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PRILOG ZA ČLANAK

„JEDNA ZANIMLJIVA ALGEBARSKA NEJEDNAKOST I NJENA PRIMJENA"

Addenda to the paper „An interesting algebraic inequality and its application"

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Sažetak: U radu [1] je dokazana sljedeća algebarska nejednakost oblika

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}, \tag{1}$$

gdje su a, b, c, x, y, z pozitivni realni brojevi kao i jedna njena efikasna primjena. Ovdje ćemo dati još jedan dokaz nejednakosti (1) i njene primjene na poboljšanju nekih nejednakosti za trougao.

Ključne riječi i izrazi: algebarska nejednakost, aritmetičko-geometrijska nejednakost, nejednakost trougla.

Abstract: In [1] shawn the following algebraic inequality

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}, \tag{1}$$

where a, b, c, x, y, z are positive real numbers and one effective application of this inequality. We give yet a proof of the inequality (1) and its applications on the improvements of the some triangle inequalities.

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Dajemo dokaz nejednakosti (1), bez primjene Koši – Švarcove i Jensenove nejednakosti. Koristeći poznatu nejednakosti između aritmetičke i geometrijske sredine pozitivnih brojeva, imamo:

$$a^3xyz + b^3x^2z + a^3y^2z \geq 3\sqrt[3]{a^2xyz \cdot b^3x^2z \cdot a^3y^2z} = 3a^2bxyz\ ,$$

$$b^3xyz + c^3xy^2 + b^3xz^2 \geq 3b^2cxyz,$$

$$c^3xyz + c^3x^2y + a^3yz^2 \geq 3c^2axyz\ ,$$

$$a^3y^2z + b^3x^2z + b^3xyz \geq 3ab^2xyz\ ,$$

$$a^3xyz + a^3yz^2 + c^3x^2y \geq 3a^2cxyz\ ,$$

$$b^3xz^2 + c^3xy^2 + c^3xyz \geq 3bc^2xyz$$

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$$b^3x^2z + a^3y^2z + c^3xy^2 + +c^3x^2y + a^3yz^2 + b^3xz^2 \geq 6abcxyz\ .$$

Ako saberemo svih sedam prethodnih nejednakosti dobijamo

$$2(a^3 + b^3 + c^3)xyz + 3(a^3y^2z + a^3yz^2 + b^3x^2z + b^3xz^2 + c^3xy^2 + c^3x^2y) \geq$$

$$3xyz(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2)$$

$$\Leftrightarrow\ 3(a^3 + b^3 + c^3)xyz + 3((a^3yz + b^3zx + c^3xy)(x + y + z) - xyz(a^3 + b^3 + c^3))$$

$$\geq\ 3xyz\big((a + b + c)(ab + bc + ca) - 3abc\big) + (a^3 + b^3 + c^3)xyz$$

$$\Leftrightarrow 3(a^3yz + b^3zx + c^3xy)(x + y + z) \geq xyz(a^3 + b^3 + c^3)$$

$$\Leftrightarrow\ \frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}\ \geq\ \frac{(a+b+c)^3}{3(x+y+z)}\ ,$$

tj. (1) vrijedi.

Sada primijenimo dokazanu nejednakost (1) na poboljšanje nekih nejednakosti trougla. Koristićemo standardne oznake za ΔABC: a, b, c – stranice, A, B, C – unutrašnji uglovi, s – poluobim,

m_a, m_b, m_c –težišnice, R i r – poluprečnici opisane i upisane kružnice trougla.

Primjer 1. Ako u nejednakost (1) stavimo $x = am_a^2, y = bm_b^2$ i $z = cm_c^2$, imamo:

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z}\ \geq\ \frac{(a+b+c)^3}{3(am_a^2 + bm_b^2 + cm_c^2)}\ ,$$

tj.

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\geq\frac{16s^2}{3(s^2+2Rr+5r^2)}$$

$$=\frac{16}{3}\left(1-\frac{2Rr+5r^2}{s^2+2Rr+5r^2}\right), \tag{2}$$

zbog $a+b+c=2s$ i $am_a^2+bm_b^2+cm_c^2=\frac{s}{2}(s^2+2Rr+5r^2)$, v. [3], str. 211. S obzirom da je ([2], str. 50)

$$s^2\geq r(16R-5r),$$

iz (2) slijedi

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\geq\frac{16}{3}\left(1-\frac{2R+5r}{18R}\right)$$

$$=\frac{16}{27}\left(8-\frac{5r}{2R}\right). \tag{3}$$

Napomena 1. Nejednakost (3) je bolja („jača“) od nejednakosti ([3], str. 229)

$$\frac{a^2}{m_a^2}+\frac{b^2}{m_b^2}+\frac{c^2}{m_c^2}\geq 4,$$

jer je

$$\frac{16}{27}\left(8-\frac{5r}{2R}\right)\geq 4 \Leftrightarrow R\geq 2r \text{ (Euler's inequality).}$$

Primjer 2. Na osnovu nejednakosti (1) je

$$\frac{a^3}{a\cos\frac{A}{2}}+\frac{b^3}{b\cos\frac{B}{2}}+\frac{c^3}{c\cos\frac{C}{2}}\geq\frac{(a+b+c)^3}{3(a\cos\frac{A}{2}+b\cos\frac{B}{2}+c\cos\frac{C}{2})},$$

tj.

$$a^2\sec\frac{A}{2}+b^2\sec\frac{B}{2}+c^2\sec\frac{C}{2}\geq\frac{8s^3}{3\cdot s\sqrt{3}}$$

$$=\frac{8s^2\sqrt{3}}{9}, \tag{4}$$

zbog ([3], str. 169)

$$a\cos\frac{A}{2}+b\cos\frac{B}{2}+c\cos\frac{C}{2}\leq s\sqrt{3}.$$

Napomena 2. Nejednakost (4) predstavlja poboljšanje nejednakosti iz [3], str. 125 :

$$a^2 sec\frac{A}{2} + b^2 sec\frac{B}{2} + c^2 sec\frac{C}{2} \geq 12Rr\sqrt{3}\,,$$

jer je

$$\frac{8s^3\sqrt{3}}{9} \geq 12Rr\sqrt{3} \Leftrightarrow 2s^2 \geq 27Rr \text{ ([2], str. 52).}$$

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