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FINITELY DUAL QUASI-NORMAL RELATION

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ABSTRACT. In this paper, following Jiang Guanghao and Xu Luoshan's concepts of finitely conjugative and finitely dual normal relations on sets, the concept of finitely dual quasi-normal relations is introduced. A characterization of that relations is obtained.

1. Introduction and Preliminaries

In this article, following concepts of finitely conjugative relations ([**1**], Jiang Guanghao and Xu Luoshan), finitely dual normal relations ([**2**], Jiang Guanghao and Xu Luoshan) and finitely quasi-conjugative relations ([**5**], D.A.Romano and M.Vinčić) introduced in their articles, we introduce and analyze notion of finitely dual quasi-normal relations on sets

For a set X, we call ρ a binary relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ be denote the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. The latter family, with the composition, is not only a semigroup, but also a monoiod. Namely, $Id_X = \{(x, x) : x \in X\}$ is its identity element. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \ \ \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds and $(\alpha^C)^{-1} = (\alpha^{-1})^C$. Specially, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:
– *dually normal* ([**2**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

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$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$

– *conjugative* ([**1**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$

– *dually conjugative* ([**1**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$

– *quasi-regular* ([**4**]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = \alpha^C \circ \beta \circ \alpha.$$

Put $\alpha^1 = \alpha$. It is easy to see that $(\alpha^{-1})^C = (\alpha^C)^{-1}$ holds. Previous description gives equality

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, C\}$. We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated. (See, for example, our article [**4**], [**6**], [**7**] and [**8**].)

Notions and notations which are not explicitly exposed but are used in this article, reader can find them from book [**3**] and articles [**1**], [**2**] and [**4**], for an example.

2. Finitely dual quasi-normal relations

In this section we introduce the concept of finitely dual quasi-normal relations as a finite extension of dually quasi-normal relation, introduced in the forthcoming article [**6**], and give a characterization of that relations. For that we need the concept of finite extension of a relation. That notion and belonging notation we borrow from articles [**1**] and [**2**]. For any set X, let

$$X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty }\}.$$

Definition 2.1 ([**1**], Definition 3.3; [**2**], Definition 3.4). *Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the* finite extension *of α, by*

$$(\forall F, G \in X^{(<\omega)})((F, G) \in \alpha^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha).$$

From this definition, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^C)^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^C),$$

$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^{-1})^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^{-1} = \alpha F)$$

and

$$(\forall F, G \in X^{(<\omega)})((F, G) \in ((\alpha^{-1})^C)^{(<\omega)} \Longleftrightarrow G \subseteq F(\alpha^C)^{-1} = \alpha^C F)$$

Notion of dually quasi-normal relation we borrow from paper [**8**].

Definition 2.2 ([**8**], Definition 2.1 (b)). *For relation $\alpha \in \mathcal{B}(X)$ we say that it is a* dually quasi-normal *relation on X if exists a relation $\beta \in \mathcal{B}(X)$ such that*

$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha^C.$$

The family of dually quasi-normal relations on not empty. Let $\alpha \in \mathcal{B}(X)$ be a relation such that $(\alpha^C)^{-1} \circ \alpha^C = Id_X$. We have

$$\alpha = Id_X \circ \alpha \circ Id_X = ((\alpha^C)^{-1} \circ \alpha^C) \circ \alpha \circ ((\alpha^C)^{-1} \circ \alpha^C) =$$

$$(\alpha^C)^{-1} \circ (\alpha^C \circ \alpha \circ (\alpha^C)^{-1}) \circ \alpha^C = (\alpha^C)^{-1} \circ \beta \circ \alpha^C.$$

Therefore, α is a dually quasi-normal relation.

Now, we can introduce concept of *finitely dual quasi-normal relation.*

Definition 2.3. *A relation α on a set X is called* finitely dual quasi-normal *if there exists a relation $\beta^{(<\omega)}$ on $X_{(<\omega)}$ such that*

$$\alpha^{(<\omega)} = ((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}.$$

Although it is seems, in accordance with Definition 2.2, it would be better to call a relation α on X to be finitely dual quasi-normal it its a finite extension to $X^{(<\omega)}$ is a dually quasi-normal relation, we will not use that option. That concept is different from our concept given by Definition 2.3.

Now we give an essential characterization of finitely dual quasi-normal relations.

Theorem 2.1. *A relation α on a set X if a finitely dual quasi-normal relation if and only if for all $F, G \in X^{(<\omega)}$, if $G \subseteq F\alpha$, then there are $U, V \in X^{(<\omega)}$, such that*
(i) $U \subseteq F\alpha^C$, $G \subseteq \alpha^C V$, *and*
(ii) *for all $S, T \in X^{(<\omega)}$, if $U \subseteq S\alpha^C$ and $T \subseteq \alpha^C V$ then $T \subseteq S\alpha$.*

Proof. (1) Let α be a finitely dual quasi-normal relation on set X. Then there is a relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that $((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$. For all $(F, G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F, G) \in \alpha^{(<\omega)}$, thus $(F, G) \in ((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$. Whence there is $(U, V) \in (X^{(<\omega)})^2$ such that $(F, U) \in (\alpha^C)^{(<\omega)}$, $(U, V) \in \beta^{(<\omega)}$ and $(V, G) \in ((\alpha^C)^{-1})^{(<\omega)}$, i.e., $U \subseteq F\alpha^C$, $G \subseteq V(\alpha^C)^{-1} = \alpha^C V$. Hence we get the condition (i).

Now we check the condition (ii). For all $(S, T) \in (X^{(<\omega)})^2$, if $U \subseteq S\alpha^C$ and $T \subseteq \alpha^C V$, i.e., $(S, U) \in (\alpha^C)^{(<\omega)}$ and $(V, T) \in ((\alpha^C)^{-1})^{(<\omega)}$, then by $(U, V) \in \beta^{(<\omega)}$, we have $(S, T) \in ((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, i.e., $(S, T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$.

(2) Let α be a relation on a set X such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(F, G) \in \beta \Longleftrightarrow (\forall S, T \in X^{(<\omega)})((F \subseteq S\alpha^C \wedge T \cap \alpha^C G \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that (a) $((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H, W) \in ((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, then there are $F, G \in X^{(<\omega)}$ with $(H, F) \in (\alpha^C)^{(<\omega)}$, $(F, G) \in \beta^{(<\omega)}$ and $(G, W) \in ((\alpha)^C)^{-1})^{(<\omega)}$. Then $F \subseteq H\alpha^C$ and $W \subseteq G(\alpha^C)^{-1} = \alpha^C G$. For all $w \in W$, let $S = H$, $T = \{w\}$. Then $F \subseteq S\alpha^C$ and $\alpha^C G \cap T \neq \emptyset$ because $w \in T$ and $w \in \alpha^C G$. Since

$(F,G) \in \beta^{(<\omega)}$, we have that $F \subseteq S\alpha^C \wedge \alpha^C G \cap T \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H,W) = (S,W) \in \alpha^{(<\omega)}$. Therefore, we have $((\alpha^C)^{-1})^{(<\omega)} \circ \beta \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$.

The second, check that (b) $\alpha^{(<\omega)} \subseteq ((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), there are $A, B \in X^{(<\omega)}$ such that:

(i') $A \subseteq H\alpha^C$, $W \subseteq \alpha^C B$, and

(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq S\alpha^C$ and $T \subseteq \alpha^C B$, then $T \subseteq S\alpha$.

Now, we have to show that $(A,B) \in \beta^{(<\omega)}$. Let be for all $(C,D) \in (X^{(<\omega)})^2$ the following $A \subseteq D\alpha^C$ and $D \cap \alpha^C B \neq \emptyset$ hold. From $D \cap \alpha^C B \neq \emptyset$ follows that there exists an element $d \in D \cap \alpha^C B (\neq \emptyset)$. So, $d \in D$ and $d \in \alpha^C B$. Put $S = C$ and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq S\alpha^C \wedge T = \{d\} \subseteq \alpha^C B) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

i.e. $\emptyset \neq \{d\} \cap S\alpha = T \cap S\alpha$. Therefore, $(A,B) \in \beta^{(<\omega)}$ by definition of $\beta^{(<\omega)}$. Finally, for $(H,A) \in (\alpha)^{(<\omega)}$, $(A,B) \in \beta^{(<\omega)}$ and $(B,W) \in ((\alpha^C)^{-1})^{(<\omega)}$ follows that $(H,W) \in ((\alpha^C)^{-1})^{(<\omega)} \circ \beta \circ (\alpha^C)^{(<\omega)}$.

By assertion (a) and (b), finally we have $\alpha^{(<\omega)} = ((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$
$\square$

Particulary, if we put $F = \{x\}$ and $G = \{y\}$ in the previous theorem, we conclude the following corollary.

Corollary 2.1. *Let α be a relation on a set X. Then α is a finitely dual quasi-normal on X if and only if for all elements $x, y \in X$ such that $(x,y) \in \alpha$ there are finite subsets $U, V \in X^{(<\omega)}$ such that*

(1^0) $(\forall u \in U)((x,u) \in \alpha^C) \wedge (\exists v \in V)((y,v) \in \alpha^C)$, *and*

(2^0) *for all $S \in X^{(<\omega)}$ and $t \in X$ holds*

$$(U \subseteq S\alpha^C \wedge (\exists v \in V)((t,v) \in \alpha^C)) \Longrightarrow (\exists s \in S)((s,t) \in \alpha)\ .$$

Proof. Let α be a finitely dual quasi-normal relation on X and let x, y be elements of X such that $(x,y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem 3.1 then there exist finite U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold.

Opposite, let for all elements $x, y \in X$ such that $(x,y) \in \alpha$ be there are U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold. Define binary relation $\beta^{<\omega} \subseteq X^{<\omega} \times X^{<\omega}$ by

$$(A,B) \in \beta^{<\omega} \Longleftrightarrow (\forall S \in X^{<\omega})(\forall t \in X)((A \subseteq S\alpha^C \wedge t \in \alpha^C B) \Longrightarrow t \in S\alpha).$$

The proof that the equality $((\alpha^C)^{-1})^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$ holds is some as in the Theorem 3.1. So, the relation α is a finitely dual quasi-normal. $\square$

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