IMVI
Open Mathematical Education Notes
Vol. 6(2016), 57-64
www.imvibl.org / JOURNALS / IMVI OMEN

ISSN (p) 2303-4882, ISSN (o) 1840-4383

College Students' Understanding of Logical Implication

Daniel A. Romano

East Sarajevo University, Faculty of Education,
b.b., Semberski ratari Street, 76300 Bijeljina, Bosnia and Herzegovina
e-mail: bato49@hotmail.com

Susanne Strachota

University of Wisconsin-Madison
1025 West Johnson Street, Office 690, Madison, WI 53706
e-mail: sstrachota@wisc.edu

Abstract. In this paper we present the results of student testing, from two technical colleges at Banja Luka University, on understanding and applying the logical implication and its contraposive in a real-world situation. The findings support our belief that the tested population has insufficient knowledge of these logical concepts. Furthermore, we argue that these results could reveal the critical link between conceptual and procedural knowledge of these logical tools.

Key words and phrases: logic, implication, contraposition

Mathematics Subject Classification (2010): **97B40, 97C60, 97E30**
Math. Didactic Subject Classification (2010): **B40, C60, E30**

Introduction

Many students from technical colleges who are enrolled in calculus, listen to lectures on mathematical logic. According to the course syllabus, 9 or more hours are devoted to mathematical logic. Through the topic 'Introduction to Mathematical Logic', students become familiar with:

- The alphabet of mathematical logic
- Axioms of classical logic
- Rules of inference in logic

Additionally, students gain insight on several formulas of special interest to this area of study.

In this research, we were interested in whether students' conditional sentences of colloquial language can be transformed into logical formulas. Do students recognize the sufficient and necessary conditions in these sentences? This question lies at the core of our research, which seeks to understand the process in which students reconstruct previously learned concepts and use that knowledge to make sense of mathematical proofs.

Through participating in the lectures, students negotiate and demonstrate their conceptual understanding and procedural skills. Due to insufficient knowledge, sometimes students are left to reconstruct mathematical ideas, relying on what they gained in the lectures and their previous understanding.

This process of reconstructing is a small part of learning Calculus. In this process, the elements of 'propositional algebra' and 'predicate algebra' are presented to students as a basis for the aforementioned reconstruction. How and how much those lessons students learned according to their reconstruction of evidence, is an important question. In order to gain insight into which proficiencies were necessary for students' reconstruction and understanding of mathematical proofs, we investigated students' understanding of the formulas in which the implications appear. In this paper, we reveal students' abilities to construct and make sense of logical implication and its contrapositive through tasks aimed at assessing whether students can construct formulas from sentences that represent real-world situations.

The Concept of Logical Implication

Implication as a logical conjunction and as a logical operation is a critical concept in logic. All mathematical propositions are expressed as follows: "If H (hypothesis), then T (conclusion)." This statement corresponds to a formal understanding of implications as: $H \Rightarrow T$, which can be read in one of the following ways:

- *H is sufficient conditions for T;*
- *T is a necessary condition for H;*
- *H implies T;*
- *If H, then T is;*
- *The H follows T;*
- *From the assumptions H one can deduce conclusion T.*

For an elaboration of the concepts of logical implication, see [8, 9, 15].

Motivation

Many researchers in mathematics education (e.g., Milbou, Deprez & Laenens [12]) argue that teaching logic supports students in developing all mathematical skills. Yet, a significant number of students have trouble understanding aspects of logic (*ibid.*). To this end, Lin, Li and Wu [11] show that 80% of students (n = 202), ranging from 17 to 20 years old, lack knowledge of quantifiers as a means to express denial and that more than 70% of these students lack conceptual understanding of proof by contradiction. Despite that, the authors express their belief that with proper support these students will understand the aforementioned concept and apply it in real situations.

Based on our experience, we argue that first year students of technical colleges struggle to distinguish between the definitions of mathematical concepts and processes, which involve those concepts, such as statements or theorems. From our observations, the latter is always expressed formal implication (e.g., $H \Rightarrow T$).

Moreover, logical implication plays a fundamental role in real-world situations. Identifying the causal connections of conditional statements in colloquial language depends on one's understanding of the elements of implication and its contraposition. In fact, this skill lies at the core of learning. Namely, in order to answer a question, students must deduce conclusions from hypotheses by interpreting ideas through written text or verbal communication.

Theoretical Foundation

Prior studies in the field have indicated a need for continued research on students' understanding of mathematical structure and the links between mathematical concepts. Many authors (e.g., [8]) have pointed out that research on how students understand the structure of logical relations, especially the logical implications, is lacking.

Although the field has not agreed upon one way to teach calculus, calculus curricula, the Mathematical Encyclopedic courses at technical colleges in particular, often include elements of formal logic. Epp (see [6], for further detail) states that, "I believe in presenting logic in a manner that continually links it to language and to both real-world and mathematical subject matter" (p. 895). In spite of Epp's remarks, some mathematics educators argue that formal reasoning and everyday language and reasoning are distinct mental activities (e.g., [1, 2]).

Furthermore, some mathematics educators argue that proving and reasoning logically are separate mental activities. Thus, instruction that explicitly introduces logic does not necessarily support students in proof construction. For example, Hanna and de Villiers [7] state: "It remains unclear what good comes from teaching students the elements of formal logic, because many mathematicians are willing to admit they rarely use formal logic in their research" (p. 311). Selden and Selden [17] also support this assertion when they state that, "logic does not occur within the proofs as often as one might expect..." (p. 347). Conversely, Rips [14] describes proof as a more sophisticated, logically structured process: "At the most general level, the formal proof is a finite sequence of sentences in which every sentence is either a premise, an axiom of the logical system, or a sentence that follows from the preceding sentences by one of the system's rules ([14], p. 34).

These varying opinions suggest that further examination of the role of logic in students' understanding of mathematical proof would be beneficial. Moreover, a review of the literature reveals that few studies address this need. To the best of our knowledge, Baker ([3]; a master's thesis) and Chin and Tall [4] are some of the rare, if not only, examples of this work. Additionally, Hoyles and Küchemann [8, 9], Deloustal-Jorrand [5] and dissertations by Lee [10] and Savic [16] investigate related questions.

Participants and Background

This research was conducted in two parts: an interview and an assessment. We interviewed 72 first year students, divided into four groups, at the Faculty of Technology, University of Banja Luka. We assessed 40 first-year students, divided into five groups, of Mechanical Engineering Faculty at the same university. Participants were organized into groups to minimize communication between students about the assessment questions during the data collection process.

Research Goals

Although we did assess the mathematical skills of participants, our primary goal is broader. Specifically, we hypothesized that a gap between formal statements indicating a student's mathematical understanding and the real-world situation. Therefore, our goal is to understand in what ways applied didactics lead to this gap.

We acknowledge that the participants may or may not be familiar with the language of formal logic. To this end, we were focused on whether the candidates have a natural inclination to understand and use logical implication in a real-world situation.

Research Methodology

On the admittance test for Calculus 1, students solve, among others, the following two tasks:

Problem 1. Given statements:
(A) The math test is evaluated.
(B) The student has passed the math test.
(C) The task of this test was solved in an acceptable manner.

Write the meanings of the following formulas:
(MF, Group 1) 1.1. $\neg(C \wedge A) \Rightarrow B$, 1.2. $\neg B \Rightarrow \neg C \vee \neg A$.
(MF, Group 2) 2.1. $\neg C \vee A \Rightarrow B$, 2.2. $\neg B \Rightarrow C \wedge \neg A$.
(MF, Group 3) 3.1. $C \Rightarrow A$, 3.2. $A \Rightarrow B$, 3.3. $(C \Rightarrow A) \Rightarrow B$, 3.4. $\neg B \Rightarrow \neg C$.
(MF, Group 4) 4.1. $C \Rightarrow B$, 4.2. $\neg C \wedge A \Rightarrow \neg B$, 4.3. $\neg B \Rightarrow \neg C \vee \neg A$.
(MF, Group 5) 5.1. $C \Rightarrow B$, 5.2. $\neg C \vee A \Rightarrow \neg B$, 5.3. $\neg B \Rightarrow \neg C \wedge \neg A$.
(TF, Group 1) 6.1. $C \wedge A \Rightarrow B$, 6.2. $\neg B \Rightarrow \neg C \vee \neg A$.
(TF, Group 2) 7.1. $C \vee A \Rightarrow B$, 7.2. $\neg B \Rightarrow \neg C \wedge \neg A$.
(TF, Group 3) 8.1. $C \Rightarrow A$, 8.2. $A \Rightarrow B$, 8.3. $(C \Rightarrow A) \Rightarrow B$, 8.4. $\neg B \Rightarrow \neg C$.

(TF, Group 4) 9.1. $C \Rightarrow B$, 9.2. $\neg C \wedge A \Rightarrow \neg B$, 9.3. $\neg B \Rightarrow \neg C \vee \neg A$.

Figure 1. Sample task from admittance test

As previously noted, the participants came from two separate programs within the same university (Faculty of Technology and Mechanical Engineering Faculty). Within those programs, participants were further divided into groups to minimize bias. In figure 1, “MF,” “TF,” and a group number indicate which subgroup of students answered that question. “MF” and TF” indicate whether the student was enrolled in the Mechanical Engineering Faculty or the Faculty of Technology, respectively.

The objective of this task was to assess students’ ability to interpret formal logic statements if the elements in these statements are given in the declarative form. We were interested in how participants would use the given implications to form conditional sentences.

We argue that this information is valuable because we can infer whether students understand and are able to interpret logical implications, such as:

$$\boldsymbol{H} \text{ (hypothesis)} \Rightarrow \boldsymbol{T} \text{ (conclusion)}.$$

That is, this knowledge allows us to infer whether or not the participants understand the concept of *hypothesis* and the concept of *conclusion*. Furthermore, we gain insight as to whether or not participants understand and are able to correctly interpret an atomic statement of implications in which the elements are negated, thereby allowing us to establish whether or not the participants understand the concept of negation. Lastly, we are able to assess students’ understanding of a contrapositive through our analysis of statements of implication in the form of $\neg T \Rightarrow \neg H$.

Problem 2. Given statements:

1. If I learned Calculus, I will pass the admittance test in mathematics.
2. If I do not pass this test in mathematics, then I'm not well prepared in Calculus.
3. I did not pass this admittance test in mathematics because I'm not well prepared in Calculus.

Mark the elements of each statement in order using variables (use capital letters of the Latin alphabet A, B, ...).*Then, translate the sentences in formulas.*

Figure 2. Sample task from admittance test

The objective of this task was to assess whether students were proficient at the following skills:
(a) Understanding the concept of atomic and non-atomic logical formula;
(b) Recognition of elements;
(c) The use of logical connectives or operators;
(d) Using elements as a means for reconstruction of non-atomic statements; and
(e) Identify the hypothesis and conclusion in the real-world examples of implications.
We believe assessing these skills allows us to determine students’ *understanding of concepts, strategic competence* and *adaptive inference.*

The tasks were evaluated using 'chunk-by-chunk' analysis. For an elaboration on 'chunk-by-chunk analysis’ see [13] or [13]. In the analysis, expected responses to the tasks were divided into independent parts that no longer carry meaning. For instance, we used the following symbols and associated meanings as our coding scheme.

1. Code $\varnothing$ (empty set) means the participant did not offer any information as a reply to the question (no response).

2. Code 0 (zero) means that the information is completely unacceptable as a valid answer.
3. Code $\frac{k}{n}$ means that the participant offered *k* of the possible *n* parts in an acceptable manner.

Results and Analysis

An analysis of problem 1 indicates that most students were successful on the simple implication and contrapositive of simple implication tasks (see tables 1 and 2), whereas, fewer students were successful on the non-linear complex implication tasks, which involved either a non-atomic hypothesis or non-atomic conclusion (see tables 4 and 5). Despite students' success on these tasks, no students were successful on the complex implication tasks (see table 3).

Results of Problem 1

Questions **P1.1**: 3.1, 3.2, 4.1, 8.1, 8.2 and 9.1. (Simple implication)

Efficacy	1	0	∅	Σ
Number	66	3	2	71
Frequency (%)	92.95	4.23	2.82	100

Table 1: Distribution of success (N = 71)

Questions **P1.2**: 3.4 and 8.4. (Contraposition of the simple implications)

Efficacy	1	0	∅	Σ
Number	20	2	1	23
Frequency (%)	86.96	8.70	4.35	100

Table 2: Distribution of success (N = 23)

Questions **P1.3**: 3.3 and 8.3. (Complex implication)

Efficacy	1	0	∅	Σ
Number	0	22	1	23
Frequency (%)	0.00	95.65	4.35	100

Table 3: Distribution of success (N = 23)

Question **P1.4**: 1.1, 2.1, 4.2, 5.2, 7.1 and 9.2. (Nonlinear complex implication – non-atomic hypothesis)

Efficacy	1	0	∅	Σ
Number	33	32	4	69
Frequency (%)	47.83	46.37	5.8	100

Table 4: Distribution of success (N = 69).

Question **P1.5**: 1.2, 2.2, 4.3, 5.3, 6.2, 7.2 and 9.3. (Nonlinear complex implication – non-atomic conclusion)

Efficacy	1	0	∅	Σ
Number	31	53	5	89
Frequency (%)	34.83	59.55	5.62	100

Table 5: Distribution of success (N = 89)

An analysis of problem 2 indicates that most students were unsuccessful at transforming colloquial statements about real-world situations into formal logical statements. In particular, students struggled to identify and interpret "because" in terms of formal logic. Specifically, few students were able to represent *'I did not pass the admittance test in mathematics'* as $\neg B$, and then relate this statement to *'I'm not well prepared in Calculus,'* which should be represented as $\neg A$, and thereby construct the complete statement is $\neg B \Rightarrow \neg A$.

Results of Problem 2

Label the statement element '*I learned Calculus 1'* with letter *A* and the statement element "*I passed the admittance test of mathematics*" with letter *B*.

Question **P2.1**: Identifying the formula $A \Rightarrow B$.

Efficacy	1	0	∅	Σ
Number	43	58	11	112
Frequency (%)	38.40	51.78	9.82	100

Table 6: Distribution of acceptable answers on the question P2.1 (N = 112)

Question **P2.2**: Identifying statements (2) in the form $\neg B \Rightarrow \neg A$ designed as contraposition of the formula $A \Rightarrow B$ comes in P1.

Efficacy	1	0	∅	Σ
Number	30	71	11	112
Frequency (%)	26.79	63.39	9.82	100

Table 7: Distribution of acceptable answers on the question P2.2 (N = 112)

Question **P2.3**: Identify 'because' in statement (3) and thus represent *'I did not learn Calculus 1'*, as $\neg B$, and as a sufficient condition for the conclusion *'I did not pass the admittance test in mathematics.'* Consequently, represent *'I'm not well prepared in Calculus'* as $\neg A$, thereby conclude that the complete statement is $\neg B \Rightarrow \neg A$.

Efficacy	1	0	∅	Σ
Number	8	93	11	112
Frequency (%)	7.14	83.04	9.82	100

Table 8: Distribution of acceptable answers on the question P.2.3 (N = 112)

We conclude translating conditional colloquial sentences, which represent real-world situations into logical formulas, depends on the complexity of the original sentences. As illustrated in table 6, students demonstrated modest success (P1: 38.40%) in recognizing logical implication when statements or formulas were in the form $A \Rightarrow B$. As illustrated in table 7, fewer students demonstrated success (P2: 26.79%) in recognizing logical implication when the statements or formulas used the contraposition. As illustrated in table 8, the most challenging task was to identify the appropriate logical formulas in declarative conditional sentences. Few students were able to complete this task with success (P3: 7.14%). These results support prior research on students' difficulties in constructing logical implication statements and understanding the connection between these statements and inverse statements.

Conclusion

In this study, we intended to gain insight of student's understanding of logical implication and related concepts. Although there are some limitations to our study, we draw two preliminary conclusions.

First, high school graduates struggle to understand the concepts of logical implication and its contraposition. Second, we argue that with proper instruction first year students of technical colleges could gain understanding of and strengthen their ability to use logical implication and its contrapositive. Moving forward, we suggest that future research builds on this work by studying ways in which first year students of technical colleges can be supported in developing their understanding of logical implication.

Advanced mathematics courses rely heavily on the concepts of logic. In close, we emphasize the value of students succeeding in advanced mathematics courses, such as calculus. As the global economy continues to develop technologically, STEM-related careers become more competitive and lucrative. If students are not explicitly introduced to the concepts and tools of mathematical implication, they will not be prepared for the mathematics courses necessary for success in STEM academic programs, and thus will be marginalized from the career and economic opportunities afforded by STEM-related careers when they graduate from college. On that note, we strongly encourage researchers to continue work in the area of logical implication.

References

[1] Ayalon, M. and Even, R. (2008a). Deductive reasoning: In the eye of the beholder. *Educational Studies in Mathematics*, **69**(3): 235-247.

[2] Ayalon, M. and Even, R. (2008b). Views of mathematical educators on the role of mathematics learning in the development of deductive reasoning. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rotano, and A. Sepulveda (Eds.), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX*, (Vol. 2, pp. 113-120). Mexico: Cinvestav-UMSHN.

[3] Baker, S. (2001). Proofs and logic: An examination of mathematics bridge course proofs. *Unpublished master's thesis.* Cookeville, TN: Tennessee Technological University.

[4] Chin, E.-T., and Tall, D. (2002). Proof as a formal procept in advanced mathematical thinking. In F.-L. Lin (Ed.), *Proceedings of the International Conference on Mathematics: Understanding Proving and Proving to Understand* (pp. 212-221). Taipei, Taiwan: National Taiwan Normal University.

[5] Deloustal-Jorrand, V. (2004). Studyng the mathematical concept of implications through a problem on written proofs, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 pp. 263–270.

[6] Epp, S. S. (2003). The role of logic in teaching proof. *MAA Monthly*, **110**(10): 886-899.

[7] Hanna, G., and de Villiers, M. (2008). ICMI Study 19: Proof and proving in mathematics education (Discussion document). *ZDM-The International Journal of Mathematics Education*, **40**(2): 329-336.

[8] Hoyles, C. and Küchemann, D. (2002). Students' Understandings of Logical Implication, *Educational Studies in Mathematics*, **51**(3): 193-223.

[9] Kuchemann, D. and Hoyles, C. (2002). Students' understanding of a logical implications and its converse, *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, 2002 Vol. 3 pp 241–248.

[10] Lee, K. (2011). *Student's logical reasoning and mathematical proving of implications*, Ph.D. Michigan State University.

[11] Lin, F.L., Lee, Y.S. and Wu, J.Y. (2003). Students' understanding of proof by contradiction. In N.A. Pateman, B.J. Doughterty and J.T. Zilliox, (Eds.) *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 443-450). Honolulu, Hawaii: University of Hawaii.

[12] Milbou, L., Deprez, J. and Laenens, R. (2013). A study on the reintroduction of logic in secondary schools, In: *International conference 'The future of education', 3th edition*, Florence, Italy, 13-14, June 2013, Libreria universitaria.it Edizioni, Available online: http://conference.pixel-online.net/FOE/conferences/foe2013/conferenceproceedings.php

[13] Miller, G. (1956). The magical number seven, plus or minus two: Some limits on our capacity for processing information. *Psychological Review*, **63**(2): 343-355.

[14] Rips, L. J. (1994). *The psychology of proof: Deductive reasoning in human thinking.* Cambridge, MA: The MIT Press.

[15] Saguillo, J. M. (2002). Conception of logical implications, *Logica Trianguli*, **6**: 41-67.

[16] Savic, M. (2012). *Proof and Proving: Logic, Impasses, and the Relationship to Problem Solving*; Ph.D., New Mexico State University, 2012.

[17] Selden, J. and Selden, A. (2009). Teaching proof by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades* (pp. 339-354). New York, NY: Rutledge.

Received by editors 19.04.2016; Revised version 23.05.2016; Available online 30.05.2016.