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ZNAČAJ METODE POMOĆNE NEJEDNAKOSTI ZA DOKAZIVANJE RAZNIH NEJEDNAKOSTI

(Importance of method helps inequality for the proving of diverse inequalities)

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Sažetak: U ovom radu je dat dokaz jedne zanimljive algebarske nejednakosti od tri promjenljive. Za njen dokaz se koristi jedna pomoćna nejednakost koju smo konstruisali uz detaljno objašnjene kako smo to uradili. Poslije dokaza date nejednakosti, data je i njena generalizacija kao i njen dokaz.
Riječ je o sljedećoj nejednakosti

$$\frac{x^3}{x^2+y+z}+\frac{y^3}{y^2+z+x}+\frac{z^3}{z^2+x+y}\ge\frac{1}{7},$$

gdje su x,y,z *realni pozitivni brojevi čija je suma jednaka* 1, *tj.* $x+y+z=1$.

Ključne riječi i izrazi: pomoćna nejednakost, generalizacija, simetrična neje-dnakost, nejednakost Čebiševa.

Abstract: In this paper is gived the proof of one interesting algebraic inequality of three variables. For hers proof we use one help inequality. After this proof is gived the generalization of this inequality and hers proof.
This is the inequality

$$\frac{x^3}{x^2+y+z}+\frac{y^3}{y^2+z+x}+\frac{z^3}{z^2+x+y}\ge\frac{1}{7},$$

where x,y,z are positive real numbers and $x+y+z=1$.

Key words and phrases: help inequality, generalization, symmetric inequality, Chebishev's inequality.

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Riječ je o sljedećoj nejednakosti

$$\frac{x^3}{x^2+y+z}+\frac{y^3}{y^2+z+x}+\frac{z^3}{z^2+x+y}\ge\frac{1}{7}, \tag{1}$$

gdje su x,y,z realni pozitivni brojevi čija je suma jednaka 1, tj. $x+y+z=1$.

Dokaz: Koristićemo pomoćnu nejednakost koja glasi:

$$\frac{x^3}{x^2+1-x} \geq \frac{22}{49}x - \frac{5}{49};\ (x>0). \tag{2}$$

Pokažimo najprije da je ova nejednakost tačna. Ona je nakon množenja sa $x^2+1-x=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0$ ekvivalenta nejednakosti:

$$\left(1-\frac{22}{49}\right)x^3+\left(\frac{22}{49}+\frac{5}{49}\right)x^2-\left(\frac{22}{49}+\frac{5}{49}\right)x+\frac{5}{49}\geq 0$$

$$\Leftrightarrow \frac{27}{49}x^3+\frac{27}{49}x^2-\frac{27}{49}x+\frac{5}{49}\geq 0 \,/\, \cdot 49$$

$$\Leftrightarrow 27x^3+27x^2-27x+5\geq 0;\ (3x=t)$$

$$\Leftrightarrow t^3+3t^2-9t+5\geq 0$$

$$\Leftrightarrow t^2(t-1)+4t(t-1)-5(t-1)\geq 0$$

$$\Leftrightarrow (t-1)\left(t^2+4t-5\right)\geq 0$$

$$\Leftrightarrow (t-1)^2(t+5)\geq 0,$$

a ova nejednakost je tačna jer je zbog $x>0$ također $t>0$. Važi jednakost ako i samo ako je $t=1$, tj. $x=\frac{1}{3}$.

Vodeći računa da je $x+y+z=1$, tj. $1-x=y+z$, dobijamo iz nejednakosti (2):

$$\frac{x^3}{x^2+y+z}\geq \frac{22}{49}x-\frac{5}{49}, \tag{3}$$

kao i analogne nejednakosti:

$$\frac{y^3}{y^2+z+x}\geq \frac{22}{49}y-\frac{5}{49} \tag{4}$$

i

$$\frac{z^3}{z^2+x+y} \geq \frac{22}{49}z - \frac{5}{49}\,. \tag{5}$$

Sabirajući sada nejednakosti (3), (4) i (5), dobijamo:

$$\frac{x^3}{x^2+y+z}+\frac{y^3}{y^2+z+x}+\frac{z^3}{z^2+x+y} \geq \frac{22}{49}(x+y+z)-\frac{15}{49}\,,$$

a odavde zbog $x+y+z=1$:

$$\frac{x^3}{x^2+y+z}+\frac{y^3}{y^2+z+x}+\frac{z^3}{z^2+x+y} \geq \frac{7}{49}=\frac{1}{7}\,,$$

a ovo je nejednakost (1).

Vrijedi jednakost u (1) ako i samo ako je $x=y=z=\frac{1}{3}$.

Sada ćemo objasniti kako smo došli do nejednakosti (2). Neka je

$$\frac{x^3}{x^2+1-x} \geq \alpha x+\beta \quad (x>0;\ \alpha,\ \beta\in\boldsymbol{R}).$$

Nakon množenja ove nejednakosti sa $x^2+1-x=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0$ dobijamo ekvivalentnu nejednakost:

$$P(x)=(1-\alpha)x^3+(\alpha-\beta)x^2+(\beta-\alpha)x-\beta \geq 0\,.$$

Imamo sada

$$P\left(\frac{1}{3}\right)=\frac{1-\alpha}{27}+\frac{\alpha-\beta}{9}+\frac{\beta-\alpha}{3}-\beta\,.$$

Odredićemo sada a i β tako da $x=\frac{1}{3}$ bude nula polinoma $P(x)$, tj. $P\left(\frac{1}{3}\right)=0$; imamo:

$$\frac{1-\alpha}{27}+\frac{\alpha-\beta}{9}+\frac{\beta-\alpha}{3}-\beta=0\,/\cdot 27$$

$$\Leftrightarrow 1-\alpha+3(\alpha-\beta)+9(\beta-\alpha)-27\beta=0$$

$$\Leftrightarrow 7\alpha+21\beta=1. \tag{6}$$

Uzećemo da je $x=\frac{1}{3}$ dvostruka nula polinoma $P(x)$, tj. mora biti $P'\left(\frac{1}{3}\right)=0$; slijedi:

$$P'(x)=3x^2(1-\alpha)+2(\alpha-\beta)x+\beta-\alpha=0 \text{, te}$$

$$P'\left(\frac{1}{3}\right)=\frac{1}{3}(1-\alpha)+\frac{2}{3}(\alpha-\beta)+\beta-\alpha=0$$

$$\Leftrightarrow 1-\alpha+2\alpha-2\beta+3\beta-3\alpha=0$$

$$\Leftrightarrow 2\alpha-\beta=1. \tag{7}$$

Rješavanjem sistema jednačina (6) i (7) dobijamo:

$$\alpha=\frac{22}{49} \quad \text{i} \quad \beta=-\frac{5}{49}.$$

Lijepa ideja, nema šta!

Sada ćemo dokazati generalisanu nejednakost (1) koja glasi:

$$\frac{x^{k+2}}{x^{k+1}+y^k+z^k}+\frac{y^{k+2}}{y^{k+1}+z^k+x^k}+\frac{z^{k+2}}{z^{k+1}+x^k+y^k}\ge\frac{1}{7}, \tag{8}$$

gdje $x,y,z>0$ i $k\in\boldsymbol{N}$.

Dokaz: Pošto je data nejednakost simetrična, to ne narušavajući opštost možemo pretpostaviti da je $x\ge y\ge z>0$. Nakon ovoga lako provjerimo da važe nejednakosti:

$$\frac{x^{k+1}}{x^{k+1}+y^k+z^k}\ge\frac{y^{k+1}}{y^{k+1}+z^k+x^k}\ge\frac{z^{k+1}}{z^{k+1}+x^k+y^k};$$

$$z^{k+1}+y^k+x^k\ge y^{k+1}+x^k+z^k\ge x^{k+1}+z^k+y^k$$

te

$$x^k\ge y^k\ge z^k.$$

Na osnovu nejednakosti Čebiševa slijedi:

$$\frac{x^{k+2}}{x^{k+1}+y^k+z^k}+\frac{y^{k+2}}{y^{k+1}+z^k+x^k}+\frac{z^{k+2}}{z^{k+1}+x^k+y^k}$$

$$\geq \frac{x+y+z}{3}\left(\frac{x^{k+1}}{x^{k+1}+y^k+z^k}+\frac{y^{k+1}}{y^{k+1}+z^k+x^k}+\frac{z^{k+1}}{z^{k+1}+x^k+y^k}\right)$$

$$=\frac{1}{3}\left(\frac{x^{k+1}}{x^{k+1}+y^k+z^k}+\frac{y^{k+1}}{y^{k+1}+z^k+x^k}+\frac{z^{k+1}}{z^{k+1}+x^k+y^k}\right)\cdot\frac{\sum\limits_{cyclic}\left(x^{k+1}+y^k+z^k\right)}{\sum\limits_{cyclic}\left(x^{k+1}+y^k+z^k\right)}$$

$$=\frac{1}{3}\left(\sum_{cyclic}\left(\frac{x^{k+1}}{x^{k+1}+y^k+z^k}\sum_{cyclic}\left(x^{k+1}+y^k+z^k\right)\cdot\frac{1}{\sum\limits_{cyclic}\left(x^{k+1}+y^k+z^k\right)}\right)\right)$$

$$\geq \frac{1}{3}\left(x^{k+1}+y^{k+1}+z^{k+1}\right)\cdot\frac{1}{\sum\limits_{cyclic}\left(x^{k+1}+y^k+z^k\right)}$$

$$=\frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+2\left(x^k+y^k+z^k\right)}. \tag{9}$$

Na osnovu nejednakosti Čebiševa također imamo:

$$3\left(x^{k+1}+y^{k+1}+z^{k+1}\right)\geq 3\cdot\frac{x+y+z}{3}\left(x^k+y^k+z^k\right),\text{ tj.}$$

$$x^k+y^k+z^k\leq 3\left(x^{k+1}+y^{k+1}+z^{k+1}\right). \tag{10}$$

Sada iz (9) i (10) slijedi:

$$\frac{x^{k+2}}{x^{k+1}+y^k+z^k}+\frac{y^{k+2}}{y^{k+1}+z^k+x^k}+\frac{z^{k+2}}{z^{k+1}+x^k+y^k}$$

$$\geq \frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+2\left(x^k+y^k+z^k\right)}$$

$$\geq \frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+6\left(x^{k+1}+y^{k+1}+z^{k+1}\right)}=\frac{1}{7}.$$

Vrijedi jednakost u (8) ako i samo ako je $x=y=z=\frac{1}{3}$.

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