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ISOGEOMETRIC ANALYSIS OF A SPATIALLY CURVED BERNOULLI-EULER BEAM SUBJECTED TO MOVING LOAD

Abstract

Dynamic analysis of a spatially curved Bernoulli-Euler beam subjected to the moving load is considered in this paper. The isogeometric approach is used for the spatial discretization of the weak form of the equation of motion. Both the reference geometry and the solution space are represented using the same NURBS basis functions that guarantee an accurate description of beam's centerline. The time integration is done by the explicit technique. The presented formulation is validated by the comparison with the existing results from the literature for the curved beam subjected to the constant load moving with the constant velocity.

Keywords: isogeometric approach, Bernoulli-Euler curved beam, moving load

ИЗОГЕМЕТРИЈСКА АНАЛИЗА УТИЦАЈА ПОКРЕНТОГ ОПТЕРЕЋЕЊА НА ПРОСТОРНОЈ КРИВОЛИНИЈСКОЈ БЕРНУЛИ-ОЈЛЕРОВОЈ ГРЕДИ

Сажетак

У овом раду је приказана динамичка анализа просторне криволинијске Бернули-Ојлерове греде под утицајем покретног оптерећења. Изогеометријски приступ је примењен у циљу просторне дискретизације слабе форме једначина кретања греде. Овај приступ се базира на примени истих базних NURBS функција за описивање геометрије и кинематике криволинијске греде, чиме је омогућен тачан приказ системне линије греде. Временска интеграција једначина је извршена применом експлицитне методе. Приказана формулација је валидирана поређењем са резултатима из литературе за случај криволинијске греде оптерећене покретном силом константног интензитета и брзине.

Кључне ријечи: изогеометријски приступ, Бернули-Ојлерова крива греда, покретна сила

1. INTRODUCTION

Beam-like structures are often subjected to dynamic loads during their lifetime. Therefore, the dynamic analysis of beam elements is necessary for an accurate prediction of their real-life behavior. One type of the dynamic loads is the mass that moves along the structure, which is the standard load case for cranes and bridges. The moving mass is usually modeled as a moving force with constant magnitude and direction. Such approach gives a moving load model where the inertial term of the moving mass is neglected. Majority of the research in this field is related to the analysis of a mass moving along a straight beam. One of the earliest investigations was carried out by Stokes in 1849 [1], where the influence of the moving mass on the plane straight Bernoulli-Euler beam was considered analytically using the moving load model.

Due to the aesthetic and functional requirements in the design process, curved spatial beam elements cannot be avoided. Geometrical model of the curved spatial beam requires the spatial curve, which can be obtained using computer-aided design (CAD) software packages. To accurately describe the free-form curves and the curves of conic sections such as circle, ellipse, parabola and hyperbola, CAD packages utilize the NURBS (Non-Uniform Rational B-Spline) basis functions.

In order to conduct the general dynamic analysis of complex spatially curved beams subjected to the moving load, numerical methods are essential. Nowadays, the Finite Element Method (FEM) is implemented in most software packages for structural analysis. A direct relation between CAD and FEM has not yet been established [2], leading to costly and time-consuming iterative design process. The isogeometric approach establishes a direct relationship between the geometry and the unknown fields of the structure [2]. This is enabled by using the NURBS functions as basis functions of the numerical model's reference and solution spaces. Therefore, the same basis functions are applied for the geometry and kinematics, which eliminates the errors due to the geometric approximation in a spatially discretized model. In order to improve the mesh, three types of mesh refinement are used in the isogeometric approach, denoted as H-, P-, and K-methods [2].

A dynamic analysis of an arbitrarily curved spatial beam subjected to the moving load is studied in this paper. A short review on the NURBS basis function is given in Section 2 and followed by the beam's geometry representation. The governing equation of motion of the Bernoulli-Euler isogeometric beam element is briefly given in Section 4, while more details can be found in authors' previous paper [3]. The moving load model is presented in Section 5 and followed by the numerical example of spatially curved beam subjected to the moving load in Section 6. At the end, the main conclusions have been drawn.

2. BASICS OF NURBS

The exact shape of an arbitrary curve $\mathbf{C}(\xi)$ in Euclidean 3D space can be represented as:

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\mathbf{C}_i \tag{1}$$

where $R_{i,p}(\xi)$ is the i-th NURBS basis function, p is the function degree, C_i is the position of the control point i, while n is the number of basis functions and control points. NURBS functions are derived from the B-spline functions:

$$R_{i,p}(\xi) = \sum_{i=1}^{n} \frac{N_{i,p}(\xi)\cdot w_i}{\sum_{j=1}^{n} N_{j,p}(\xi)\cdot w_i} \tag{2}$$

where w_i is the *i*-th function weight. In order to define B-spline functions, Cox de Boor algorithm is often applied [4].

For the case of a zero degree ($p = 0$), the B-spline functions are defined as:

$$N_{i,0}(\xi) = \begin{cases} 1, & \text{if } \xi \in [\xi_i, \xi_{i+1}[\\ 0, & \text{otherwise} \end{cases} \tag{3}$$

while for the polynomial degree greater than zero (p > 0):

$$N_{i,p}(\xi)=\begin{cases}\dfrac{\xi-\xi_i}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi)+\dfrac{\xi_{i+p-1}-\xi}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi), & \text{if } \xi\in[\xi_i,\xi_{i+p+1}[\\ 0, & \text{otherwise}\end{cases} \tag{4}$$

The B-spline functions are polynomial functions defined in parametric domain (ξ) using the knot vector. This vector represents a set of non-decreasing real numbers, the knots.
Important properties of the B-spline function, as well as the NURBS functions, used in the following derivations, are the non-negativity and the partition of unity over the parametric domain. More about the B-spline and NURBS basis functions can be found in [4].

3. BEAM GEOMETRY

Due to the assumptions of beam theories, all kinematic and stress quantities of a beam can be given as a function of beam's centerline. In general, the beam's centerline has an arbitrary shape in the Euclidean three-dimensional space, forming a curved line. The formulation of a curved beam is conducted using the curvilinear coordinate system attached to the beam's centerline.

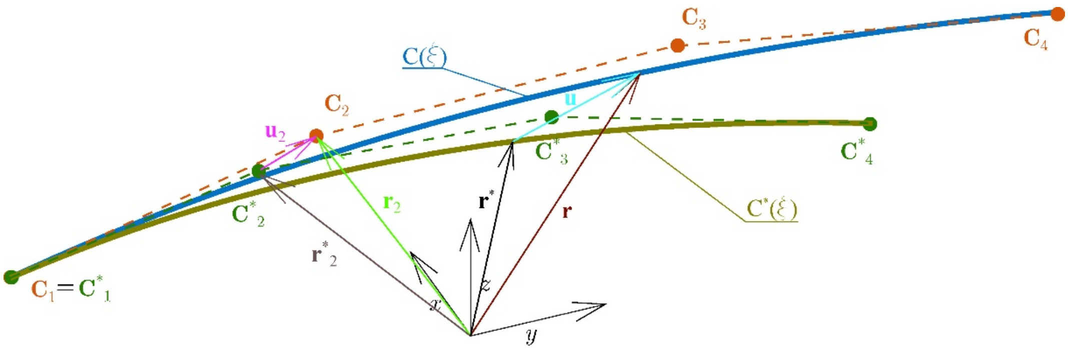

Figure 1. *Centerline of a curved beam with corresponding control points*

Using the NURBS parameterization, the position vector of a curved line is defined as:

$$\mathbf{r}(\xi)=\sum_{i=1}^{n}R_{i,p}(\xi)\mathbf{r}_i \tag{5}$$

where $\mathbf{r}_i$ is the position of the i-th control point, Figure 1. To fully define the beam continuum, a unique triad must be attached to each point of a curve. Here, this triad is aligned with the Frenet-Serret frame. The basis vectors are defined using the well-known relations of differential geometry [5] and relations between the arc-length and NURBS parameterizations:

$$\mathbf{g}_1=\mathbf{r}_{,1}=\frac{d\mathbf{r}}{d\xi}=\frac{d\mathbf{r}}{ds}\frac{ds}{d\xi}=\mathbf{t}\frac{ds}{d\xi}=\mathbf{t}\sqrt{g_{11}}$$

$$\mathbf{g}_2=\mathbf{n}=\frac{1}{K}\frac{d\xi}{ds}\frac{d}{d\xi}\left(\frac{\mathbf{g}_1}{|\mathbf{g}_1|}\right) \tag{6}$$

$$\mathbf{g}_3=\mathbf{b}=\frac{\mathbf{g}_1\times\mathbf{n}}{|\mathbf{g}_1\times\mathbf{n}|}$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are orthonormal basis vectors of beam's centerline obtained using arc-length parameterization (Frenet-Serret frame of reference), while $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ form orthogonal vector basis with respect to the parametric coordinate. The vector $\mathbf{g}_1$ is collinear with the tangent $\mathbf{t}$, while the vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ are in the beam's cross- section plane. In the previous relations, K is the modulus of curvature, while g_{11} is the component of the metric tensor of the beam's centerline:

$$g_{ij}=\begin{bmatrix}g_{11}&0&0\\0&1&0\\0&0&1\end{bmatrix},\ \det\left(g_{ij}\right)=g_{11}=g \tag{7}$$

By using the well-known Frenet-Serret relations and Eq. (6), the derivatives of the basis vectors with respect to the parametric coordinate are:

$$\begin{bmatrix} \mathbf{g}_{1,1} \\ \mathbf{g}_{2,1} \\ \mathbf{g}_{3,1} \end{bmatrix} = \begin{bmatrix} \Gamma^1_{11} & gK & 0 \\ -gK & 0 & \sqrt{g}\tau \\ 0 & -\sqrt{g}\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} \tag{8}$$

where $()_{,1}$ represents the derivative with respect to the parameter ξ, $\Gamma^1{}_{11}$ is the Christoffel symbol of the second kind, and τ is the torsion of the beam's centerline.

In this paper, the beam cross-section principal axes coincide with the basis vectors $\mathbf{g}_2$ and $\mathbf{g}_3$. If this condition is not satisfied, the basis vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ need to be rotated around the basis vector $\mathbf{g}_1$ to align them with the principal axes, forming a new moving frame of reference [6].

Using the introduced basis vectors, the position vector of an arbitrary point of the beam can be defined as:

$$\hat{\mathbf{r}} = \mathbf{r} + \eta \mathbf{g}_2 + \zeta \mathbf{g}_3 \tag{9}$$

where η and ζ are the coordinates along the principal axes. Consequently, the first basis vector of an arbitrary point is defined as:

$$\hat{\mathbf{g}}_1 = \frac{d\hat{\mathbf{r}}}{d\xi} = \mathbf{g}_{1,1} + \eta \mathbf{g}_{2,1} + \zeta \mathbf{g}_{3,1} = g_0 \mathbf{g}_1 + \eta K_1 \mathbf{g}_2 + \zeta K_1 \mathbf{g}_3 \tag{10}$$

Due to the assumption of the rigid cross-section, the vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ are translated from the beam's centerline to an arbitrary point. By observing the metric tensor of an arbitrary point, it is evident that the basis vector $\hat{\mathbf{g}}_1$ is not perpendicular to the vectors $\mathbf{g}_2$ and $\mathbf{g}_3$. To keep the same orthogonal vector basis over the whole cross section, a new parameterization is performed by introducing ξ_λ coordinate. More about this new frame of reference can be found in [6].

4. ISOGEOMETRIC BERNOULLI-EULER BEAM FORMULATION

Due to the external impact, the beam's centerline has a new position defined with the current position vector:

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u} \tag{11}$$

where $\mathbf{u}$ represents the displacement vector of the beam's centerline. Using the isogeometric approach, the displacement vector can be represented as:

$$\mathbf{u}(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)\mathbf{u}_i = \sum_{i=1}^{n} R_{i,p}(\xi) u_i^m \mathbf{i}_m \tag{12}$$

where $\mathbf{u}_i$ is the displacement vector of the i-th control point. As can be noticed, the displacement vector and the reference geometry of beam's centerline are represented using the same basis functions, which is the fundamental property of the isogeometric approach.

Formulation of the spatial Bernoulli-Euler isogeometric beam is conducted by applying the convective coordinate system, therefore the position vector of an arbitrary point of a deformed beam is:

$$\hat{\mathbf{r}}^* = \hat{\mathbf{r}} + \eta \mathbf{g}_2^* + \zeta \mathbf{g}_3^* \tag{13}$$

The basis vectors of the deformed configuration can be expressed as:

$$\mathbf{g}_m^* = \mathbf{g}_m + \mathbf{u}_m \tag{14}$$

where $\mathbf{u}_m$ represents the increment of the m-th basis vector.

Using Eqs. (9), (13) and (14), the displacement vector of an arbitrary point of a beam is defined as:

$$\hat{\mathbf{u}} = \mathbf{u} + \eta \mathbf{u}_2 + \zeta \mathbf{u}_3 \tag{15}$$

Using Eq. (15), the acceleration vector of an arbitrary point is obtained as the second material derivative:

$$\hat{\mathbf{a}} = \left(\ddot{\hat{\mathbf{u}}}\right) = \ddot{\mathbf{u}} + \eta\ddot{\mathbf{u}}_2 + \zeta\ddot{\mathbf{u}}_3 \tag{16}$$

In addition, the variation of displacement of an arbitrary point is obtained from Eq. (15) as:

$$\delta\hat{\mathbf{u}} = \delta\mathbf{u} + \eta\delta\mathbf{u}_2 + \zeta\delta\mathbf{u}_3 \tag{17}$$

The components of the Green-Lagrange strain tensor in convective coordinate system are:

$$\hat{\varepsilon}_{ij} = \frac{1}{2}\left(\hat{\mathbf{g}}_i^* \cdot \hat{\mathbf{g}}_j^* - \hat{\mathbf{g}}_i \cdot \hat{\mathbf{g}}_j\right) = \frac{1}{2}\left(\hat{g}_{ij}^* - \hat{g}_{ij}\right) \tag{18}$$

In the case of beams, the non-zero components of strain tensor are:

$$\hat{\varepsilon}_{11} = \frac{1}{2}\left(\hat{g}_{11}^* - \hat{g}_{11}\right)$$

$$\hat{\varepsilon}_{12} = \frac{1}{2}\left(\hat{g}_{12}^* - \hat{g}_{12}\right) \tag{19}$$

$$\hat{\varepsilon}_{13} = \frac{1}{2}\left(\hat{g}_{13}^* - \hat{g}_{13}\right)$$

By substituting the Bernoulli-Euler assumptions into the previous equations, the required kinematic relations are obtained. Degrees of freedom of the isogeometric Bernoulli-Euler beam are the displacements of the beam's centerline and the torsional rotation of the beam's cross-section. The derivations of the kinematic relations can be found in detail in [3].
Assuming the linear elastic material behavior, the constitutive relations can be written as:

$$\hat{S}_i^j = 2\mu\hat{\varepsilon}_i^j + \lambda\delta_i^j\hat{\varepsilon}_m^m \tag{20}$$

where $\hat{S}_j^i$ represents the mixed components of the second Piola-Kirchoff stress tensor, while μ and λ are Lame's constants.
In order to obtain the equations of motion, the principle of virtual work is used:

$$\int_{V_0} \rho\hat{\mathbf{a}} \cdot \delta\hat{\mathbf{u}} dV_0 + \int_{V_0} \mathbf{S} : \delta\boldsymbol{\varepsilon} dV_0 = \int_{V_0} \hat{\mathbf{f}}\delta\hat{\mathbf{u}} dV_0 \tag{21}$$

where ρ is the mass density, while $\hat{\mathbf{f}}$ is the external load. By substituting Eqs. (16), (17), (19) and (20) into Eq. (21), the governing equation of motion of Bernoulli-Euler isogeometric curved beam subjected to the moving load is obtained:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Q} \tag{22}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{Q}$ is the vector of equivalent control forces, while $\mathbf{q}$ is the displacement vector of the control points. Solution of this equation requires application of a time integration procedure. The explicit step by step integration has been applied using the finite differences method [7]. The reduced integration has been applied in Eq. (21) [8], and implemented into the original MATLAB code [9].

5. MOVING LOAD

A mass moving along the structure generates a dynamic response. This load can be modeled as a single load with constant magnitude and direction, $\mathbf{f}_0$, that moves along a beam with the constant velocity:

$$\mathbf{f}(t) = \mathbf{f}_0 \cdot \delta\left(\xi - V_\xi t\right)$$

$$V_\xi = \frac{d\xi}{dt} = \frac{ds}{dt}\frac{d\xi}{ds} = \frac{V}{\sqrt{g}} \tag{23}$$

where V_ξ and V are the magnitudes of velocity with respect to the parametric and arc-length coordinates, respectively.

The vector of equivalent forces of the i-th control point in the case of a point load is:

$$\mathbf{Q}_i = \int_{d\xi} \mathbf{f} \cdot R_{i,p}(\xi)\sqrt{g}\,d\xi = \mathbf{f} \cdot R_{i,p}(\xi_m)\sqrt{g} \tag{24}$$

where ξ_m is the position of the moving load on a beam.

6. NUMERICAL EXAMPLE

The validation study of the proposed formulation is given in this section. The horizontally curved arch with the length L = 24 m and the subtended angle α = 30° is subjected to the out-of-plane and the in-plane moving load with constant speed V = 40 m/s. The displacements and the torsional rotations at both ends of the beam are restrained. The beam's geometry has been modeled with the cubic NURBS, as given in Figure 2.

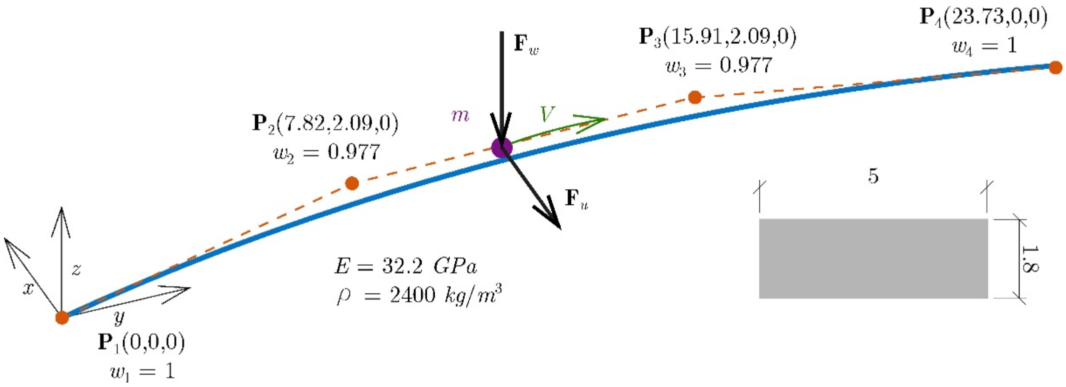

Figure 2. *The arch subjected to the moving load*

The material is homogeneous and defined using the Young's modulus E = 32.2 GPa, the Poisson's ratio ν = 0.2 and the mass density ρ = 2400 kg/m^3, while the cross-section is rectangular with the dimensions b/h = 5/1.8 m. The beam is subjected to the out-of-plane load $\mathbf{F}_w$ = -293.32 kN and the in-plane load $\mathbf{F}_u$ = 1043.71 kN directed towards the arch center. The displacements of the beam's midpoint obtained using the isogeometric approach have been compared with the semi-analytical results from the literature, applicable only for the simply supported arches [10]. It is important to point out that the beam model presented in [10] is based on the Timoshenko beam theory. In this example, the validation study is conducted as well as the convergence study using the P- refinement procedure.

The in-plane (u) and the out-of-plane (w) displacement components of the midpoint obtained using the P-refinement procedure are presented respectively in Figure 3 and Figure 4.

In addition, the same example is used to calculate the influence line of the beam's midpoint displacement components by neglecting the inertial part of the beam in the principle of virtual work.

By comparing the results of the beam's midpoint displacements obtained using the dynamic and static analysis, significant difference can be observed, especially for the case of the out-of-plane displacement.

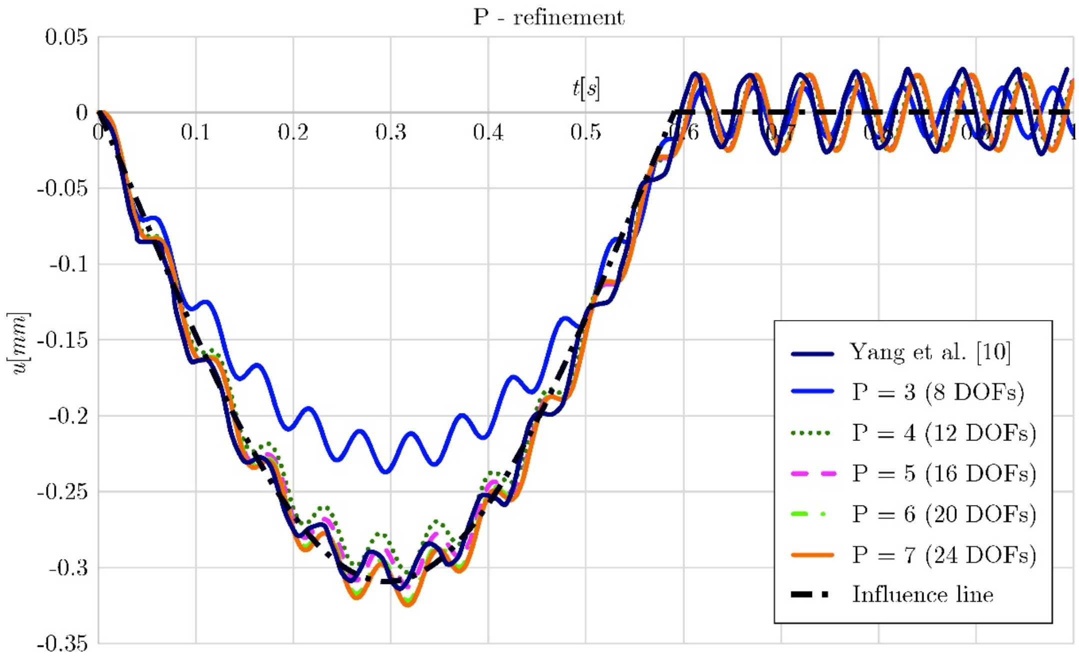

Figure 3. *Comparison of in-plane displacement component (u) of the beam's midpoint*

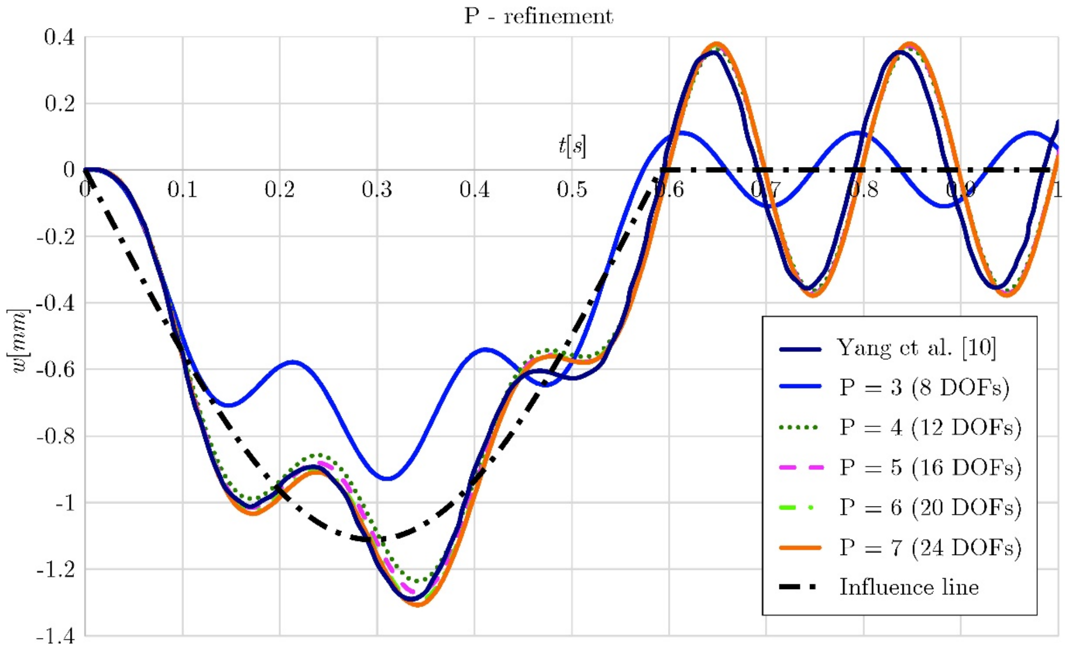

Figure 4. *Comparison of the out-of-plane displacement component (w) of the beam's midpoint*

7. CONCLUSIONS

The dynamic analysis of spatially curved Bernoulli-Euler beam subjected to the moving load is briefly presented. Spatial discretization is performed by the isogeometric approach, while the explicit procedure is used for the time integration. To validate the proposed method, the numerical study of the curved spatial beam subjected to the point load has been carried out. Satisfactory agreement has been observed between the results obtained using the proposed method and the results from the literature. In addition, the influence line of the beam's midpoint has been calculated, and the difference between the static and dynamic results is shown. The accurate modeling of the moving load has significant influence on the response of a beam. In future work, the inertial part of the moving load will be taken into account, modeling the moving mass more accurately. Also, implicit procedures and effects of the higher-order metric will be considered [11, 12].

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