function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$*, the following properties are equivalent.*

(1) f *is* $g\omega$*-continuous.*
(2) f *is* gp-$\mathcal{I}_\omega$*-continuous and* $g\omega_t$-$\mathcal{I}$*-continuous.*
(3) f *is* αg-$\mathcal{I}_\omega$*-continuous and* $g\omega_{\alpha^*}$-$\mathcal{I}$*-continuous.*
(4) f *is* gs-$\mathcal{I}_\omega$*-continuous and* $g\omega_S$-$\mathcal{I}$*-continuous.*

Remark 6.1.

(1) *Theorem 6.1 together with Remark 5.4 gives a decomposition of* $g\omega$*-continuity into* gp-$\mathcal{I}_\omega$*-continuity and* $g\omega_t$-$\mathcal{I}$*-continuity.*

(2) *Theorem 6.1 together with Remark 5.5 gives a decomposition of* $g\omega$*-continuity into* αg-$\mathcal{I}_\omega$*-continuity and* $g\omega_{\alpha^*}$-$\mathcal{I}$*-continuity.*

(3) *Theorem 6.1 together with Remark 5.6 gives a decomposition of* $g\omega$*-continuity into* gs-$\mathcal{I}_\omega$*-continuity and* $g\omega_S$-$\mathcal{I}$*-continuity.*

Corollary 6.1. [**19**] *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $\mathcal{I} = \{\emptyset\}$*. For a function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$*, the following properties are equivalent.*

(1) *f is $g\omega$-continuous.*
(2) *f is gp-ω-continuous and $g\omega_t$-continuous.*
(3) *f is αg-ω-continuous and $g\omega_{\alpha^*}$-continuous.*

Proof. Since $\mathcal{I} = \{\emptyset\}$, $A^* = cl(A)$ and $cl^*(A) = A^* \cup A = cl(A)$ for any subset A of X. Thus:

A is αg-$\mathcal{I}_\omega$-open (resp. gp-$\mathcal{I}_\omega$-open) if and only if it is αg-ω-open (resp. gp-ω-open) and

A is $g\omega_t$-$\mathcal{I}$-set (resp. $g\omega_{\alpha^*}$-$\mathcal{I}$-set) if and only if it is $g\omega_t$-set (resp. $g\omega_{\alpha^*}$-set).

The proof follows immediately from Theorem 6.1. □

References

[1] Acikgoz, A., Noiri, T. and Yuksel, S. On α-I-continuous and α-I-open functions. *Acta Math. Hungar.*, **105**(1-2)(2004), 27-37.
[2] Al-Omari, A. and Noorani, M. S. M. Regular generalized ω-closed sets. *Intern. J. Math. Math. Sci.*, Volume 2007, Article ID 16292, 11 pages, doi: 10.1155/2007/16292.
[3] Hdeib, H. Z. ω-closed mappings. R*evista Colomb. De Matem.*, **16**(1-2)(1982), 65-78.
[4] Khalid, Y. Al-Zoubi. On generalized ω-closed sets. *Intern. J. Math. Math. Sci.*, **13**(2005), 2011-2021.
[5] Kuratowski, K. *Topology, Vol. 1,* Academic Press, New York 1966.
[6] Levine, N. Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly*, **70**(1963), 36-41.
[7] Levine, N. A decomposition of continuity in topological spaces. *Amer. Math. Monthly*, **68**(1961), 44-46.
[8] Mashhour, A. S., Abd El-Monsef, M. E. and El-Deeb, S. N. On precontinuous and weak precontinuous mappings. *Proc. Math. Phys. Soc. Egypt*, **53**(1982), 47-53.
[9] Njastad, O. On some classes of nearly open sets. *Pacific J. Math.*, **15**(1965), 961-970.
[10] Noiri, T., Al-Omari, A. and Noorani, M. S. M. Weak forms of ω-open sets and decompositions of continuity, *Eur. J. Pure Appl. Math.*, **2**(1)(2009), 73-84.
[11] Noiri, T., Rajamani, M. and Inthumathi, V. On decompositions of g-continuity via idealization. *Bull. Cal. Math. Soc.*, **99**(4)(2007), 415-424.
[12] Ravi, O., Rajasekaran, I., Satheesh Kanna, S. and Paranjothi, M. New generalized classes of τ_ω. *Eur. J. Pure Appl. Math.*, **9**(2)(2016), 152-164.
[13] Ravi, O., Murugesan, S. and Paranjothi, M. $\star_\omega$-operfect sets and α-$\star_\omega$-closed sets. S*outh Asian J. Math.*, **6**(6)(2016), 300-314.
[14] Ravi, O., Sekar, P. and Vidhyalakshmi, K. New generalized classes of τ_ω in ideal spaces, *South Asian J. Math.*, **6**(3)(2016), 126-139.
[15] Ravi, O., Sekar, P. and Vidhyalakshmi, K. ω-topology and $\star$-topology. *Bull. Inter. Math. Virtual Inst.*, **7**(1)(2017), 139-151.
[16] Rose, D. On Levine's decomposition of continuity. *Canad. Math. Bull.*, **21**(1978), 477-481.
[17] Sundaram, P. and Rajamani, M. On decompositions of generalized continuous maps in topological spaces, *Acta Ciencia Indica*, **26**(2)(2000), 101-104.
[18] Tong, J. A decomposition of continuity. *Acta Math. Hungar.*, **48**(1986), 11-15.
[19] Umamaheswari, R., Nethaji, O. and Ravi, O. $g\omega$-continuity and its decompositions. *South Asian J. Math.*, **7**(1)(2017), 53-63.
[20] Vaidyanathaswamy, R. The localization theory in set topology. *Proc. Indian Acad. Sci., Sect A*, **20**(1944), 51-61.
[21] Willard, S. *General Topology*, Addison-Wesley, Reading, Mass, USA, 1970.

Received by editors 12.01.2017; Available online 10.04.2017.

K. Meena: Department of Mathematics, Arulmigu Palaniandavar Arts College for Women, Palani, Dindigul District, Tamil Nadu, India
E-mail address: meenasenthil267@gmail.com

R. Premkuma: Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai District, Tamil Nadu, India
E-mail address: prem.rpk27@gmail.com

O. Ravi: Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai District, Tamil Nadu, India
E-mail address: siingam@yahoo.com